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PENALTY METHOD IN DESIGN OPTIMIZATION OF SYSTEMS GOVERNED BY A UNILATERAL BOUNDARY VALUE PROBLEM

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Résumé : Dans ce travail on étudie le cas de l'identification du domaine dans les problèmes, l'état dont lesquel est décrit par les inéquations variationnelles elliptiques du type de Signorini-Fichera. En utilisant la méthode de pénalisation, l'inéquation d'état est remplacée par une famille des équations elliptiques non-linéaires. On démontre que les solutions des problèmes de l'identification gouvernés par des équations d'état pénalisées sont dans une certaine liaison au problème original de l'identification.

Summary : A method of penalization is used to transform an optimal design problem governed by variational inequalities to the optimal design problem governed by equations. It is shown that the corresponding optimal designs (associated with penalized problems) are in an appropriate sense close to the optimal design of the original problem.

1. - INTRODUCTION

Many practical problems lead to finding an optimal design of a mechanical system, the behaviour of which is described by equations, corresponding to some law of physics. In recent years much attention has ben given to this type of the design optimization - from the computational point of view as well as from the theoretical point of view ([1], [5], [6], [13], [14], [15] and bibliography therein).

On the other hand, there are many physical situations described by variational inequalities (unilateral of free boundary value problems, see for example [2], [8], [9]). A relatively small amount of papers is devoted to optimal control problems governed by inequalities. Moreover, most of them analyse the case when the control variables appear on the right hand side of inequalities ([10] , [11] , [12]). Recently, in [7] the mathematical analysis of design optimization of systems governed by a unilateral boundary value problem is given.

In the present paper the same type of problem of optimal design as in [7] is considered. Hlaváček and Nečas give in [7] a proof of existence of a solution for different cost functionals and for one common state problem, which is formulated in terms of variational inequality on a variable domain. In this paper a different approach is used. The variational inequality is replaced by a family of the penalized problems, each of them is given by the classical elliptic boundary value problem. We show that the corresponding optimal designs (associated with the penalized problems) are close (in an appropriate sense) to an optimal design of the original problem.

The main advantage of the approach of this paper consists in the fact that only a small modification of existing algorithms enables us to apply them for solving numerically the problem in question. In [4] numerical realization of methods presented here is given.

2. - SETTING OF THE PROBLEM

Let $\Omega \subset \mathbb{R}^2$ be a domain with Lipschitz boundary $\partial\Omega$. By $H^k(\Omega)$ ($k \geq 0$, integer) we denote the classical Sobolev space of functions, the generalized derivates of which up to order k are square integrable in Ω ($L^2(\Omega) := H^0(\Omega)$). The norm on $H^k(\Omega)$ will be denoted by $\| . \|_{k,\Omega}$. $H^{-k}(\Omega)$ ($k > 0$, integer, i.e. $k \in \mathbb{N}$) is the dual space to $H^k(\Omega)$. Furthermore, $H^k(\Gamma)$ (Γ non-empty open set of $\partial\Omega$) denotes the Sobolev space of functions, whose domain is Γ ; $\| . \|_{k,\Gamma}$ is the corresponding norm in $H^k(\Gamma)$. By $E(\Omega)$ we mean the set of all infinite times continuous differentiable functions, which can be continuously extended with all their derivatives, up to the boundary. The set of all functions from $E(\overline{\Omega})$, vanishing in some neighbourhood of $\partial\Omega$ will be denoted by $D(\Omega)$.

In this paper we shall study the problem :

$$(P) \qquad \begin{cases} \text{Find } w \in U_{ad} \text{ such that} \\ J(w) \leq J(v) \quad \text{for all} \quad v \in U_{ad}. \end{cases}$$

Here U_{ad} is the set of admissible functions (controls), defined by

$$U_{ad} := \left\{ v \in C^{0,1}([0,1]) \mid 0 < \alpha \leqslant v(x_2) \leqslant \beta,\ \left| \frac{dv}{\partial x_2} \right| \leqslant C_1, \right.$$

$$\left. \int_0^1 v(x_2) dx_2 = C_2,\ \alpha, \beta, C_1, C_2 > 0 \text{ given constants} \right\}.$$

The cost functional J is given by

$$J(v) = \int_{\Omega(v)} |y(v) - z_d|^2 \, dx_1 dx_2 = \| y(v) - z_d \|^2_{0,\Omega(v)},$$

where z_d is an element of $L^2(\Omega_{\tilde{\beta}})$,

$$\Omega_{\tilde{\beta}} := (0,\tilde{\beta}) \times (0,1),\ \tilde{\beta} > \beta$$

and $y = y(v)$ is the solution of the unilateral boundary value problem (the state inequality) :

$$(2.1) \quad \begin{cases} \text{Find } y = y(v) \in K(v) \text{ such that} \\ (\text{grad } y, \text{grad } (\xi - y))_{0,\Omega(v)} \geqslant (f, \xi - y)_{0,\Omega(v)} \text{ for all } \xi \in K(v). \end{cases}$$

