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FINITE ELEMENT ANALYSIS OF FREE EDGE STRESSES IN NON-LINEAR VISCOELASTIC COMPOSITES UNDER UNIAXIAL EXTENSION, BENDING AND TWISTING LOADINGS

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This paper is dedicated to my mentor
Dr. H. H. Hilton
Professor Emeritus of Aeronautical and Astronautical
Engineering and of Supercomputing Applications, University of Illinois
at Urbana-Champaign, U.S.A., for his seventieth birthday anniversary

ABSTRACT

A finite element procedure for the analysis of time-dependent interlaminar stresses in non-linear viscoelastic laminated composites subjected to arbitrary combinations of axial extension, bending and/or twisting loads has been developed based on Schapery's non-linear constitutive relations and Pipes and Pagano's displacement field for laminates under a generalized plane deformation state. Parametric studies are presented to demonstrate the accuracy of the numerical procedures. As illustrative examples, time-dependent non-linear interlaminar stresses for cross-ply and angle-ply laminates subjected to bending and twisting are present. Other layup orientations can be conveniently analysed using the presently developed numerical procedure.

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KEY WORDS: FEA of non-linear viscoelasticity; interlaminar stresses; generalized plane deformation

INTRODUCTION

Interlaminar stresses which exist near free edges of laminates may result in the delamination onset and growth and arise due to mismatches in layer properties. Numerous studies[1–22,30] have been undertaken to investigate interlaminar stresses and failures of laminated composites. A variety of approximate elastic solution methods for inter-laminar stress is now available. Based on global equilibrium, Kassapoglou and Lagace[1,8] developed a simple and effective method of approximately predicting interlaminar stresses. Rose and Herakovich[16] extended Kassapoglou and Lagace's work including additional terms in the assumed stress field. The accuracy of the method is demonstrated through comparison of results with the KL solution and finite element results for angle-ply and cross-ply laminates. However, little is known about interlaminar stresses due to

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bending and/or twisting loads. Yin[21,22] analysed the interlaminar stresses in a laminate subjected to axial extension, bending, and twisting loads using a variational method involving Lekhnitskii's stress functions.[23]

Environmental factors such as temperature, moisture content, oxygen, and ultraviolet radiation are significant contributors to material degradation of polymer matrix composites and these effects have received substantial attention in the literatures. Linear and non-linear viscoelastic behaviour has been observed in laboratory tests of polymer matrix composites.[24–29] Under elevated load conditions, history-dependent effects can also lead to accumulation of residual stresses. In this case, it has been found that the viscoelastic effect can lead to a state of stress which is higher than that obtained by an elastic analysis. Therefore, in composite structural design, time-dependent effects of polymer matrix composite materials must be considered in order to ensure the environmental durability over the entire life of composite structures. Elastic approaches cannot accurately predict residual stress and strain fields since material properties and strengths of polymeric matrix composites are strongly time-dependent.

A limited number of studies[6,11,17,20,30] have been conducted for rate-dependent interlaminar stresses and delamination initiations. Flaggs and Vinson[17] studied the combined effects of temperature and humidity based on a general laminated composite plate buckling theory. Lin and Yi[11] and Yi *et al.*[30] developed the numerical procedure to analyse the linear and non-linear viscoelastic interlaminar stresses. Yi[20] and Hilton and Yi[6] also proposed the modified Quadratic Delamination Criterion to account for time dependent strengths in order to predict the delamination initiations in viscoelastic composite laminates as functions of time and loading history. Their analysis includes stochastic processes due to combined random loads and random delamination failure stresses as well as random anisotropic viscoelastic material properties. Oden[31] proposed the finite element formulation for non-linear viscoelastic materials. Henriksen[32] developed a two-dimensional non-linear viscoelastic finite element analysis and recursive relations for isotropic materials. Based on Henriksen's recursive formulation, Roy and Reddy[33] proposed a two-dimensional finite element procedure for the non-linear viscoelastic analysis of adhesively bonded joints and Kennedy and Wang[9] presented a three-dimensional finite element analysis for non-linear viscoelastic composites. Yi *et al.*[30] developed the finite element formulation and solution procedures for analysing non-linear viscoelastic response of composites under extensional loading. However, no work has been found in time-dependent interlaminar stresses due to bending and/or twisting loads.

In this study, based on Pipes and Pagano's displacement field[15] for laminates under a generalized plane deformation state and Schapery's non-linear constitutive relations,[27,34,35] a finite element procedure is developed for the analysis of non-linear viscoelastic interlaminar stresses in composite laminates subjected to arbitrary combinations of axial extension, bending, and twisting loads. Parametric studies are presented to demonstrate the accuracy of the numerical procedures. Numerical results have been obtained for cross-ply and angle-ply laminates subjected to bending and twisting in order to demonstrate the feasibility of the present approach.

