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Stress functions for a heterogeneous section of a tree

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Abstract

Consider a tree that is composed of wood, which is orthotropic with respect to the cylindrical coordinate axes, where the z -axis is directed up the center of the tree. When a section of a tree is considered as a Relaxed Saint-Venant's Problem the stresses in the plane of a transverse cross section will equal zero. By allowing some of, or all of the compliance coefficients to be functions of the radial coordinate r , the structure of the stress functions can be fundamentally altered. If the compliance coefficient in the z -direction (S_{3333}) is allowed to remain constant, the S_{rz} and S_{0z} shear stresses will be functions of the coordinates, dimensions, applied loads, and compliance coefficients, while the other stresses will only be functions of the coordinates, dimensions and applied loads. If S_{3333} is also allowed to be a function of r , then all the nonzero stresses will be functions of the coordinates, dimensions, applied loads, and compliance coefficients.

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1. Introduction

Lyons et al. (2002a) derived the stress and displacement equations for a cylindrical section of a tree subject to loads independent of x_3 , and with compliance coefficients (in cylindrical coordinates) that were constant throughout the section. The constraints placed on the compliance coefficients when assuming they were constant, produced a compliance tensor that acted similarly to one for a transversely isotropic material. Lyons et al. (2002b) noted that the published values for the compliance tensor for *Pseudotsuga menziesii* (Douglas fir) precludes the possibility of the compliance tensor being constant in the cross section of a tree. Lyons et al. (2002b) proposed a compliance tensor that was linearly dependent on the radial coordinate r .

The objectives of this paper are to derive the stress equations for a cylindrical section of a tree for two different constitutive equations that allow for different levels of heterogeneity. The first constitutive equation considered will be similar to the one proposed by Lyons et al. (2002b) where only the minimum

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dependence on r was assumed in order to allow for the nonunique strains at $r = 0$. The second constitutive equation to be considered is similar to the constitutive equation proposed by Lyons et al. (2002b), except that the compliance and elasticity coefficients in the z -direction will also be linear functions of the radial coordinate r .

2. Constitutive equations and problem statement

The following constitutive equation in cylindrical coordinates was proposed by Lyons et al. (2002b) and will be used to model a cylindrical section of a tree (prime denotes in cylindrical coordinates).

$$\left. \begin{aligned} E'_{ij} &= S'_{ijkl} S'_{kl} = [S_{ijkl} + r^* M_{ijkl}] S'_{kl} \\ S'_{ij} &= C'_{ijkl} E'_{kl} = [C_{ijkl} + r^* K_{ijkl}] E'_{kl} \end{aligned} \right\} \text{in cylindrical coordinates} \quad (2.1)$$

Here E'_{ij} and S'_{ij} are the infinitesimal strain tensor and Cauchy's stress tensor, S'_{ijkl} and C'_{ijkl} are the compliance and elasticity tensors, and S_{ijkl} , M_{ijkl} , C_{ijkl} , and K_{ijkl} are constants.

Lyons et al. (2002a) transformed the elasticity and compliance tensors from a cylindrical basis to a Cartesian basis,

$$\begin{aligned} C_{ijkl} &= Q_{mi} Q_{nj} Q_{rk} Q_{sl} C'_{mnrsl} \\ S_{ijkl} &= Q_{mi} Q_{nj} Q_{rk} Q_{sl} S'_{mnrsl} \end{aligned} \quad (2.2)$$

where $Q_{ij} = Q_{ij}(\theta)$ is a clockwise rotation about the x_3 -axis. The complete list of transformation equations is included in Appendix A. Given (2.2) the constitutive equations can be written in Cartesian coordinates

$$S_{ij} = C_{ijkl} E_{kl}, \quad E_{ij} = S_{ijkl} S_{kl} \quad (2.3)$$

Lyons et al. (2002b) showed the transformed compliance tensor could be simplified so that some of the coefficients could remain constant, while still satisfying the constraints imposed by the nonunique strains in the \mathbf{e}_θ and \mathbf{e}_r directions. The simplified equations for the transformed compliance tensor are included in Appendix B.

With the compliance coefficients transformed into Cartesian coordinates, Eq. (B.1), it is possible to use Iesan's (1987) formulation to solve the relaxed Saint-Venant's problem. Lyons et al. (2002a) considered a cylindrical section of a tree as a relaxed Saint-Venant's problem with loads independent of x_3 . The statement of the relaxed Saint-Venant's problem will be repeated here for convenience.

From now on in this paper, Greek indices will range from 1 to 2, while Latin indices range from 1 to 3 unless otherwise specified. Summation notation is used for repeated indices, and a comma followed by a subscript will indicate a partial derivative with respect to the coordinate. Note the following special notations will be used, the Kronecker delta function (δ_{ij}), and the two-dimensional alternator symbol ($\epsilon_{\alpha\beta}$).

Consider a cylindrical section of a tree as a cantilever beam with constant cross sections (Fig. 1). Let Σ_1 be the open cross section at $x_3 = 0$, let Σ_2 be the open cross section at $x_3 = h$, and let Σ be an arbitrary open cross section with normal x_3 . The lateral surface of the cylinder will be Π , while the boundary of an arbitrary cross section is Γ . The closure of an arbitrary cross section will be $\bar{\Sigma} = \Sigma \cup \Gamma$.