Here $f \in L^2(\Omega_{\tilde{\beta}})$ and (see Figure 2.1)

$$\Omega(v) := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 \in (0,1),\ 0 < x_1 < v(x_2),\ v \in U_{ad}\},$$

$$\Gamma(v) := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 \in (0,1),\ x_1 = v(x_2),\ v \in U_{ad}\},$$

$$V(\Omega(v)) := \{\omega \in H^1(\Omega(v)) \mid \omega = 0 \text{ on } \partial\Omega(v) \setminus \Gamma(v)\}$$

$$K(v) := \{\omega \in V(\Omega(v)) \mid \omega \geqslant 0 \text{ on } \Gamma(v)\}.$$

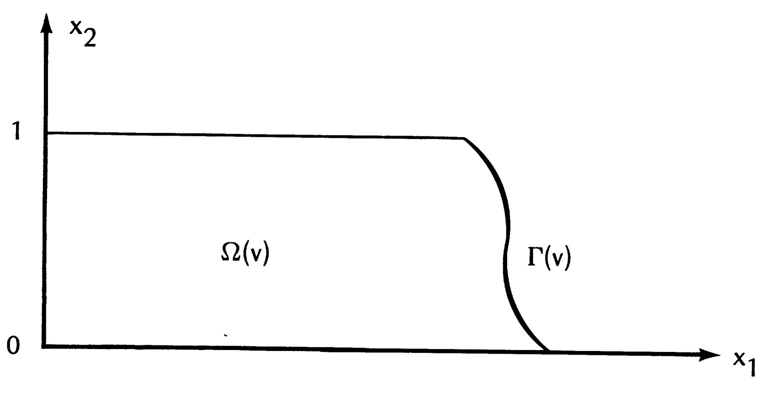

Figure 2.1

Remark 2.1. By applying the Green's formula to (2.1) one can easily prove that $y \in K(v)$ satisfies in $\Omega(v)$ the Poisson equation with the boundary conditions of Signorini's type. On a given part of the boundary the homogeneous Dirichlet condition is prescribed and the remaining part - with unilateral conditions - has to be determined :

$$\begin{cases} -\Delta y = f & \text{in} \quad \Omega(v) \\ y = 0 & \text{on} \quad \partial\Omega \setminus \Gamma(v) \\ y \geq 0 , \ \dfrac{\partial y}{\partial n} \geq 0, \ y \dfrac{\partial}{\partial n} y = 0 & \text{on} \quad \Gamma(v) \end{cases}$$

As the solution y of (.1) depends on $\Omega(v)$, among others, i.e. $v \in U_{ad}$, we shall write $y = y(v)$ to emphasize this dependence. Thus, (P) is the problem on *optimal shape* of Ω, minimizing the cost functional J. The existence of a solution for Problem (P) is proved in [7] .

It is well-known that the penalty method enables us to replace the unilateral boundary value problem (2.1) by a family of classical elliptic boundary value problems. Let

$$P : H^1(\Omega(v)) \to H^{-1}(\Omega(v))$$

be a *penalty operator,* that is

$$\text{(2.2)} \qquad \begin{cases} \text{Ker}(P) = \text{kernel of} \quad P = K(v) \\ P \text{ is Lipschitz continuous} \\ P \text{ is monotone.} \end{cases}$$

Instead of (2.1) we shall consider a family of problems :

$$(2.1)_\epsilon \qquad \begin{cases} \text{Find } y_\epsilon = y_\epsilon(v) \in V(\Omega(v)) \text{ such that} \\ (\text{grad } y_\epsilon \, , \text{ grad } \xi)_{0,\Omega(v)} + \dfrac{1}{\epsilon}(P(y_\epsilon), \xi)_V = (f,\xi)_{0,\Omega(v)} \quad \text{for all} \quad \xi \in V(\Omega(v)). \end{cases}$$

The symbol $(. \, , .)_V$ denotes the duality pairing between $H^{-1}(\Omega(v))$ and $H^1(\Omega(v))$. It is known (see [3] and [9]) that $y_\epsilon(v) \to y(v)$, $\epsilon \to 0_+$ in the $H^1(\Omega(v))$-norm.

The above given penalty approach suggests us the following idea : Let us study the optimal design problem, in which the state inequality (2.1) is replaced by a family of state equations $(2.1)_\epsilon$. More precisely :

$$(P)_\epsilon \qquad \begin{cases} \text{Find } w_\epsilon \in U_{ad} \text{ such that} \\ J(w_\epsilon) \leq J(v) \quad \text{for all} \quad v \in U_{ad} \, , \end{cases}$$

where

$$J(v) = \| y_\epsilon(v) - z_d \|^2_{0,\Omega(v)},$$

and $y_\epsilon(v)$ is the solution of $(2.1)_\epsilon$.

A natural question arises : What is the relation between solutions for Problem (P) and Problem $(P)_\epsilon$ if $\epsilon \to 0_+$? Before going closer to this question we prove that Problem $(P)_\epsilon$ has a solution for any $\epsilon > 0$.

3. - ANALYSIS OF PROBLEM $(P)_\epsilon$

Henceforth, we shall assume that the penalty operator $P : H^1(\Omega(v)) \to H^{-1}(\Omega(v))$ is of the form

$$(3.1) \qquad (P(\omega),\xi)_V = -\int_0^1 \omega^-(v(x_2),x_2)\,\xi(v(x_2),x_2)dx_2\,,$$

where ω^- denotes the negative part of ω $(a^- := (|a| - a)/2)$. It is easily seen that operator P, defined by (3.1), has the properties (2.2). The main result of this section is

THEOREM 3.1. *For any* $\epsilon > 0$ *there exists a solution* $w_\epsilon \in U_{ad}$ *of Problem* $(P)_\epsilon$.