ANALYSIS

Generalized plane deformation problems

As shown in Figure 1, the laminate is considered to be under a generalized plane deformation state. Pipes and Pagano[15] presented the following displacement field for laminates under

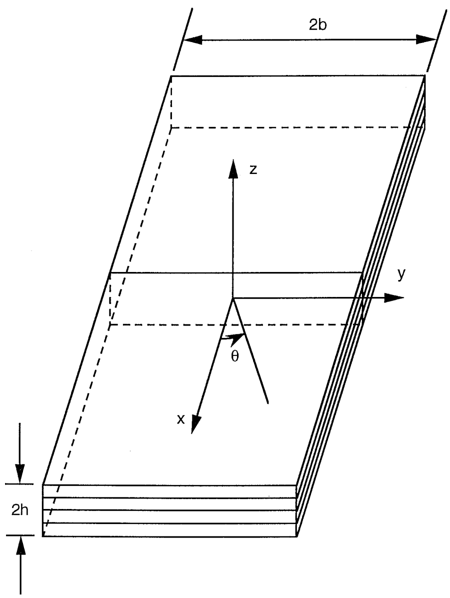

Figure 1. Co-ordinate and geometry of a laminate under the generalized plane deformation state

a generalized plane deformation state:

$$
\begin{aligned}
u(x,y,z,t) &= -[c_1(t)z + c_2(t)]y + [c_4(t)y + c_5(t)z + c_6(t)]x + U(y,z,t)\\
v(x,y,z,t) &= [c_1(t)z + c_2(t)]x - c_4(t)\frac{x^2}{2} + V(y,z,t)\\
w(x,y,z,t) &= -c_1(t)xy + c_7(t)x - c_5(t)\frac{x^2}{2} + c_8(t) + W(y,z,t)
\end{aligned}
\tag{1}
$$

In the above, U, V and W are the unknown portions of the displacements. The constants c_2, c_7 and c_8 are the rigid-body translation and rotation of the composite laminate, c_6 is the uniform axial extension and the twisting curvature c_1 is the relative angle of rotation about the x-axis. c_4 and c_5 represent the bending of the laminate in the y–z and x–z planes.

Small displacements are considered and the strain-displacement relationships are

$$
\begin{gathered}
\varepsilon_x = u_{,x} \quad \varepsilon_y = v_{,y} \quad \varepsilon_z = w_{,z}\\
\gamma_{yz} = (v_{,z} + w_{,y}) \quad \gamma_{zx} = (w_{,x} + u_{,z}) \quad \gamma_{xy} = (u_{,y} + v_{,x})
\end{gathered}
\tag{2}
$$

where a comma denotes partial differentiation, ε_i are engineering strains in the laminate coordinates and u, v and w are displacements.

Governing equations for anisotropic hygro-thermo-viscoelastic composites

Based on the time–temperature–moisture superposition principle, the relaxation curves can be shifted and master relaxation curves can be obtained at the reference temperature and humidity, where linear anisotropic viscoelastic relaxation moduli can be represented as

$$Q_{ij}(T_{\mathrm{f}}, M_{\mathrm{f}}, \zeta_{ij}) = Q_{ij}^{\infty} + \Delta Q_{ij}(T_{\mathrm{f}}, M_{\mathrm{f}}, \zeta_{ij}) \quad (3)$$

(no summation over repeated i or j)

with the reduced time ζ_{ij} defined by

$$\zeta_{ij}(t) = \int_0^t a_{ij}[T(s), M(s)]\,\mathrm{d}s \quad (4)$$

In the above, $i, j = 1, 2, \ldots, 6$, T_{f} and M_{f} are the reference temperature and moisture content, Q_{ij}^{∞} and ΔQ_{ij} are the equilibrium moduli at constant strain and transient components, respectively, and ζ_{ij} are reduced times which are related to the shift factors a_{ij}.

By using a generalized Maxwell model, the relaxation moduli can be represented in terms of exponential series such that

$$Q_{ij}(T_{\mathrm{f}}, M_{\mathrm{f}}, \zeta_{ij}) = Q_{ij}^{\infty} + \sum_{\omega=1}^{\mathrm{NT}_{ij}} Q_{ij\omega}\,\mathrm{e}^{-\zeta_{ij}/\lambda_{ij\omega}} \quad (5)$$

(no summation over repeated i or j)

where $\lambda_{ij\omega}$ are relaxation times and NT_{ij} are the numbers of terms used in the series expansion.

Introduction of the abbreviated notation leads to the following relaxation moduli Q_{ij} and reduced times ζ_{ij}

$$Q_r = Q_{ij} \quad \zeta_r = \zeta_{ij} \quad (6)$$

where $r = 1, \ldots, 9$ for orthotropic composites.

The transformed relaxation moduli $\bar{Q}_{ij}$ with respect to the laminate co-ordinate can be obtained by the co-ordinate transformation:

$$[\bar{Q}] = [T]^{-1}[Q][T]^{-\mathrm{T}} \quad (7)$$

where $[T]$ is the co-ordinate transformation matrix. Then, by using the abbreviated notation, the above transformed $\bar{Q}_{ij}$ becomes

$$\bar{Q}_{ij}(t) = \sum_{r=1}^{9} A_{ijr} Q_r(t) \quad (8)$$

where $\overline{(.)}$ denotes the laminate co-ordinate and A_{ijr} are the transformation coefficients.[11]

Based on Schapery's single integral formulation, the constitutive relations for non-linear thermo-viscoelastic composite materials with respect to the laminate co-ordinate can be expressed as

$$\sigma_i(t) = \sum_{r=1}^{9} \left[\mathscr{A}_{ijr} h_r^{\infty} Q_r^{\infty} \tilde{\varepsilon}_j(\mathrm{t}) + \mathscr{A}_{ijr} h_r^1 \cdot \int_0^t \Delta Q_r[\zeta_r(t) - \zeta_r'(\tau)] \frac{\partial h_r^2 \tilde{\varepsilon}_j(\tau)}{\partial \tau}\,\mathrm{d}\tau \right] \quad (9)$$

with

$$\tilde{\varepsilon}_j(t) = \varepsilon_j(t) - \varepsilon_j^*(t) \quad (10)$$

In the above, σ_i are stresses and ε_j and ε_j^* are total and hygrothermal strains. Q_r^{∞} and ΔQ_r are the equilibrium moduli at constant strain and transient components defined by linear viscoelasticity. The free hygrothermal strain $\varepsilon_j^*(t)$ is related to the temperature and moisture changes by