The resultant loads applied to the cross section at $x_3 = 0$ are the forces \mathbf{F} and the moments \mathbf{M} , and these are represented by integral functions of the displacement vector \mathbf{u} , where $f(\mathbf{u}) = \mathbf{F}$ and $m(\mathbf{u}) = \mathbf{M}$. The lateral surface of the cylinder is unloaded, the cross section at Σ_2 is fixed, and body loads will be ignored in this analysis. The problem in Fig. 1 is of the class P_1 as defined by Iesan (1987), where the resultant loads acting on Σ are independent of x_3 and $F_x = 0$.

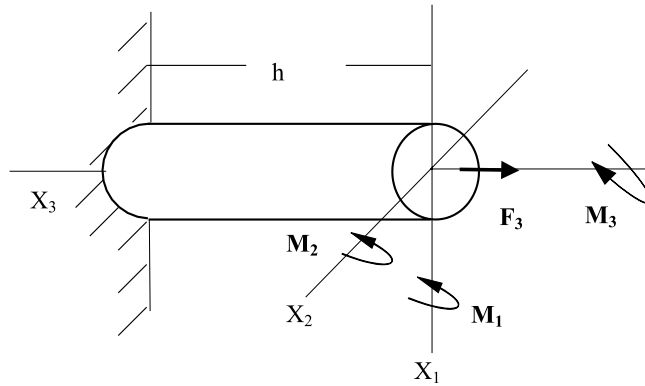


Fig. 1. Cylindrical cantilever beam subject to loads independent of x_3 .

Lyons et al. (2002a) presented the total displacements as

$$u_i^0 = u_i + u_i^l \tag{2.4}$$

where u_i are the displacements resulting from strain, and u_i^l are displacement resulting from a rigid body motion. The displacements resulting from strain, derived in a manner similar to that used by Iesan (1987), are

$$u_i = \delta_{iz} \left[-a_\alpha \frac{x_3^2}{2} + e_{\beta\alpha} a_4 x_\beta x_3 \right] + \delta_{i3} [a_\rho x_\rho + a_3] x_3 + W_i \tag{2.5}$$

where a_ρ are constants that will have to be determined using the boundary conditions, and $\mathbf{W} = \mathbf{W}(x_1, x_2)$ is a vector composed of the functions of integration.

Since the body forces are being ignored and the lateral surface of the cylinder is unloaded, the necessary conditions for a solution imply that the sum of the stress fields acting on Σ_2 must be in equilibrium with the resultant loads acting on Σ_1 therefore,

$$\begin{aligned} \int_{\Sigma_2} S_{23}(\mathbf{u}) \, da &= -f_\alpha(\mathbf{u}) = 0, & \int_{\Sigma_2} S_{33}(\mathbf{u}) \, da &= -f_3(\mathbf{u}) = -F_3, \\ \int_{\Sigma_2} e_{\alpha\beta} x_\alpha S_{3\beta}(\mathbf{u}) \, da &= -m_3(\mathbf{u}) = -M_3, & \int_{\Sigma_2} x_\alpha S_{33}(\mathbf{u}) \, da &= e_{\alpha\beta} m_\beta(\mathbf{u}) = e_{\alpha\beta} M_\beta \end{aligned} \tag{2.6}$$

Substitute (2.5) into the definition of the infinitesimal strain tensor, the resulting strains are

$$\begin{aligned} E_{11}(\mathbf{u}) &= W_{1,1}, & E_{22}(\mathbf{u}) &= W_{2,2}, & E_{33}(\mathbf{u}) &= (a_\rho x_\rho + a_3), & E_{23}(\mathbf{u}) &= \frac{1}{2}[a_4 x_1 + W_{3,2}], \\ E_{13}(\mathbf{u}) &= \frac{1}{2}[-a_4 x_2 + W_{3,1}], & E_{12}(\mathbf{u}) &= \frac{1}{2}[W_{1,2} + W_{2,1}] \end{aligned} \tag{2.7}$$

Substitute the strain tensor (2.7) into the constitutive equation (2.3), then the stress tensor in Cartesian coordinates becomes

$$S_{ij}(\mathbf{u}) = C_{ij33}(a_\rho x_\rho + a_3) - a_4 C_{ij\alpha 3} e_{\alpha\beta} x_\beta + T_{ij}(\mathbf{W}) \tag{2.8}$$

The $T_{ij}(\mathbf{W}) = C_{ijk\alpha} W_{k,\alpha}$, are the stresses resulting from the displacement vector \mathbf{W} , which is independent of x_3 and so forms a generalized plain strain problem. Iesan (1987) found that the generalized plane strain problem could be separated into four auxiliary problems $T_{ij}^{(p)}$ ($p = 1, 2, 3, 4$), which are defined by the following equilibrium equations (1)–(3) and boundary conditions (4)–(6).

$$\begin{aligned}
 (1) \quad & T_{ix}^{(\beta)}(\mathbf{W})_{, \alpha} + (C_{ix33}x_{\beta})_{, \alpha} = 0 \\
 (2) \quad & T_{ix}^{(3)}(\mathbf{W})_{, \alpha} + (C_{ix33})_{, \alpha} = 0 \\
 (3) \quad & T_{ix}^{(4)}(\mathbf{W})_{, \alpha} - e_{\rho\beta}(C_{ix\rho3}x_{\beta})_{, \alpha} = 0 \\
 (4) \quad & T_{ix}^{(\beta)}(\mathbf{W})n_{\alpha} = -C_{ix33}x_{\beta}n_{\alpha} \\
 (5) \quad & T_{ix}^{(3)}(\mathbf{W})n_{\alpha} = -C_{ix33}n_{\alpha} \\
 (6) \quad & T_{ix}^{(4)}(\mathbf{W})n_{\alpha} = e_{\rho\beta}C_{ix\rho3}x_{\beta}n_{\alpha}
 \end{aligned} \tag{2.9}$$