Proof. To simplify notations we set $\epsilon = 1$ and we shall write y instead of y_ϵ. Let $\{v_n\}$, $v_n \in U_{ad}$, be a minimizing sequence for the cost functional J, i.e.

$$q := \inf_{v \in U_{ad}} J(v) = \lim_{n\to\infty} J(v_n) = \lim_{n\to\infty} \| y_n - z_d \|^2_{0,\Omega_n}\,,$$

where $y_n := y(v_n) \in V_n := V(\Omega_n)$, $\Omega_n := \Omega(v_n)$, is the unique solution of the equation

$$(3.2) \qquad (\text{grad}\, y_n\,, \text{grad}\,\xi)_{0,\Omega_n} + (P(y_n),\xi)_{V_n} = (f,\xi)_{0,\Omega_n} \quad \text{for all} \quad \xi \in V_n\,.$$

We first prove that $\{ \| y_n \|_{1,\Omega_n} \}$ is bounded. By setting $\xi = y_n$ in (3.2) we obtain

$$(3.3) \qquad \begin{aligned} \| \text{grad}\, y_n \|^2_{0,\Omega_n} &= (\text{grad}\, y_n\,, \text{grad}\, y_n)_{0,\Omega_n} \\ &\leqslant (\text{grad}\, y_n\,, \text{grad}\, y_n)_{0,\Omega_n} + (P(y_n),y_n)_{V_n} \\ &= (f,y_n)_{0,\Omega_n} \leqslant \| f \|_{0,\Omega_{\widetilde{\beta}}} \| y_n \|_{0,\Omega_n}\,. \end{aligned}$$

According to the Poincaré-Friedrichs inequality

$$\| y_n \|_{0,\Omega_n} \leqslant \beta \| \operatorname{grad} y_n \|_{0,\Omega_n} \text{ for all } y_n \in V_n .$$

Thus, by (3.3) there exists a positive constant C such that

$$\| y_n \|_{1,\Omega_n} \leqslant C . \tag{3.4}$$

Taking into account the definition of U_{ad} we see that there exists a subsequence $\{v_{n'}\} \subset \{v_n\}$ and an element $w \in U_{ad}$ such that

$$v_{n'} \rightrightarrows w \text{ (uniformly) in } [0,1] .$$

To simplify the notations, we shall write $\{v_n\}$ instead of $\{v_{n'}\}$. Let $m \in \mathbb{N}$ be fixed. Then there exists $n_0 = n_0(m)$ such that for any $n \geqslant n_0(m)$

$$\Omega_n \supset \overline{G}_m ,$$

where (Figure 3.1)

$$G_m := \{ (x_1,x_2) \in \mathbb{R}^2 \mid x_2 \in (0,1), 0 < x_1 < w(x_2) - 1/m \} .$$

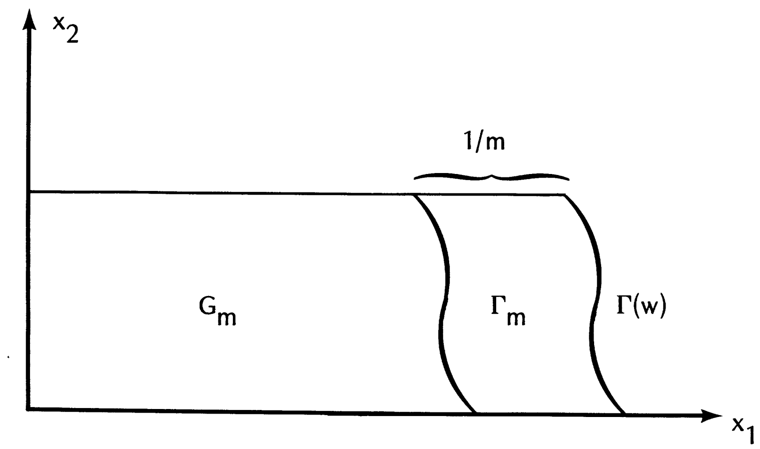

Figure 3.1

As

$$\| y_n \|_{1,G_m} \leqslant \| y_n \|_{1,\Omega_n} \leqslant C,^{(1)}$$

one may extract a subsequence $\{y_{n_1}\} \subset \{y_n\}$ such that

$$y_{n_1} \rightharpoonup y^{(m)} \text{ (weakly) in } H^1(G_m),$$

where $y^{(m)} \in H^1(G_m)$ and $y^{(m)} = 0$ on $\partial G_m \setminus \Gamma_m$, $\Gamma_m = w(x_2) - 1/m$.