$$\varepsilon_j^*(t) = \bar{\alpha}_j \theta^{\mathrm{T}}(t) + \bar{\beta}_j \theta^{\mathrm{H}}(t) \quad (11)$$

where $\bar{\alpha}_j$ and $\bar{\beta}_j$ are, respectively, thermal and hygroscopic expansion coefficients transformed with respect to the laminate co-ordinate system and θ^{T} and θ^{H} are temperature and moisture changes related to an unstressed reference state. The quantities h_r^{∞}, h_r^1, h_r^2 and a_r are strain-dependent material properties. The reduced time ζ_r can be defined as a function of shift factor

$$\zeta_r(t) = \int_0^t a_r(T, M, \varepsilon)\,\mathrm{d}s \quad \zeta_r'(\tau) = \int_0^\tau a_r(T, M, \varepsilon)\,\mathrm{d}s \tag{12}$$

The shift functions a_r may depend on strain, temperature and moisture contents. When the non-linear material parameters are set equal to one, equation (9) reduces to the statement of the linear Boltzman superposition principle.

Material parameters of laminated composites are evaluated by uniaxial tests. However, under uniaxial test conditions, an individual ply within the laminate is in a multiaxial stress state and the influence of other stresses on material parameters must be considered. Lou and Schapery,[26] Hiel *et al.*,[27] Tuttle and Brinson[28] and Walrath[29] have introduced the average matrix octahedral shear stress in order to account for such multiaxial conditions. Similarly, Yi *et al.*[30] introduced the octahedral shear strain $\varepsilon_{\mathrm{oct}}^m$ and then non-linear material parameters can be expressed as functions of single invariant

$$\varepsilon_{\mathrm{oct}}^{m2} = \tfrac{1}{9}[(\varepsilon_1^m - \varepsilon_2^m)^2 + (\varepsilon_2^m - \varepsilon_3^m)^2 + (\varepsilon_3^m - \varepsilon_1^m)^2] \tag{13}$$

where ε_1^m, ε_2^m and ε_3^m are principal strains of the matrix.

Finite element formulation

The finite element equilibrium equations for analyses of non-linear viscoelastic composite laminates are derived from the constitutive integral equation and virtual work. The displacement components U, V, W in equations (1) may be approximated as

$$\mathbf{U}^{(e)} = \boldsymbol{\Psi}\mathbf{d}^{(e)} \tag{14}$$

where

$$\mathbf{U}^{(e)} = \begin{Bmatrix} U(y, z, t) \\ V(y, z, t) \\ W(y, z, t) \end{Bmatrix} \tag{15}$$

$$\boldsymbol{\Psi} = \begin{bmatrix} \psi_1 & 0 & 0 & \dots & \psi_l & 0 & 0 \\ 0 & \psi_1 & 0 & \dots & 0 & \psi_l & 0 \\ 0 & 0 & \psi_1 & \dots & 0 & 0 & \psi_l \end{bmatrix} \tag{16}$$

$$\mathbf{d}^{(e)} = \lfloor\, u_1, v_1, w_1, u_2, v_2, w_2, \dots, u_l, v_l, w_l \,\rfloor^{\mathrm{T}} \tag{17}$$

In the above, l is the number of nodes for each element, $\boldsymbol{\Psi}$ is the shape function matrix and $\mathbf{d}^{(e)}$ is the element nodal displacement vector.

By substituting equation (14) into (1), the displacements with an element can be expressed in terms of curvatures, axial and twisting strains, and nodal displacements

$$\mathbf{u}^{(e)} = \mathbf{M}\mathbf{q}^{(e)} = [\mathbf{L}\boldsymbol{\Psi}]\,\mathbf{q}^{(e)} \tag{18}$$

where

$$\mathbf{u}^{(e)} = \begin{Bmatrix} u(x, y, z, t) \\ v(x, y, z, t) \\ w(x, y, z, t) \end{Bmatrix} \tag{19}$$

$$\mathbf{L} = \begin{bmatrix} -yz & xy & xz & x \\ xz & -\frac{x^2}{2} & 0 & 0 \\ xy & 0 & -\frac{x^2}{2} & 0 \end{bmatrix} \tag{20}$$

and

$$\mathbf{q}^{(e)} = \begin{Bmatrix} c_1(t) \\ c_4(t) \\ c_5(t) \\ c_6(t) \\ \mathbf{d}^{(e)} \end{Bmatrix} \tag{21}$$

The following strain–displacement relationship can be obtained by differentiating equation (18) with respect to x_i

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{q}^{(e)} \tag{22}$$

where $\boldsymbol{\varepsilon}$ is the strain tensor and $\mathbf{B}$ is the strain–displacement matrix.

Using virtual work and the constitutive integral equations, finite element equilibrium equations for non-linear viscoelastic composite laminates can now be formulated. In the absence of body forces, the virtual work principle for element (e) becomes

$$\delta\pi^{(e)} = \int_{V^{(e)}} \delta\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{\sigma}\, \mathrm{d}V^{(e)} - \int_{S^{(e)}} \delta\mathbf{u}^{(e)^{\mathrm{T}}}\, \mathbf{T}^{(e)}\, \mathrm{d}S^{(e)} = 0 \tag{23}$$

where $\delta\boldsymbol{\varepsilon}$ is the virtual strain tensor, $\boldsymbol{\sigma}$ the stress tensor, $\mathbf{T}^{(e)}$ the boundary tractions, $\delta\mathbf{u}^{(e)}$ the virtual displacement vector, $V^{(e)}$ the body volume and $S^{(e)}$ is the surface on which boundary tractions are prescribed.