Here \mathbf{n} is the unit normal to Γ . The auxiliary problems combine as follows:

$$T_{ij}(\mathbf{W}) = \sum_{p=1}^4 a_p T_{ij}^{(p)}(\mathbf{W}) \tag{2.10}$$

After substituting the stresses (2.8) into the necessary conditions for a solution (2.6), and taking note of the simplifications resulting from (B.1), the following system of equations can be found for determining a_p .

$$\begin{bmatrix} \int_{\Sigma_2} x_1^2 C_{3333} da & \int_{\Sigma_2} x_1 x_2 C_{3333} da & \int_{\Sigma_2} x_1 C_{3333} da & 0 \\ \int_{\Sigma_2} x_1 x_2 C_{3333} da & \int_{\Sigma_2} x_2^2 C_{3333} da & \int_{\Sigma_2} x_2 C_{3333} da & 0 \\ \int_{\Sigma_2} x_1 C_{3333} da & \int_{\Sigma_2} x_2 C_{3333} da & \int_{\Sigma_2} C_{3333} da & 0 \\ 0 & 0 & 0 & \int_{\Sigma_2} [G_5] da \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} \tag{2.11}$$

where

$$\begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} = \begin{bmatrix} M_1 - \int_{\Sigma_2} x_1 T_{33} da \\ M_2 - \int_{\Sigma_2} x_2 T_{33} da \\ -F_3 - \int_{\Sigma_2} T_{33} da \\ -M_3 - \int_{\Sigma_2} x_1 T_{32} + x_2 T_{31} da \end{bmatrix} \quad \text{and} \quad G_5 = [-2x_1 x_2 C_{1323} + x_1^2 C_{2323} + x_2^2 C_{1313}]$$

Note from (B.1) that $C_{3333} = C'_{3333}$, where C_{3333} is a constant, and recall that the integrals are taken over a circular cross-section. Therefore, equation (2.11) becomes

$$\begin{bmatrix} C'_{3333}I & 0 & 0 & 0 \\ 0 & C'_{3333}I & 0 & 0 \\ 0 & 0 & C'_{3333}A & 0 \\ 0 & 0 & 0 & \int_{\Sigma_2} [G_5] da \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} \tag{2.12}$$

Here I is the moment of inertia, and A is the cross-sectional area. Since C'_{3333} , A and I are never zero, then a_p ($p = 1, 2, 3$) are uniquely defined by (2.12). The integrand G_5 can be simplified as follows when taking (B.1) into account,

$$\begin{aligned}
 G_5 &= [-2x_1 x_2 C_{1323} + x_1^2 C_{2323} + x_2^2 C_{1313}] \\
 &= r^3 [2 \sin^2(\theta) \cos^2(\theta) + \cos^4(\theta) + \sin^2(\theta)] \underline{M}_{2323} + r^2 [\cos^2(\theta) + \sin^2(\theta)] \underline{C}_{1313} \\
 &= r^3 \underline{M}_{2323} + r^2 \underline{C}_{1313}
 \end{aligned} \tag{2.13}$$

Note in Eq. (2.13) that it is possible for \underline{M}_{2323} to be negative, where as \underline{C}_{1313} is always positive. However, trees in second growth stands are commonly less than one meter in diameter, then $r^3 < r^2$. Therefore, when $|\underline{M}_{2323}| \leq |\underline{C}_{1313}|$, (2.13) will be greater than zero except at $r = 0$, and so a_4 is uniquely defined by (2.12) except possibly at $r = 0$.

3. Stress functions when considering S_{3333} a constant

Expand Eq. (2.8),

$$\begin{aligned}
 (1) \quad S_{11} &= C_{1133}(a_1x_1 + a_2x_2 + a_3) + T_{11}(\mathbf{W}) \\
 (2) \quad S_{22} &= C_{2233}(a_1x_1 + a_2x_2 + a_3) + T_{22}(\mathbf{W}) \\
 (3) \quad S_{33} &= C_{3333}(a_1x_1 + a_2x_2 + a_3) + T_{33}(\mathbf{W}) \\
 (4) \quad S_{23} &= C_{2332}(a_4x_1) - C_{2313}(a_4x_2) + T_{23}(\mathbf{W}) \\
 (5) \quad S_{13} &= C_{1332}(a_4x_1) - C_{1313}(a_4x_2) + T_{13}(\mathbf{W}) \\
 (6) \quad S_{12} &= C_{1233}(a_1x_1 + a_2x_2 + a_3) + T_{12}(\mathbf{W})
 \end{aligned} \tag{3.1}$$

Lyons et al. (2002b) showed that the in-plane stresses in cylindrical coordinates ($S'_{\alpha\beta}$) are functions of the in-plane stresses in Cartesian coordinates ($S_{\alpha\beta}$).

$$S'_{\alpha\beta} = Q_{\alpha\rho} Q_{\beta\gamma} S_{\rho\gamma} \tag{3.2}$$

Note in (3.2) that $S'_{\alpha\beta}$ and $S_{\alpha\beta}$ are 2×2 second order tensors, and that $Q_{\alpha\beta}$ is the transformation

$$Q_{\alpha\beta} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \tag{3.3}$$