Similarly, as $G_{m+1} \supset \bar{G}_m$, there exists a subsequence $\{y_{n_2}\} \subset \{y_{n_1}\}$ and an element $y^{(m+1)} \in H^1(G_{m+1})$, $y^{(m+1)} = 0$ on $\partial G_{m+1} - \Gamma_{m+1}$ such that

$$y_{n_2} \rightharpoonup y^{(m+1)} \text{ in } H^1(G_{m+1}).$$

Evidently,

$$y^{(m)} = y^{(m+1)} \text{ in } G_m.$$

Proceeding in this way for any m, $m \to \infty$, we can construct a subsequence $\{y_{n_k}\} \subset \{y_{n_{k-1}}\}$ such that

$$y_{n_k} \rightharpoonup y^{(m+k-1)} \text{ in } H^1(G_{m+k-1}),\ k = 1,2,..., \tag{3.5}$$

where $y^{(m+k-1)} \in H^1(G_{m+k-1})$, $y^{(m+k-1)} = 0$ on $\partial G_{m+k-1} \setminus \Gamma_{m+k-1}$ and

$$y^{(m+k-1)} = y^{(j)} \text{ in } G_j,\ j < m + k - 1. \tag{3.6}$$

Let $\{y_n^{(D)}\}$ be a diagonal sequence, constructed by means of $\{y_{n_k}\}$, k = 1,2,... . From (3.5) and (3.4) it follows that for any m,

$$y_n^D \rightharpoonup y\,|_{G_m} \text{ in } H^1(G_m), \tag{3.7}$$

where $y\,|_{G_m} := y^{(m)}$. Clearly,

$$y \in H^1(\Omega(w)) \text{ and } y = 0 \text{ on } \partial\Omega(w) \setminus \Gamma(w).$$

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(1) In what follows, C *will denote a generic strict positive constant with different values on different places. Moreover,* C *will be independent on* n *and* m.

We shall show that $y = y(w)$, i.e. y solves the equation

(3.8) $$(\text{grad } y, \text{grad } \xi)_{0,\Omega(w)} + (P(y), \xi)_w = (f,\xi)_{0,\Omega(w)} \quad \text{for all } \xi \in V(\Omega(w)).$$

Next, we shall write shortly y_n and v_n instead of y_n^D and v_n^D. Let $\xi \in H_0^1(\Omega_{\tilde{\beta}})$ be an arbitrary element. From the definition of $\{y_n\}$ it follows that

(3.9) $$(\text{grad } y_n, \text{grad } \xi)_{0,\Omega_n} + (P(y_n), \xi)_{v_n} = (f,\xi)_{0,\Omega_n} \quad \text{for all } \xi \in H_0^1(\Omega_{\tilde{\beta}}).$$

Passing to the limit on the right hand side of (3.9) we have

(3.10) $$(f,\xi)_{0,\Omega_n} \to (f,\xi)_{0,\Omega(w)} \quad \text{for } n \to \infty.$$

Let m be fixed and $n \geqslant n_0(m)$ (that is, let n be such that $\Omega_n \supset \overline{G}_m$). Then

(3.11) $$(\text{grad } y_n, \text{grad } \xi)_{0,\Omega_n} = (\text{grad } y_n, \text{grad } \xi)_{0,G_m} + (\text{grad } y_n, \text{grad } \xi)_{0,\Omega_n \setminus \Omega(w)}$$
$$+ (\text{grad } y_n, \text{grad } \xi)_{0,(\Omega(w) \setminus G_m) \cap \Omega_n}.$$

Now, by virtue of (3.7), it holds that

(3.12) $$(\text{grad } y_n, \text{grad } \xi)_{0,G_m} \to (\text{grad } y, \text{grad } \xi)_{0,G_m}, \text{ for } n \to \infty.$$

Furthermore, we have that

(3.13) $$|(\text{grad } y_n, \text{grad } \xi)_{0,\Omega_n \setminus \Omega(w)}| \leqslant \| y_n \|_{1,\Omega_n} \| \xi \|_{1,\Omega_n \setminus \Omega(w)} \to 0,$$

for $n \to \infty$ and by (3.4) that

(3.14) $$|(\text{grad } y_n, \text{grad } \xi)_{0,(\Omega(w) \setminus G_m) \cap \Omega_n}| \leqslant \| y_n \|_{1,\Omega_n} \| \xi \|_{1,\Omega(w) \setminus G_m} \leqslant$$
$$\leqslant C \| \xi \|_{1,\Omega(w) \setminus G_m}.$$

By applying in turn (3.12), (3.13) and (3.14) to decomposition (3.11) we obtain that the convergence

(3.15) $$(\text{grad } y_n, \text{grad } \xi)_{0,\Omega_n} \to (\text{grad } y, \text{grad } \xi)_{0,\Omega(w)}$$

holds for $m \to \infty$ (and, consequently, for $n \to \infty$).