Similar to the stress–strain relationships, finite element equilibrium equations for non-linear viscoelastic bodies can also be stated as hereditary integral equations. By using equations (18) and (22) and the virtual work principle, the finite element equilibrium equations are obtained for each element. The finite element equilibrium equations for viscoelastic composites can be stated as the following integral equations:

$$\sum_{r=1}^{9} \left[h_r^{\infty^{(e)}} k_{mnr}^{\infty^{(e)}} q_n^{(e)}(t) + h_r^{1^{(e)}} \int_0^t k_{mnr}^{t^{(e)}}(\mathbf{x}, \zeta_r - \zeta_r') \frac{\partial h_r^{2^{(e)}} q_n^{(e)}(s)}{\partial s}\, \mathrm{d}s \right] = f_m^{(e)}(t) + f_m^{h^{(e)}}(t) \tag{24}$$

In the above, $k_{mnr}^{\infty^{(e)}}$ and $k_{mnr}^{t^{(e)}}$ are the element stiffness matrices, $q_n^{(e)}$ is the vector of element nodal displacements and $f_m^{(e)}$ and $f_m^{h^{(e)}}$ are element nodal force vectors due to applied surface tractions, uniaxial strain, pure bending or twisting and hygrothermal loadings, respectively. Non-linear material parameters $h_r^{\infty^{(e)}}$, $h_r^{1^{(e)}}$, $h_r^{2^{(e)}}$ and $a_r^{(e)}$ can be described as functions of displacements. The element stiffness matrix and the element nodal force vectors can be defined as follows:

$$k_{mnr}^{\infty^{(e)}} = \iint_{\Gamma^{(e)}} B_{im} \mathscr{A}_{ijr} Q_r^{\infty} B_{jn} \,\mathrm{d}y\,\mathrm{d}z \tag{25}$$

$$\begin{aligned} k_{mnr}^{t^{(e)}}(\mathbf{x}, \zeta_r - \zeta_r') &= \iint_{\Gamma^{(e)}} B_{im} \mathscr{A}_{ijr} \Delta Q_r(\zeta_r - \zeta_r') B_{jn} \,\mathrm{d}y\,\mathrm{d}z \\ &= \sum_{\omega=1}^{\mathrm{NT}_r} \iint_{\Gamma^{(e)}} B_{im} \mathscr{A}_{ijr} Q_{r\omega}^t \exp[-(\zeta_r - \zeta_r')/\lambda_{r\omega}] B_{jn} \,\mathrm{d}y\,\mathrm{d}z \\ &= \sum_{\omega=1}^{\mathrm{NT}_r} k_{mnr\omega}^{t^{(e)}} \exp[-(\zeta_r - \zeta_r')/\lambda_{r\omega}] \end{aligned} \tag{26}$$

$$f_m^{(e)}(t) = \int_{S^{(e)}} M_{lm} T_l(t) \,\mathrm{d}S^{(e)} \tag{27}$$

(no summation over repeated r and $l = 1, 2, 3$)

and the residual nodal force vector due to hygrothermal loads becomes

$$\begin{aligned} f_m^{h^{(e)}}(t) = \sum_{r=1}^{9} \Bigg[& h_r^{\infty^{(e)}} \iint_{\Gamma^{(e)}} B_{im} \mathscr{A}_{ijr} Q_r^{\infty} \varepsilon_j^*(t) \,\mathrm{d}y\,\mathrm{d}z + h_r^{1^{(e)}} \\ & \times \int_{\Gamma^{(e)}} \int_{\tau=0}^{\tau=t} B_{im} \mathscr{A}_{ijr} Q_r^t(\zeta_r - \zeta_r') \frac{\partial h_r^{2^{(e)}} \varepsilon_j^*(\tau)}{\partial \tau} \,\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}z \Bigg] \end{aligned} \tag{28}$$

where $\Gamma^{(e)}$ is the area of element.

Using an exponential series for relaxation moduli, the force vector can be rewritten as

$$\begin{aligned} f_m^{h^{(e)}}(t) = \sum_{r=1}^{9} \Bigg[& h_r^{\infty^{(e)}} \cdot \{ f_{mr}^{\infty^{\alpha(e)}} \cdot \theta^{\mathrm{T}}(t) + f_{mr}^{\infty^{\beta(e)}} \cdot \theta^{\mathrm{H}}(t) \} \\ & + h_r^{1^{(e)}} \sum_{\omega=1}^{\mathrm{NT}_r} \Bigg\{ \int_{\tau=0}^{\tau=t} f_{mr\omega}^{t^{\alpha(e)}} \exp[-(\zeta_r - \zeta_r')/\lambda_{r\omega}] \cdot \frac{\partial h_r^{2^{(e)}} \theta^{\mathrm{T}}(\tau)}{\partial \tau} \\ & + f_{mr\omega}^{t^{\beta(e)}} \exp[-(\zeta_r - \zeta_r')/\lambda_{r\omega}] \frac{\partial h_r^{2(e)} \theta^{\mathrm{H}}(\tau)}{\partial \tau} \Bigg\} \,\mathrm{d}\tau \Bigg] \end{aligned} \tag{29}$$