Therefore, since (3.3) is an orthonormal transformation the in-plane stresses in Cartesian coordinates ($S_{\alpha\beta}$) can be written as functions of the in-plane stresses in cylindrical coordinates ($S'_{\alpha\beta}$),

$$\begin{aligned}
 Q_{\beta\gamma}^{-1} Q_{\alpha\rho}^{-1} S'_{\alpha\beta} &= Q_{\beta\gamma}^{-1} Q_{\alpha\rho}^{-1} Q_{\alpha\rho} Q_{\beta\gamma} S_{\rho\gamma} \\
 S_{\rho\gamma} &= Q_{\beta\gamma}^{-1} Q_{\alpha\rho}^{-1} S'_{\alpha\beta}
 \end{aligned} \tag{3.4}$$

where $Q_{\alpha\beta}^{-1}$ is the inverse of $Q_{\alpha\beta}$.

Lyons et al. (2002b) proved for problems where the applied loads are independent of x_3 that $S'_{\alpha\beta} = 0$ and so by (3.4)

$$S_{\rho\gamma} = 0 \tag{3.5}$$

Substitute (3.5) into (3.1), then the first, second and sixth equations become

$$\begin{aligned}
 T_{11} &= -C_{1133}(a_\rho x_\rho + a_3) \\
 T_{22} &= -C_{2233}(a_\rho x_\rho + a_3) \\
 T_{12} &= -C_{1233}(a_\rho x_\rho + a_3)
 \end{aligned} \tag{3.6}$$

Recall the displacement vector $\mathbf{W} = \mathbf{W}(x_1, x_2)$, then

$$E_{33}(\mathbf{W}) = S_{3311}T_{11} + S_{3322}T_{22} + S_{3333}T_{33} + 2S_{3312}T_{12} = 0 \tag{3.7}$$

Solve (3.7) for T_{33} , then

$$T_{33} = -\frac{1}{S_{3333}}[S_{3311}T_{11} + S_{3322}T_{22} + 2S_{3312}T_{12}] \tag{3.8}$$

Recall for generalized plane strain that $E_{ij} = S_{ijmn}T_{mn}$. Chirita (1979) notes this process must be reversible, therefore, $T_{kl} = C_{klrs}E_{rs}$. This results in $E_{ij} = S_{ijmn}C_{mnr s}E_{rs}$, and

$$S_{ijmn}C_{mnr s} = \frac{1}{2}[\delta_{ir}\delta_{js} + \delta_{ij}\delta_{jr}] = \begin{cases} (i = s, j = r) \Rightarrow 1/2 \\ (i = r, j = s) \Rightarrow 1/2 \\ (i = r, j = s; i = s, j = r) \Rightarrow 1 \\ \text{all other } i, j, r, s \Rightarrow 0 \end{cases} \tag{3.9}$$

Substitute (3.6) into (3.8) while taking (3.9) into account, then

$$T_{33} = [a_\rho x_\rho + a_3] [S_{3333}^{-1} - C_{3333}] \quad (3.10)$$

Following procedures similar to those used by Lyons et al. (2002a) to determine the auxiliary generalized plane strain stress functions, it can be shown that

$$T_{\alpha 3}^{(p)} = 0 \quad \text{for } p = 1, 2, 3 \quad (3.11)$$

Therefore, by (2.10)

$$T_{\alpha 3}(\mathbf{W}) = a_4 T_{\alpha 3}^{(4)}(\mathbf{W}) \quad (3.12)$$

Recall the third and sixth equations of (2.9)

$$\begin{aligned} T_{13}^{(4)}{}_{,1} + T_{23}^{(4)}{}_{,2} &= -x_1(C_{1323,1} + C_{2323,2}) + x_2(C_{1313,1} + C_{1323,2}) \\ T_{13}^{(4)} n_1 + T_{23}^{(4)} n_2 &= (C_{1313}x_2 - C_{1323}x_1)n_1 + (C_{1323}x_2 - C_{2323}x_1)n_2 \end{aligned} \quad (3.13)$$

Substitute $C_{\alpha 3 \beta 3}$ from (B.1) into (3.13), then

$$\begin{aligned} T_{13}^{(4)}{}_{,1} + T_{23}^{(4)}{}_{,2} &= 4r [M_{1313} - M_{2323}] \cos(\theta)^3 \sin(\theta) \\ T_{13}^{(4)} n_1 + T_{23}^{(4)} n_2 &= \frac{r}{2} [M_{1313} - M_{2323}] \sin(4\theta) \end{aligned} \quad (3.14)$$

Let $\theta = \frac{m\pi}{2}$, where $m = 1, 2, \dots$, then (3.14) becomes

$$\begin{aligned} T_{13}^{(4)}{}_{,1} + T_{23}^{(4)}{}_{,2} &= 0 \\ T_{13}^{(4)} n_1 + T_{23}^{(4)} n_2 &= 0 \end{aligned} \quad (3.15)$$

The auxiliary generalized plane strain stresses in cylindrical coordinates are

$$\begin{aligned} T_{13}^{(4)'} &= \cos(\theta) T_{13}^{(4)} + \sin(\theta) T_{23}^{(4)} \\ T_{23}^{(4)'} &= -\sin(\theta) T_{13}^{(4)} + \cos(\theta) T_{23}^{(4)} \end{aligned} \quad (3.16)$$