For a moment, let us suppose that

$$(P(y_n), \xi)_{v_n} \to (P(y), \xi)_w, \text{ for } n \to \infty. \tag{3.16}$$

This will be proved in Lemma 3.2. Then, by (3.10), (3.15) and (3.16) y satisfies the equation

$$(\text{grad } y, \text{grad } \xi)_{0,\Omega(w)} + (P(y), \xi)_w = (f,\xi)_{0,\Omega(w)} \text{ for all } \xi \in H_0^1(\Omega_{\tilde{\beta}}). \tag{3.17}$$

Let $\xi \in V(\Omega(w))$, and let $\tilde{\xi} \in H_0^1(\Omega_{\tilde{\beta}})$ be its continuous prolongation. Then there exists a sequence $\{\tilde{\xi}_n\} \subset D(\Omega_{\tilde{\beta}})$ such that

$$\tilde{\xi}_n \to \tilde{\xi} \text{ in } H^1(\Omega_{\tilde{\beta}}). \tag{3.18}$$

Replacing in (3.17) ξ by $\tilde{\xi}_n$ and passing to the limit for $n \to \infty$ we obtain

$$(\text{grad } y, \text{grad } \tilde{\xi})_{0,\Omega(w)} + (P(y), \tilde{\xi})_w = (f, \tilde{\xi})_{0,\Omega(w)} \text{ for all } \tilde{\xi} \in H_0^1(\Omega_{\tilde{\beta}}), \tag{3.19}$$

or equivalently, (3.8) holds for any $\xi \in V(\Omega(w))$. In other words, we have $y = y(w)$. Consequently, it remains to verify that $w \in U_{ad}$ is a minimizer of J on U_{ad}.

Let $\{v_n\}$, $v_n \in U_{ad}$, be a minimizing sequence of J, i.e.

$$q = \inf_{v \in U_{ad}} J(v) = \lim_{n \to \infty} J(v_n).$$

Then

$$J(v_n) = \| y_n - z_d \|^2_{0,\Omega_n} = \| y_n - z_d \|^2_{0,G_m} + \| y_n - z_d \|^2_{0,\Omega_n \setminus G_m} \geqslant \| y_n - z_d \|^2_{0,G_m}$$

holds for any m and $n \geqslant n_0(m)$ (the meaning of $n_0(m)$ and G_m is the same as before).

If we restrict to diagonal sequences $\{y_n^D\}$ and $\{v_n^D\}$, denote them again by $\{y_n\}$ and by $\{v_n\}$, and take into account the convergence (3.7), we finally obtain

$$q := \lim_{n \to \infty} J(v_n) \geqslant \lim_{n \to \infty} \| y_n - z_d \|^2_{0,G_m} =: \| y(w) - z_d \|^2_{0,G_m}, \tag{3.20}$$

for any m. By taking the limit $m \to \infty$ in (3.20), we get the assertion of Theorem 3.1. □

To complet the proof of the previous Theorem, it remains to verify (3.16).

LEMMA 3.2. *It holds that*

$$(P(y_n), \xi)_{v_n} \to (P(y), \xi)_w \text{, for } n \to \infty.$$

Proof. To simplify the notations we shall write instead of

$$\int_0^1 y^-(w(x_2), x_2)\xi(w(x_2), x_2)dx_2 \text{ etc.., shortly } \int_0^1 y^-(w)\xi(w)dx_2 \text{, etc..}$$

Let $m \in \mathbb{N}$ be fixed. Then

$$(3.21) \qquad \left| \int_0^1 y^-(w)\xi(w)dx_2 - \int_0^1 y_n^-(v_n)\xi(v_n)dx_2 \right|$$

$$\leq \int_0^1 |y^-(w)\xi(w) - y^-(\Gamma_m)\xi(w)|dx_2 +$$

$$+ \int_0^1 |y^-(\Gamma_m)\xi(w) - y_n^-(\Gamma_m)\xi(v_n)|dx_2 +$$

$$+ \int_0^1 |y_n^-(\Gamma_m)\xi(v_n) - y_n^-(v_n)\xi(v_n)|dx_2 .$$

We shall estimate each term on the right hand side of (3.21). Firstly,

$$(3.22) \qquad \int_0^1 |y^-(w) - y^-(\Gamma_m)||\xi(w)|dx_2 \leq C\left(\int_0^1 |y(w) - y(\Gamma_m)|^2 dx_2\right)^{1/2}$$

$$\leq C \left(\int_0^1 \left[\int_{\Gamma_m}^w \frac{\partial y}{\partial x_1}(x_1,x_2)dx_1\right]^2 dx_2\right)^{1/2} \leq C\, m^{-1/2} \|y\|_{1,\Omega(w)}.$$

Secondly,

$$(3.23) \qquad \int_0^1 |y^-(\Gamma_m)\xi(w) - y_n^-(\Gamma_m)\xi(v_n)|dx_2$$

$$\leq \int_0^1 |y_n^-(\Gamma_m) - y^-(\Gamma_m)||\xi(v_n)|dx_2$$

$$+ \int_0^1 |y^-(\Gamma_m)||\xi(v_n) - \xi(w)|dx_2 \to 0,$$

if $n \to \infty$. Indeed, by virtue of (3.7) and by the compactness of the trace mapping $\gamma : H^1(G_m) \to L^2(\Gamma_m)$, it holds that

$$\int_0^1 |y_n^-(\Gamma_m) - y^-(\Gamma_m)|\, |\xi(v_n)|\, dx_2$$

$$\leqslant C \left(\int_0^1 |y_n^-(\Gamma_m) - y^-(\Gamma_m)|^2\, dx_2 \right)^{1/2}$$

$$\leqslant C \left(\int_0^1 |y_n(\Gamma_m) - y(\Gamma_m)|^2\, dx_2 \right)^{1/2} \leqslant C \| y_n - y \|_{0,\Gamma_m} \to 0,$$

if $n \to \infty$. Furthermore,

$$\int_0^1 |y^-(\Gamma_m)|\, |\xi(v_n) - \xi(w)|\, dx_2 \leqslant C \| y \|_{1,G_m} \left(\int_0^1 |\xi(v_n) - \xi(w)|^2\, dx_2 \right)^{1/2}$$

$$\leqslant C \| y \|_{1,G_m} \left(\int_0^1 \left| \int_{v_n}^{w} \frac{\partial \xi}{\partial x_1} (x_1,x_2) dx_1 \right|^2 dx_2 \right)^{1/2}$$

$$\leqslant C \left(\max_{x_2 \in [0,1]} |v_n(x_2) - w(x_2)| \right)^{1/2} \to 0,$$

for $n \to \infty$. Thus, (3.23) holds.