where

$$\left.\begin{aligned} f_{mr}^{\infty^{\alpha(e)}} &= \iint_{\Gamma^{(e)}} B_{im} \mathscr{A}_{ijr} Q_r^{\infty} \bar{\alpha}_j \,\mathrm{d}y\,\mathrm{d}z \\ f_{mr}^{\infty^{\beta(e)}} &= \iint_{\Gamma^{(e)}} B_{im} \mathscr{A}_{ijr} Q_r^{\infty} \bar{\beta}_j \,\mathrm{d}y\,\mathrm{d}z \\ f_{mr\omega}^{t^{\alpha(e)}} &= \iint_{\Gamma^{(e)}} B_{im} \mathscr{A}_{ijr} Q_{r\omega}^t \bar{\alpha}_j \,\mathrm{d}y\,\mathrm{d}z \\ f_{mr\omega}^{t^{\beta(e)}} &= \iint_{\Gamma^{(e)}} B_{im} \mathscr{A}_{ijr} Q_{r\omega}^t \bar{\beta}_j \,\mathrm{d}y\,\mathrm{d}z \end{aligned}\right\} \tag{30}$$

(no summation over repeated r)

The global matrices can be assembled from the element matrices and then the finite element equilibrium equations for the global system become

$$\sum_{r=1}^{9}\left[h_r^{\infty}\cdot k_{mnr}^{\infty}q_n(t)+h_r^1\cdot\int_0^t k_{mnr}^t(\mathbf{x},\zeta_r-\zeta_r')\frac{\partial h_r^2 q_n(s)}{\partial s}\,\mathrm{d}s\right]=f_m(t)+f_m^h(t) \tag{31}$$

Since the above equations are hereditary integrals, a direct integration of equation (31) requires enormous storage memory and computational time. To overcome these difficulties, a numerical algorithm similar to that used for linear viscoelastic materials by Yi *et al.*[30] was developed for the solution of equations (31). The formulation requires storage of only the previous time solution instead of all the solutions throughout the loading time history.

Let

$$\hat{q}_{nr}(t)=h_r^2\cdot q_n(t),\quad \hat{\theta}_r^{\mathrm{T}}(t)=h_r^2\cdot\theta^{\mathrm{T}}(t),\quad \hat{\theta}_r^{\mathrm{H}}(t)=h_r^2\cdot\theta^{\mathrm{H}}(t) \tag{32}$$

then the governing equations can be integrated step by step using a finite difference recurrence relationship for approximate calculations of derivatives of equations (31). By assuming that the $\hat{q}_{nr}$ vary linearly over each time step Δt_j, the variables $\hat{q}_{nr}$ and their time derivatives are given by

$$\frac{\partial\hat{q}_{nr}(t)}{\partial t}\simeq\frac{\Delta\hat{q}_{nr}(t_j)}{\Delta t_j}=\frac{\hat{q}_{nr}(t_j)-\hat{q}_{nr}(t_{j-1})}{\Delta t_j} \tag{33}$$

with

$$\Delta t_j=t_j-t_{j-1}\quad t_{j-1}\leqslant t\leqslant t_j$$

If no loading is applied at time $t<0$ then

$$\Delta\hat{q}_{nr}^{\mathrm{T}}(0)=\hat{q}_{nr}^{\mathrm{T}}(0) \tag{34}$$

Similarly, the derivatives of $\hat{\theta}_r^{\mathrm{T}}$ and $\hat{\theta}_r^{\mathrm{H}}$ can be obtained.

By using finite difference approximations in equations (33), equations (31) can be expressed in a recursive form as

$$\sum_{r=1}^{9}\left\{h_r^{\infty}k_{mnr}^{\infty}+h_r^1h_r^2\sum_{\omega=1}^{\mathrm{NT}_r}k_{mnr\omega}\cdot S_{r\omega}(\Delta t_p)\right\}\Delta q_n(t_p)=f_m(t_p)+f_m^h(t_p)$$
$$-\sum_{r=1}^{9}\left\{h_r^{\infty}k_{mnr}^{\infty}q_n(t_{p-1})+\sum_{\omega=1}^{\mathrm{NT}_r}h_r^1\cdot R_{mr\omega}(t_p)\right\} \tag{35}$$

with

$$f_m^h(t_p)=\sum_{r=1}^{9}\left[h_r^{\infty}\cdot\{f_{mr}^{\infty^{\alpha}}\theta^{\mathrm{T}}(t_p)+f_{mr}^{\infty^{\beta}}\theta^{\mathrm{H}}(t_p)\}\right.$$
$$\left.+h_r^1\cdot\sum_{\omega=1}^{\mathrm{NT}_r}\{f_{mr\omega}^{t^{\alpha}}\cdot\Delta\hat{\theta}_r^{\mathrm{T}}(t_p)+f_{mr\omega}^{t^{\beta}}\cdot\Delta\hat{\theta}_r^{\mathrm{H}}(t_p)\}\,S_{r\omega}(\Delta t_p)\right] \tag{36}$$

$$R_{mr\omega}(t_p) = \mathrm{e}^{-\Delta\zeta_r(t_p)/\lambda_{r\omega}}[R_{mr\omega}(t_{p-1}) + \{k_{mnr\omega}\Delta\hat{q}_{nr}(t_{p-1}) - f^{t^\alpha}_{mr\omega}\Delta\hat{\theta}^{\mathrm{T}}_r(t_{p-1}) - f^{t^\beta}_{mr\omega}\Delta\hat{\theta}^{\mathrm{H}}_r(t_{p-1})\}S_{r\omega}(\Delta t_{p-1})] \quad (37)$$

$$S_{r\omega}(\Delta t_p) = \frac{1}{\Delta t_p}\int_{t_{p-1}}^{t_p} \exp[-\Delta\zeta_r(t_p)/\lambda_{r\omega}]\,\mathrm{d}\tau$$

$$\Delta\zeta_r(t_p) = \zeta_r(t_p) - \zeta_r(t_{p-1}) \quad (38)$$

$$R_{mr\omega}(0) = 0$$

$$S_{r\omega}(0) = 1$$

(no summation over r)

Note that equations (35) are recursive and that it is possible to solve iteratively for the displacements at time t_p using only the previous solution at time t_{p-1}.