Solve the second equation of (3.15) for $T_{13}^{(4)}$ and substitute this into the first of (3.16), and note that $n_1 = \cos(\theta)$ and $n_2 = \sin(\theta)$, then on Γ for $\theta = (m\pi/2)$

$$T_{13}^{(4)'} = \cos(\theta) \left(\frac{-T_{23}^{(4)} \sin(\theta)}{\cos(\theta)} \right) + \sin(\theta) T_{23}^{(4)} = 0 \quad (3.17)$$

Substitute (3.17) back into (3.15), then on Γ for $\theta = (m\pi/2)$

$$T_{23}^{(4)'} = 0 \quad (3.18)$$

Substitute (3.17) and (3.18) into (3.16), then on Γ for $\theta = (m\pi/2)$

$$T_{13}^{(4)} = T_{23}^{(4)} = 0 \quad (3.19)$$

Substitute (3.19) into (3.12), then on Γ for $\theta = (m\pi/2)$

$$T_{13} = T_{23} = 0 \quad (3.20)$$

Lyons et al. (2002b) proved that the generalized plane strain stresses T_{ij} are independent of the cylindrical coordinate θ , therefore, by (3.20) $T_{13} = T_{23} = 0$ on Γ . Lyons et al. (2002b) proved that the generalized plane strains could be represented by functions of the complex variable $w = x_1 + ix_2$. Therefore, by Cauchy's integral formula, if $T_{\alpha 3} = 0$ on Γ , then

$$T_{\alpha 3} = 0 \text{ on } \bar{\Sigma} = \Sigma \cup \Gamma \tag{3.21}$$

The S_{ij} stress functions may be formed by taking note of (3.5) and by substituting (3.10) and (3.21) into the third, fourth, and fifth equations of (3.1), then

$$\begin{aligned} S_{11} &= S_{22} = S_{12} = 0 \\ S_{33} &= S_{3333}^{-1} [a_\rho x_\rho + a_3] \\ S_{23} &= a_4 C_{2323} x_1 - a_4 C_{1323} x_2 \\ S_{13} &= a_4 C_{1323} x_1 - a_4 C_{1313} x_2 \end{aligned} \tag{3.22}$$

Substitute (3.22) into the necessary conditions for a solution, then the first two equations of (2.6) can be shown to equal zero for all a_4 .

For example

$$\begin{aligned} \int_0^{2\pi} \int_0^R S_{13} r \, dr \, d\theta &= \int_0^{2\pi} \int_0^R a_4 C_{1323} x_1 - a_4 C_{1313} x_2 r \, dr \, d\theta \\ &= a_4 [K_{2323} - C_{1313}] \int_0^{2\pi} \int_0^R r^3 \sin(\theta) \, dr \, d\theta = 0 \end{aligned} \tag{3.23}$$

where R is the radius of Γ .

Substitute (3.22) into the third equation of (2.6), then

$$\begin{aligned} \int_0^{2\pi} \int_0^R S_{33} r \, dr \, d\theta &= \int_0^{2\pi} \int_0^R [a_\rho x_\rho + a_3] S_{3333}^{-1} r \, dr \, d\theta = -F_3 \\ a_3 &= \frac{-F_3 S_{3333}}{\pi R^2} \end{aligned} \tag{3.24}$$

Substitute (3.22) into the fourth equation of (2.6), then

$$\begin{aligned} \int_0^{2\pi} \int_0^R [x_1 S_{23} - x_2 S_{13}] r \, dr \, d\theta &= -M_3 \\ a_4 \int_0^{2\pi} \int_0^R [x_1 (x_1 C_{2323} - x_2 C_{1323}) - x_2 (-x_2 C_{1313} + x_1 C_{1323})] r \, dr \, d\theta &= -M_3 \end{aligned} \tag{3.25}$$

Substitute (B.1) into (3.25) and integrate, then

$$a_4 = \frac{-10M_3}{\pi R^4 (2RK_{1313} + 2RK_{2323} + 5C_{1313})} \tag{3.26}$$

Substitute (3.22) into the fifth and sixth equations of (2.6), then

$$\begin{aligned} \int_0^{2\pi} \int_0^R x_1 S_{33} r \, dr \, d\theta &= \int_0^{2\pi} \int_0^R x_1 [a_\rho x_\rho + a_3] S_{3333}^{-1} r \, dr \, d\theta = M_2 \\ \int_0^{2\pi} \int_0^R x_2 S_{33} r \, dr \, d\theta &= \int_0^{2\pi} \int_0^R x_2 [a_\rho x_\rho + a_3] S_{3333}^{-1} r \, dr \, d\theta = -M_1 \end{aligned} \tag{3.27}$$

Substitute (B.1) into (3.27) and integrate, then

$$a_1 = \frac{4}{\pi R^4} M_2 S_{3333}, \quad a_2 = \frac{-4}{\pi R^4} M_1 S_{3333} \tag{3.28}$$

Substitute (3.24), (3.26) and (3.28) into (3.22), then the stress functions become

$$\begin{aligned}
 S_{11} &= S_{22} = S_{12} = 0 \\
 S_{33} &= \frac{4x_1M_2}{\pi R^4} - \frac{4x_2M_1}{\pi R^4} - \frac{F_3}{\pi R^2} \\
 S_{23} &= \frac{-10M_3}{\pi R^4(2RK_{1313} + 2RK_{2323} + 5C_{1313})} (C_{2323}x_1 - C_{1323}x_2) \\
 S_{13} &= \frac{-10M_3}{\pi R^4(2RK_{1313} + 2RK_{2323} + 5C_{1313})} (C_{1323}x_1 - C_{1313}x_2)
 \end{aligned} \tag{3.29}$$