Finally,

$$(3.24)\quad \int_0^1 |y_n^-(\Gamma_m) - y_n^-(v_n)|\, |\xi(v_n)|\, dx_2 \leqslant C \left(\int_0^1 |y_n^-(\Gamma_m) - y_n^-(v_n)|^2\, dx_2 \right)^{1/2}$$

$$\leqslant C \left(\int_0^1 (y_n(\Gamma_m) - y_n(v_n))^2\, dx_2 \right)^{1/2}$$

$$= C \left(\int_0^1 \left[\int_{\Gamma_m}^{v_n} \frac{\partial y_n}{\partial x_1} (x_1,x_2) dx_1 \right]^2 dx_2 \right)^{1/2}$$

$$\leqslant C \left(\max_{x_2 \in [0,1]} |v_n(x_2) - w(x_2) + 1/m| \right)^{1/2}.$$

Let $\eta > 0$ be an arbitrary number. By (3.22) there exists m_0 with

$$(3.25)\quad \int_0^1 |y^-(w) - y^-(\Gamma_m)|\, |\xi(w)|\, dx_2 \leqslant \eta/3$$

for all $m \geqslant m_0$. Choosing $n \geqslant m_0$ sufficiently large we find by (3.23) that

$$(3.26) \qquad \int_0^1 | y^-(\Gamma_m)\xi(w) - y_n^-(\Gamma_m)\xi(v_n) | \, dx_2 \leqslant \eta/3$$

and by (3.24) that

$$(3.27) \qquad \int_0^1 | y_n^-(\Gamma_m) - y_n^-(v_n) | \, | \xi(v_n) | \, dx_2 \leqslant \eta/3.$$

By (3.21) the assertion of Lemma 3.2 follows from (3.25) - 3.27). □

4. - THE RELATION BETWEEN PROBLEM $(P)_\epsilon$ AND PROBLEM (P)

For a given sequence $\{\epsilon_k\}$ of positive numbers with $\epsilon_k \to 0$, $k \to \infty$, we consider a family of problems :

$$(P)_{\epsilon_k} \qquad \begin{cases} \text{Find } w_k \in U_{ad} \text{ such that} \\ J(y_k(w_k)) \leqslant J(y_k(v)) \text{ for all } v \in U_{ad}, \end{cases}$$

where

$$J(y_k(v)) = \| y_k(v) - z_d \|^2_{0,\Omega(v)}$$

and $y_k(v) \in V(\Omega(v))$ is the solution of the penalized problem

$$(4.1) \qquad (\text{grad } y_k(v), \text{grad } \xi)_{0,\Omega(v)} + \frac{1}{\epsilon_k} (P(y_k(v), \xi)_v = (f,\xi)_{0,\Omega(v)} \text{ for all } \xi \in V(\Omega(v)).$$

According to Theorem 3.1 there exists for any ϵ_k at least one optimal solution for Problem $(P)_{\epsilon_k}$ which will be denoted by w_k and the corresponding state by $y_k(w_k)$.

In the next theorem we show that some solutions of Problem $(P)_{\epsilon_k}$ are close to a solution of Problem *(P)*. Indeed, it holds :

THEOREM 4.1. *There exist a subsequence* $\{w_{k_j}, y_{k_j}(w_{k_j})\}$ *of* $\{w_k, y_k(w_k)\}$ *and elements* $w \in U_{ad}$, $y(w) \in K(w)$ *such that*

$$w_{k_j} \rightrightarrows w \text{ (uniformly) in } [0,1], \textit{ for } j \to \infty$$

$$y_{k_j}(w_{k_j}) \rightharpoonup y(w) \text{ (weakly) } \textit{in } H^1(G_m), \textit{ for } j \to \infty, \textit{ and for any } m,$$

where

$$G_m = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 \in (0,1),\ 0 < x_1 < w(x_2) - 1/m\},$$

w *is a solution of Problem (P) and* y(w) *is the corresponding state, solving the unilateral boundary value problem* (2.1) *in* $\Omega(w)$.

Proof. The proof will be given in several steps.