NUMERICAL RESULTS

To verify the present formulation, linear elastic free edge stresses due to twisting or bending evaluated by the present formulation are compared with the approximate elastic solutions obtained by Yin.[21,22] It is noticed that the present finite element formulation reduces to the statement of the linear Boltzman superposition principle as the non-linear material parameters are set equal to one and the linear viscoelastic solution at $t = 0$ is equal to an elastic one. Four-layer symmetric laminates with the thickness h are considered. The elastic properties for composites are $E_{11} = 20 \times 10^6$ psi, $E_{22} = E_{33} = 2{\cdot}1 \times 10^6$ psi and $G_{12} = G_{13} = G_{23} = 0{\cdot}85 \times 10^6$ psi. The Poisson's ratio ν_{13} and ν_{23} are also taken to be the same as $\nu_{12} = 0{\cdot}21$. A laminate width to thickness ratio is four. As shown in Figure 2, the finite element mesh consists of 28×8 meshes in the yz cross-section. Interlaminar stresses have been obtained for $(0/90)_s$ cross-ply and $(45/-45)_s$ angle-ply laminates subjected to bending and twisting. A twisting deformation produces a very large interlaminar stress since the shear strain near the free edge caused by the twisting load is an order of magnitude greater than ones by bending or extensional loads. Comparison studies between non-linear viscoelastic finite element solutions and experimental results were also reported by Yi *et al.*[30] In their study, excellent agreement between these results was also obtained. In this study, a comparison of the present FEM results with the approximate elastic solutions obtained by Yin[21,22] is made and the results are plotted in Figures 3 and 4. The good agreement between the two solutions has been obtained.

Several studies were conducted to analyse the non-linear time-dependent responses near the free edges of laminated composites. In this study, the relaxation moduli were evaluated by Schapery's non-linear stress–strain relationship and by relaxation/relaxation recovery analysis as

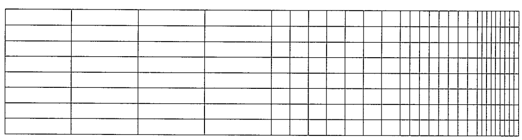

Figure 2. Finite element mesh

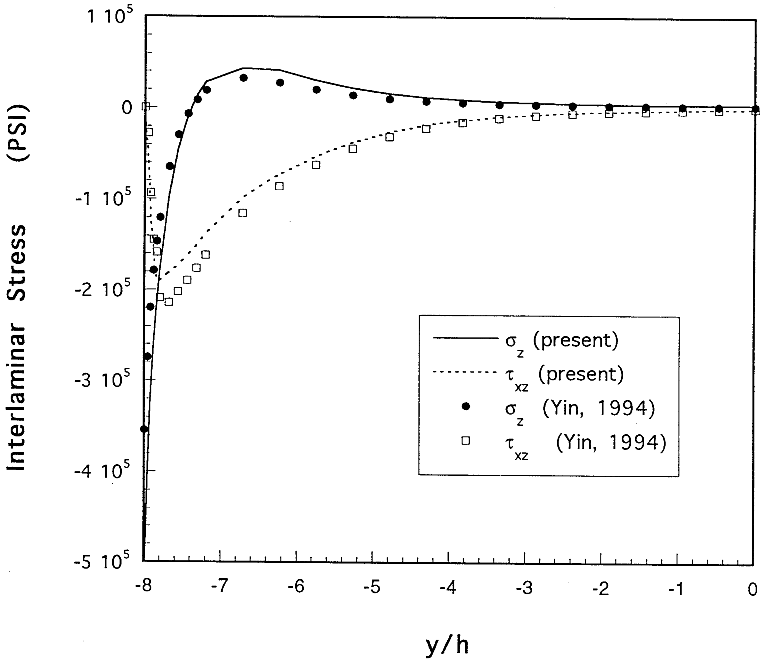

Figure 3. Elastic interlaminar stresses along the interface between 0 and 90° layers in a $(0/90)_s$ laminate under bending

extended to anisotropic relations, equation (9). T300/5208 graphite/epoxy composites are considered. Using creep and creep recovery tests, the master compliance curves and shift factors corresponding to various loading conditions were obtained from Tuttle and Brinson.[28] It is assumed that time function for Q_{33} is equal to the one for Q_{22}, and that $Q_{66} = Q_{44} = Q_{55}$. Q_{11} is taken to be elastic since it is generally controlled by fibre properties. Also the time functions for Q_{12}, Q_{13} and Q_{23} are taken to be the same as that for Q_{22}. We consider four-layer symmetric cross-ply and angle and angle-ply laminates subjected to bending and twisting. The laminate width to thickness ratio is four and the ply thickness is 0·0056 in. T300/5208 laminates with $(0/90)_s$ and $(45/-45)_s$ layups were considered and nine-node isoparametric elements were used. The step-size Δt is set to 20 s initially and Δt increases with time. There are 72 time steps involved in the calculation of time-dependent interlaminar stresses over a period of 35 days. The non-linear time-dependent interlaminar stresses in composite laminates were studied as a function of time and loading magnitude. Three bending loadings such as $c_5 = -0{\cdot}2$, $c_5 = -0{\cdot}25$ and $c_5 = -0{\cdot}3$ were considered under the isothermal conditions ($T = 147°$F). The bending loadings were held constant for the entire loading period. The interlaminar stresses σ_z near the free edge of laminate ($y/h = -0{\cdot}995$) were plotted in Figure 5. Appreciable stress relaxation occurred during the loading period. Such relaxation resulted in decreasing the magnitude of residual stresses. As shown in Figure 5, non-linear behaviour was observed at $c_5 = -0{\cdot}3$. The results of interlaminar stresses along the interface between the 0 and 90° layers are shown in Figure 6. In $(0/90)_s$ laminates, the interlaminar shear stress τ_{xz} is equal to zero. However, in the case of angle-ply laminates, both σ_z and τ_{xz} are present near the free edge. A second study was conducted for a $(45/-45)_s$ laminate subjected to twisting loads. A step loading was used by applying