4. Stress functions when considering S_{3333} a function of r

There is a fundamental difference between the functions representing the shear stresses (S_{23}, S_{13}) and the normal stress (S_{33}) in Eq. (3.29). In Eq. (3.29) S_{33} is independent of the material coefficients, while S_{23} and S_{13} depend on the material coefficients. Therefore, provided the constitutive equation is of the form (B.1), S_{33} will only depend on the initial coordinates of a point and the applied loads, and not on the magnitudes of the material coefficients. However, S_{23} and S_{13} depend on the initial coordinates, the applied loads, and the magnitudes of the material coefficients, and so will vary for different materials even if the constitutive equation is similar to (B.1). The dependence of S_{23} and S_{13} on the material coefficients is a result of dependence of S_{1313} and S_{2323} (or C_{1313} and C_{2323}) on the radial coordinate r . Thus, Section 4 will consider the constitutive equation where S'_{3333} and C'_{3333} also depend on r .

Let

$$\begin{aligned}
 C'_{3333} &= \underline{C_{3333}} + r^* \underline{K_{3333}} \\
 S'_{3333} &= \underline{S_{3333}} + r^* \underline{K_{3333}}
 \end{aligned} \tag{4.1}$$

Recall that the transformation (2.2) is a rotation about the x_3 axis, therefore,

$$\begin{aligned}
 C_{3333} &= \underline{C_{3333}} + r^* \underline{K_{3333}} \\
 S_{3333} &= \underline{S_{3333}} + r^* \underline{K_{3333}}
 \end{aligned} \tag{4.2}$$

Two important criteria must be satisfied in order to use Eq. (4.2) in the formulation developed in Section 3. First, the off diagonal terms in (2.11) were shown to equal zero when considering C_{3333} a constant. Substituting (4.2) into (2.11) it can be seen that the off diagonal terms remain equal to zero, and so Eq. (2.12) is still valid. Second, in order to prove that $S_{\alpha\beta} = 0$ in (3.22) it was necessary for $C_{3333}(a_\rho x_\rho + a_3)$ to be analytic. By Eq. (4.2) C_{3333} is only a function of r . Therefore, $C_{3333}(a_\rho x_\rho + a_3)$ is composed of analytic functions, coordinates, and constants and so must be analytic as well. Thus, the stress functions may be found using the formulation in Section 3, when substituting (4.2) into (B.1).

Recall the S_{ij} stress functions

$$\begin{aligned}
 S_{11} &= S_{22} = S_{12} = 0 \\
 S_{33} &= S_{3333}^{-1} [a_\rho x_\rho + a_3] \\
 S_{23} &= a_4 C_{2323} x_1 - a_4 C_{1323} x_2 \\
 S_{13} &= a_4 C_{1323} x_1 - a_4 C_{1313} x_2
 \end{aligned} \tag{3.22}$$

As in Section 3, substitution of (3.22) into the necessary conditions for a solution results in the first two equations of (2.6) being equal to zero for all a_4 . For example

$$\int_0^{2\pi} \int_0^R S_{13} r \, dr \, d\theta = \int_0^{2\pi} \int_0^R a_4 C_{1323} x_1 - a_4 C_{1313} x_2 r \, dr \, d\theta$$

$$= a_4 [K_{2323} - C_{1313}] \int_0^{2\pi} \int_0^R r^3 \sin(\theta) \, dr \, d\theta = 0 \tag{4.3}$$

Substitute (4.2) into (3.22) and this into the third equation of (2.6), then

$$\int_0^{2\pi} \int_0^R S_{33} r \, dr \, d\theta = \int_0^{2\pi} \int_0^R [a_\rho x_\rho + a_3] [S_{3333} + rM_{3333}]^{-1} r \, dr \, d\theta = -F_3$$

$$a_3 = \frac{-F_3 M_{3333}^2}{2\pi [RM_{3333} - S_{3333} \ln(S_{3333} + RM_{3333}) + S_{3333} \ln(S_{3333})]} \tag{4.4}$$

Substitute (3.22) into the fourth equation of (2.6), then

$$\int_0^{2\pi} \int_0^R [x_1 S_{23} - x_2 S_{13}] r \, dr \, d\theta = -M_3$$

$$a_4 \int_0^{2\pi} \int_0^R [x_1 (x_1 C_{2323} - x_2 C_{1323}) - x_2 (-x_2 C_{1313} + x_1 C_{1323})] r \, dr \, d\theta = -M_3$$

Substitute (B.1) into (4.5) and integrate, then

$$a_4 = \frac{-10M_3}{\pi R^4 (2RK_{1313} + 2RK_{2323} + 5C_{1313})} \tag{4.6}$$

Substitute (3.22) into the fifth and sixth equations of (2.6), then

$$\int_0^{2\pi} \int_0^R x_1 S_{33} r \, dr \, d\theta = \int_0^{2\pi} \int_0^R x_1 [a_\rho x_\rho + a_3] [S_{3333} + rM_{3333}]^{-1} r \, dr \, d\theta = M_2$$

$$\int_0^{2\pi} \int_0^R x_2 S_{33} r \, dr \, d\theta = \int_0^{2\pi} \int_0^R x_2 [a_\rho x_\rho + a_3] [S_{3333} + rM_{3333}]^{-1} r \, dr \, d\theta = -M_1$$