1) If we take $\xi = y_k(w_k)$ in (4.1), we especially have by the Poincaré-Friedrichs inequality that

$$C \, \| y_k(w_k) \|^2_{0,\Omega_k} \leqslant (\text{grad}\, y_k(w_k), \text{grad}\, y_k(w_k))_{0,\Omega_k} + \frac{1}{\epsilon_k} (P(y_k(w_k)), y_k(w_k))_{w_k}$$

$$= (f, y_k(w_k))_{0,\Omega_k} \leqslant \| f \|_{0,\Omega_{\tilde{\beta}}} \, \| y_k(w_k) \|_{1,\Omega_k},$$

where the abbreviation $\Omega_k := \Omega(w_k)$ is used. This implies the existence of $C > 0$ such that

$$\| y_k(w_k) \|_{1,\Omega_k} \leqslant C. \tag{4.2}$$

Having in mind the definition of U_{ad}, we can extract a subsequence from $\{w_k\}$ (and denoted again by $\{w_k\}$) such that

$$w_k \rightrightarrows w \in U_{ad} \text{ in } [0,1]. \tag{4.3}$$

Exactly the same procedure as was used in Section 3 leads to the existence of a subsequence $\{ y_{k_j}(w_{k_j}) \} \subset \{ y_k(w_k) \}$ and of an element $y \in H^1(\Omega(w))$, $y \in H^1(\Omega(w))$, $y = 0$ on $\partial\Omega(w) \setminus \Gamma(w)$ such that

$$y_{k_j}(w_{k_j}) \rightharpoonup y \mid_{G_m} \text{ in } H^1(G_m), \text{ for } j \to \infty \tag{4.4}$$

and for any $m \in \mathbb{N}$.

2) Let us show that $y \in K(w)$. For $\xi \in H^1_0(\Omega_{\tilde{\beta}})$, (4.1) and (4.2) imply that

$$| (P(y_{k_j}(w_{k_j})), \xi)_{w_{k_j}} | \leqslant C \, \epsilon_{k_j} \to 0, \text{ for } j \to \infty. \tag{4.5}$$

On the other hand, (4.3), (4.4) and Lemma 3.1 yield

$$(P(y(w)), \xi)_w = \lim_{j \to \infty} (P(y_{k_j}(w_{k_j})), \xi)_{w_{k_j}}. \tag{4.6}$$

By comparing (4.5) with (4.6) we find that $y \in K(w)$.

3) We now prove that $y = y(w)$ (defined by (4.4)) solves the unilateral boundary value problem (2.1) in $\Omega(w)$. Let $\xi \in K_m$, where

$$K_m := \{ \varphi \in H_0^1(\Omega_{\widetilde{\beta}}) \mid \varphi \geq 0 \quad \text{in} \quad \Omega_{\widetilde{\beta}} \setminus G_m \} .$$

Then $\xi \in K(w_{k_j})$ for sufficiently large j. By abbreviating $y_{k_j} := y_{k_j}(w_{k_j})$ we can write due to the monotonicity of P that

$$(\text{grad } y_{k_j}, \text{grad}(y_{k_j} - \xi))_{0,\Omega_{k_j}} \leq$$
$$\leq (\text{grad } y_{k_j}, \text{grad}(y_{k_j} - \xi))_{0,\Omega_{k_j}} + \frac{1}{\epsilon_{k_j}} (P(y_{k_j}) - P(\xi), y_{k_j} - \xi)_{w_{k_j}}$$
$$= (\text{grad } y_{k_j}, \text{grad}(y_{k_j} - \xi))_{0,\Omega_{k_j}} + \frac{1}{\epsilon_{k_j}} (P(y_{k_j}), y_{k_j} - \xi)_{w_{k_j}}$$
$$= (f, y_{k_j} - \xi)_{0,\Omega_{k_j}},$$

i.e.

$$(\text{grad } y_{k_j}, \text{grad}(\xi - y_{k_j}))_{0,\Omega_{k_j}} \geq (f, \xi - y_{k_j})_{0,\Omega_{k_j}}, \quad \text{for all } \xi \in K_m.$$

The next considerations proceed in the same way as the proof of Lemma 1.2 in [7] :

$$\begin{aligned}(4.7)\qquad &(\text{grad } y_{k_j}, \text{grad}(\xi - y_{k_j}))_{0,\Omega_{k_j}} = (\text{grad } y_{k_j}, \text{grad}(\xi - y_{k_j}))_{0,G_m} + \\ &+ (\text{grad } y_{k_j}, \text{grad}(\xi - y_{k_j}))_{0,\Omega_{k_j} \setminus \Omega(w)} + \\ &+ (\text{grad } y_{k_j}, \text{grad}(\xi - y_{k_j}))_{0,(\Omega(w) \setminus G_m) \cap \Omega_{k_j}} \leq \\ &\leq (\text{grad } y_{k_j}, \text{grad}(\xi - y_{k_j}))_{0,G_m} + (\text{grad } y_{k_j}, \text{grad } \xi)_{0,\Omega_{k_j} \setminus \Omega(w)} \\ &+ (\text{grad } y_{k_j}, \text{grad } \xi)_{0,\Omega(w) \setminus G_m}.\end{aligned}$$

From this, (4.2) and from (4.4) we have

$$\begin{aligned}(4.8)\qquad &\lim_{j \to \infty} \sup \, (\text{grad } y_{k_j}, \text{grad}(\xi - y_{k_j}))_{0,\Omega_{k_j}} \\ &\leq (\text{grad } y, \text{grad}(\xi - y))_{0,G_m} + C \, \| \xi \|_{1,\Omega(w) \setminus G_m} .\end{aligned}$$