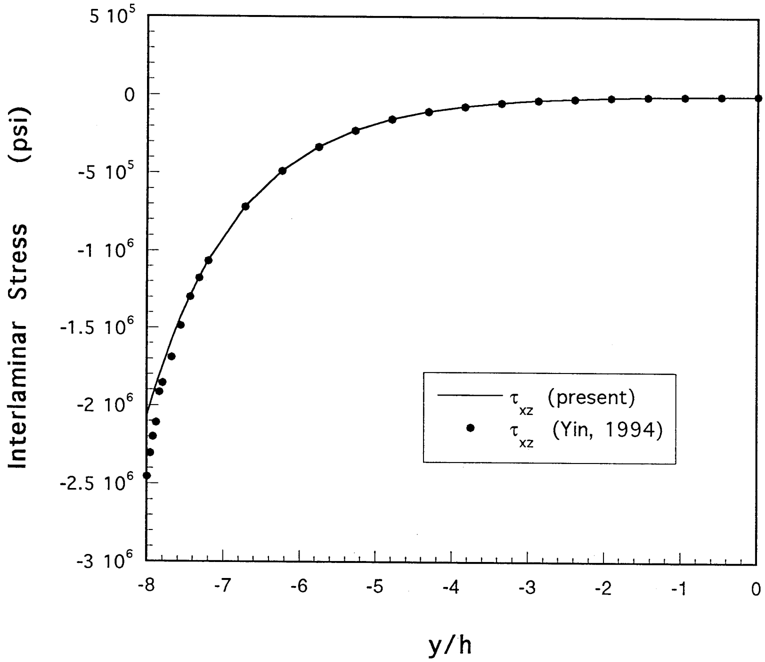

Figure 4. Elastic interlaminar stresses along the interface between 45 and $-45°$ layers in a $(45/-45)_s$ laminate under twisting deformation

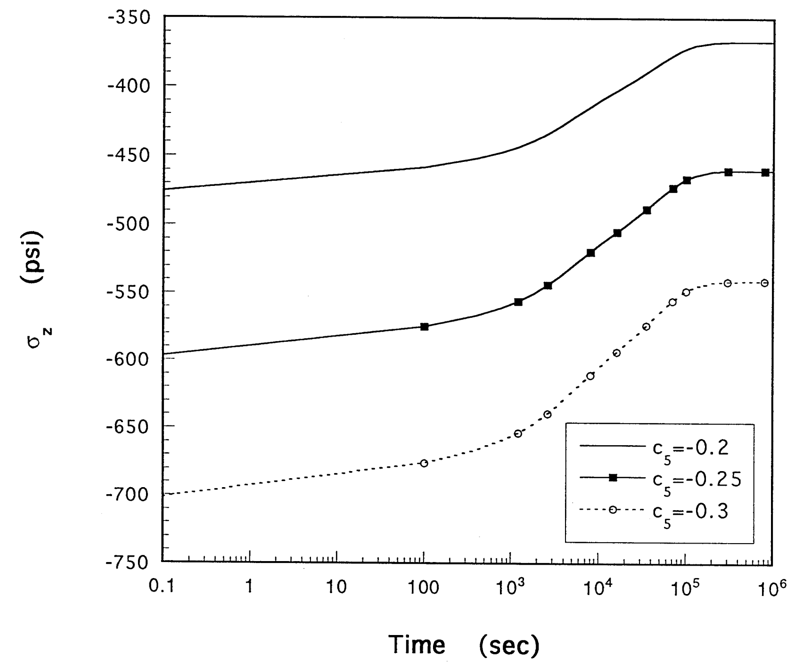

Figure 5. Time-dependent interlaminar stress σ_z near the free edge of a $(0/90)_s$ laminate under bending ($y/b = 0{\cdot}995$)

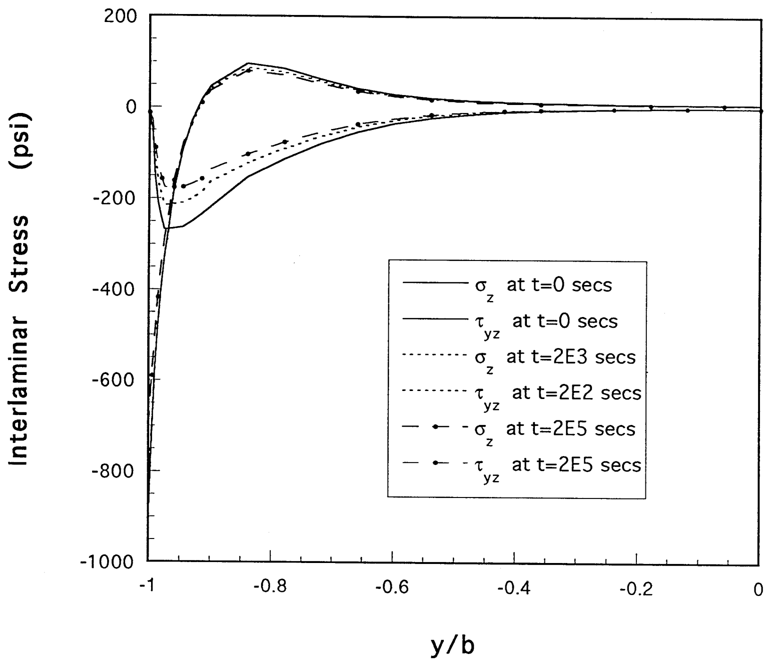

Figure 6. Time-dependent interlaminar stresses in a $(0/90)_s$ laminate under bending