Substitute (4.2) into (4.7) and integrate, then

$$a_1 = \frac{6M_2 M_{3333}^4}{\pi \left[2R^3 M_{3333}^3 - 3S_{3333} R^2 M_{3333}^2 + 6RS_{3333}^2 M_{3333} + 6S_{3333}^3 \ln\left(\frac{S_{3333}}{S_{3333} + RM_{3333}}\right) \right]}$$

$$a_2 = \frac{-6M_1 M_{3333}^4}{\pi \left[2R^3 M_{3333}^3 - 3S_{3333} R^2 M_{3333}^2 + 6RS_{3333}^2 M_{3333} + 6S_{3333}^3 \ln\left(\frac{S_{3333}}{S_{3333} + RM_{3333}}\right) \right]} \tag{4.8}$$

The stress functions can be found by substituting (4.4), (4.6) and (4.8) into (3.22). All the nonzero stress functions in (3.22) are now dependent on the magnitudes of the material coefficients.

5. Summary

Consider the following four forms of the compliance tensor: (1) isotropic homogeneous, (2) orthotropic with respect to the cylindrical coordinates and homogeneous, (3) orthotropic with respect to the cylindrical coordinates and heterogeneous except for S'_{3333} , S'_{1212} and S'_{1122} being constant, and (4) orthotropic with respect to the cylindrical coordinates and heterogeneous except for S'_{1212} and S'_{1122} being constant.

The stress functions for a cylindrical section of a tree subject to loads independent of x_3 can be classified by the assumptions implicit in the compliance tensor (Table 1). When the compliance coefficients are

Table 1
The dependence of the stress functions on the compliance coefficients

Constitutive equation	Magnitude of the in plane stresses S_{11} , S_{22} , and S_{12}	Dependence of the nonzero stresses on the material coefficients
Isotropic homogeneous (Muskhelishvili, 1963)	Zero	All the stress functions are independent of the compliance coefficients
Orthotropic in cylindrical coordinates, homogeneous (Lyons et al., 2002a)	Zero	All the stress functions are independent of the compliance coefficients
Orthotropic in cylindrical coordinates, heterogeneous with S'_{3333} , S'_{1212} and S'_{1122} constant	Zero	All the stress functions, except for S_{23} and S_{13} are independent of the compliance coefficients
Orthotropic in cylindrical coordinates, heterogeneous with S'_{1212} and S'_{1122} constant	Zero	All the nonzero stress functions are dependent on the compliance coefficients

allowed to depend on the cylindrical coordinate r the stress functions become dependent on the compliance coefficients. Consider two beams with similar shape and size, and subject to similar loads. If the constitutive equations for both beams are dependent on r , but with dissimilar compliance coefficients, then the magnitudes of the stresses will be dissimilar. This is a dramatic change from the case where the compliance coefficients are homogeneous. If the compliance coefficients are homogeneous then the stresses will have similar magnitudes in the two beams.

Appendix A. Transformation equations

The transformation equations taking the elasticity coefficients in cylindrical coordinates (C'_{ijkl}) to Cartesian coordinates (C_{ijkl}) (Lyons et al., 2002a).

$$C_{1111} = C_{\theta}^4 C'_{1111} + 2C_{\theta}^2 S_{\theta}^2 C'_{1122} + 4C_{\theta}^2 S_{\theta}^2 C'_{1212} + S_{\theta}^4 C'_{2222}$$

$$C_{2222} = S_{\theta}^4 C'_{1111} + 2C_{\theta}^2 S_{\theta}^2 C'_{1122} + 4C_{\theta}^2 S_{\theta}^2 C'_{1212} + C_{\theta}^4 C'_{2222}$$

$$C_{3333} = C'_{3333}$$

$$C_{2323} = S_{\theta}^2 C'_{1313} + C_{\theta}^2 C'_{2323}$$

$$C_{1313} = C_{\theta}^2 C'_{1313} + S_{\theta}^2 C'_{2323}$$

$$C_{1212} = C_{\theta}^2 S_{\theta}^2 [C'_{1111} - 2C'_{1122} + C'_{2222} - 2C'_{1212}] + [C_{\theta}^4 + S_{\theta}^4] C'_{1212}$$

$$C_{1122} = C_{\theta}^2 S_{\theta}^2 C'_{1111} + C_{\theta}^4 C'_{1122} - 4C_{\theta}^2 S_{\theta}^2 C'_{1212} + S_{\theta}^4 C'_{2211} + C_{\theta}^2 S_{\theta}^2 C'_{2222}$$

$$C_{1133} = C_{\theta}^2 C'_{1133} + S_{\theta}^2 C'_{2233}$$

$$C_{1123} = 0$$

$$C_{1113} = 0$$

$$C_{1112} = -C_{\theta} S_{\theta} [C_{\theta}^2 C'_{1111} - C_{\theta}^2 C'_{1122} - 2C_{\theta}^2 C'_{1212} + 2S_{\theta}^2 C'_{1212} + S_{\theta}^2 C'_{1122} - S_{\theta}^2 C'_{2222}]$$

$$C_{2233} = S_{\theta}^2 C'_{1133} + C_{\theta}^2 C'_{2233}$$

$$C_{2223} = 0$$

$$C_{2213} = 0$$

$$C_{2212} = -C_{\theta} S_{\theta} [S_{\theta}^2 C'_{1111} - S_{\theta}^2 C'_{1122} - 2S_{\theta}^2 C'_{1212} + 2C_{\theta}^2 C'_{1212} + C_{\theta}^2 C'_{1122} - C_{\theta}^2 C'_{2222}]$$

$$C_{3323} = 0$$

$$C_{3313} = 0$$

$$\begin{aligned}
C_{3312} &= -C_\theta S_\theta [C'_{3311} - C'_{3322}] \\
C_{2313} &= -C_\theta S_\theta [C'_{1313} - C'_{2323}] \\
C_{2312} &= 0 \\
C_{1312} &= 0
\end{aligned} \tag{A.1}$$

Note, $C_\theta = \cos(\theta)$ and $S_\theta = \sin(\theta)$.