In a similar way as above we find that

$$(f,\xi - y_{k_j})_{0,\Omega_{k_j}} = (f,\xi - y_{k_j})_{0,G_m} + (f,\xi - y_{k_j})_{0,\Omega_{k_j} \setminus \Omega(w)} +$$
$$+ (f,\xi - y_{k_j})_{0,(\Omega(w) \setminus G_m) \cap \Omega_{k_j}} .$$

Making use of (4.2) and (4.4) this implies that

$$\text{(4.9)} \quad \lim_{j \to \infty} \inf(f,\xi - y_{k_j})_{0,\Omega_{k_j}} \geqslant (f,\xi - y)_{0,G_m} - C(\| f \|_{0,\Omega(w) \setminus G_m} + \| \xi \|_{0,\Omega(w) \setminus G_m}).$$

A combination of (4.8) and (4.9) gives

$$\text{(4.10)} (\text{grad } y, \text{grad}(\xi - y))_{0,G_m} \geqslant (f,\xi - y)_{0,G_m} - C(\| \xi \|_{1,\Omega(w) \setminus G_m} + \| f \|_{0,\Omega(w) \setminus G_m})$$

for any $\xi \in K_m$.

Let a $\xi \in K(w)$ be given. Then we can construct a function $\psi \in H^1(\Omega_{\tilde{\beta}})$ such that

$$\psi \,|_{\partial\Omega(w)} = \xi$$

and

$$\psi \geqslant 0 \text{ in } \Omega_{\tilde{\beta}} .$$

Clearly, $\eta := \xi - \psi \in H_0^1(\Omega(w))$ and there exists a sequence $\{\eta_\ell\}$, $\eta_\ell \in \mathcal{D}(\Omega(w))$ such that

$$\eta_\ell \to \eta \text{ in } H^1(\Omega(w)) \text{ for } \ell \to \infty.$$

Let us define function ξ_ℓ as follows

$$\text{(4.11)} \qquad \xi_\ell = \begin{cases} \psi + \eta_\ell & \text{in} \quad \Omega(w) \\ \psi & \text{in} \quad \Omega_{\tilde{\beta}} \setminus \Omega(w) . \end{cases}$$

From (4.11) it is readily seen that

$$\text{(4.12)} \qquad \xi_\ell \to \psi + \eta = : \xi \text{ in } H^1(\Omega(w)) \text{ for } \ell \to \infty.$$

Moreover, ξ_ℓ are non-negative in a neighbourhood of $\Gamma(w)$, i.e.

$$\text{(4.13)} \qquad \xi_\ell \in K_m ,$$

provided that ℓ is sufficiently large. Thus, writing ξ_ℓ instead of ξ in (4.10) (this is justified according to (4.13) and letting $\ell \to \infty$ we obtain that

$$(4.14) \qquad (\text{grad } y, \text{grad}(\xi - y))_{0,G_m}$$

$$\geqslant (f, \xi - y)_{0,G_m} - C(\| \xi \|_{1,\Omega(w) \setminus G_m} + \| f \|_{0,\Omega(w) \setminus G_m})$$

holds for any $\xi \in K(w)$. Finally, taking the limit $m \to \infty$, we find that $y = y(w)$ is the solution of the unilateral boundary value problem (2.1) in $\Omega(w)$.

4) It remains to show that w is a solution of Problem *(P)* and that y(w) is the corresponding state. Let $\{ w^*, y^*(w^*) \} \in U_{ad} \times K(w^*)$ be a solution of Problem *(P)* and let

$$q^* := J(y^*(w^*)) = \inf_{v \in U_{ad}} J(v).$$

As is proved in [7], Problem *(P)* has at least one solution. Since $y^*(w^*)$ is a solution of the unilateral boundary value problem, we can write

$$y^*(w^*) = \lim_{k \to \infty} y_k(w^*) \text{ in } H^1(\Omega(w^*))\text{-topology}.$$

Here $y_k(w^*)$ denotes the solution of (4.1) on $\Omega(w^*)$. From the definition of w_k it follows that

$$(4.15) \qquad J(y_k(w_k)) \leqslant J(y_k(w^*)) \to J(y^*(w^*)) = q^*, \text{ for } k \to \infty.$$

On the other hand, as a consequence of (4.4) we have

$$J(y_{k_j}(w_{k_j})) = \| y_{k_j}(w_{k_j}) - z_d \|^2_{0,G_m} + \| y_{k_j}(w_{k_j}) - z_d \|^2_{0,\Omega_{k_j} \setminus G_m}$$

$$\geqslant \| y_{k_j}(w_{k_j}) - z_d \|^2_{0,G_m} \to \| y(w) - z_d \|^2_{0,G_m},$$

for $j \to \infty$. This implies that

$$(4.16) \qquad \liminf_{j \to \infty} J(y_{k_j}(w_{k_j})) \geqslant \| y(w) - z_d \|^2_{0,\Omega(w)} .$$

Since $w \in U_{ad}$ and since $y = y(w) \in K(w)$ is a solution of (2.1) in $\Omega(w)$ it follows from (4.16) that

$$\liminf_{j \to \infty} J(y_{k_j}(w_{k_j})) \geqslant q^* .$$

This together with (4.15) gives the assertation of Theorem 4.1 . □

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