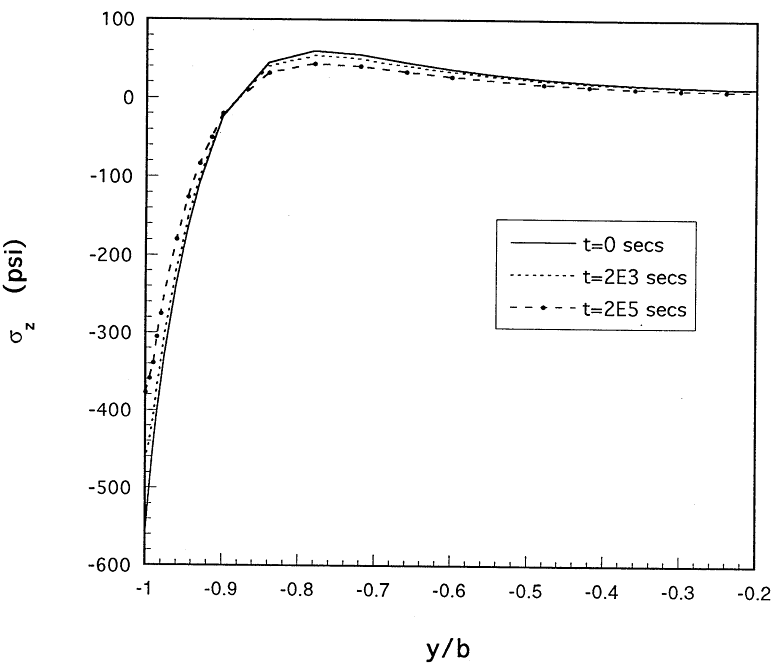

Figure 7. Time-dependent interlaminar stress σ_z in a $(45/-45)_s$ laminate under twisting deformation

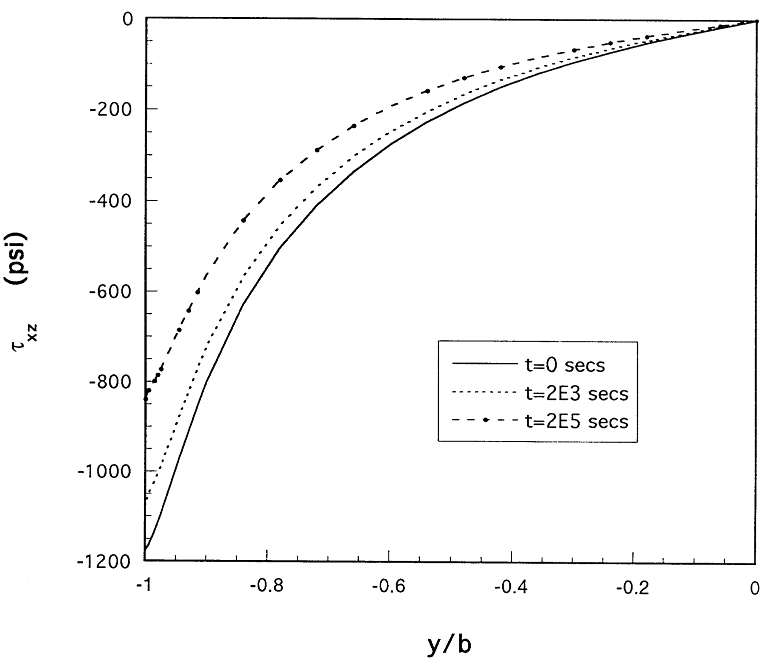

Figure 8. Time-dependent interlaminar stresses τ_{xz} in a $(45/-45)_s$ laminate under twisting deformation

$c_2 = -6 \times 10^{-2}$ in in^{-1}. The time-dependent interlaminar stresses at the $(45/-45)$ interface are illustrated in Figures 7 and 8. Normal interlaminar stresses σ_z are depicted in Figure 7 while shear stresses τ_{xz} are plotted in Figure 8. Over a period of 2·3 days, the stresses σ_z and τ_{xz} relaxed about 34 and 28·5 per cent, respectively.

CONCLUSIONS

Quasi-static non-linear anisotropic viscoelastic responses of laminated composites subjected to arbitrary combinations of axial extension, bending, twisting load and/or hygrothermal loads are formulated in terms of finite element algorithms based on Schapery's non-linear constitutive relations and Pipes and Pagano's displacement field for laminates under a generalized plane deformation state. Direct time integrations are used in this formulation. The recursive numerical algorithm is introduced to reduce the computer storage. This combination of the finite element method for spatial discretization and recursive formula makes it possible to have efficient solutions of non-linear thermo-viscoelastic problems. Numerical results have been obtained for cross-ply and angle-ply laminates, which demonstrate the feasibility of the present formulation.

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