For the transformation equations taking the compliance coefficients in cylindrical coordinates (S'_{ijkl}) to Cartesian coordinates (S_{ijkl}), replace C'_{ijkl} with S'_{ijkl} and C_{ijkl} with S_{ijkl} in Eq. (A.1).

Appendix B. Transformed compliance coefficients

The transformation equations that transform the nonzero compliance coefficients in Eq. (2.1) (S'_{ijkl}) to Cartesian coordinates (S_{ijkl}) (Lyons et al., 2002b).

$$\begin{aligned}
S_{1111} &= [C_\theta^4 + S_\theta^4] \underline{S_{1111}} + r [C_\theta^4 \underline{M_{1111}} + S_\theta^4 \underline{M_{2222}}] + 2C_\theta^2 S_\theta^2 \underline{S_{1122}} + 4C_\theta^2 S_\theta^2 \underline{S_{1212}} \\
S_{1122} &= \underline{S_{1122}} \\
S_{1133} &= \underline{S_{1133}} + r [C_\theta^2 \underline{M_{1133}} + S_\theta^2 \underline{M_{2233}}] \\
S_{1112} &= -S_\theta C_\theta [C_\theta^2 \underline{S_{1111}} + r \underline{M_{1111}}] - C_\theta^2 \underline{S_{1122}} - 2C_\theta^2 \underline{S_{1212}} + 2S_\theta^2 \underline{S_{1212}} + S_\theta^2 \underline{S_{1122}} - S_\theta^2 [\underline{S_{1111}} + r \underline{M_{2222}}] \\
S_{2222} &= [C_\theta^4 + S_\theta^4] \underline{S_{1111}} + r [S_\theta^4 \underline{M_{1111}} + C_\theta^4 \underline{M_{2222}}] + 2C_\theta^2 S_\theta^2 \underline{S_{1122}} + 4C_\theta^2 S_\theta^2 \underline{S_{1212}} \\
S_{2233} &= \underline{S_{1133}} + r [S_\theta^2 \underline{M_{1133}} + C_\theta^2 \underline{M_{2233}}] \\
S_{2212} &= -S_\theta C_\theta [S_\theta^2 \underline{S_{1111}} + r \underline{M_{1111}}] - S_\theta^2 \underline{S_{1122}} - 2C_\theta^2 \underline{S_{1212}} + 2S_\theta^2 \underline{S_{1212}} + C_\theta^2 \underline{S_{1122}} - C_\theta^2 [\underline{S_{1111}} + r \underline{M_{2222}}] \\
S_{3333} &= \underline{S_{3333}} \\
S_{3312} &= -S_\theta C_\theta r [\underline{M_{1133}} - \underline{M_{2233}}] \\
S_{2323} &= \underline{S_{1313}} + r [S_\theta^2 \underline{M_{1313}} + C_\theta^2 \underline{M_{2323}}] \\
S_{2313} &= -S_\theta C_\theta r [\underline{M_{1313}} - \underline{M_{2323}}] \\
S_{1313} &= \underline{S_{1313}} + r [C_\theta^2 \underline{M_{1313}} + S_\theta^2 \underline{M_{2323}}] \\
S_{1212} &= C_\theta^2 S_\theta^2 [2\underline{S_{1111}} + r [\underline{M_{1111}} + \underline{M_{2222}}]] - 2[\underline{S_{1122}} + \underline{S_{1212}}] + [C_\theta^4 + S_\theta^4] \underline{S_{1212}}
\end{aligned} \tag{B.1}$$

Note, $C_\theta = \cos(\theta)$ and $S_\theta = \sin(\theta)$.

Similar equations can be formed for the elastic coefficients by replacing the S_{ijkl} by C_{ijkl} , $\underline{S_{ijkl}}$ by $\underline{C_{ijkl}}$, and $\underline{M_{ijkl}}$ by $\underline{K_{ijkl}}$, in Eq. (B.1).

References

- Chirita, S., 1979. Saint-Venant's problem for anisotropic circular cylinders. *Acta Mechanica* 34, 243–250.
- Iesan, D., 1987. Saint-Venant's problem. *Lecture Notes in Mathematics*, vol. 1279. Springer-Verlag, New York, p. 161.
- Lyons, C. K., Guenther, R.B., Pyles, M.R., 2002a. Elastic equations for a cylindrical section of a tree. *International Journal of Solids and Structures*, accepted.
- Lyons, C. K., Guenther, R.B., Pyles, M.R., 2002b. Considering heterogeneity in a cylindrical section of a tree. *International Journal of Solids and Structures*, accepted.
- Muskhelishvili, N.I., 1963. *Some Basic Problems of the Mathematical Theory of Elasticity*, fourth ed. P. Noordhoff LTD, Groningen, p. 718.