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THERMOMECHANICAL ANALYSIS OF VISCOELASTIC SOLIDS

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SUMMARY

This paper is concerned with the development of a computational algorithm for the solution of the uncoupled, quasi-static boundary value problem for a linear viscoelastic solid undergoing thermal and mechanical deformation. The method evolves from a finite element discretization of a stationary value problem, leading to the solution of a system of linear integral equations determining the motion of the solid. An illustrative example is included.

INTRODUCTION

In observing the mechanical behaviour of viscoelastic solids, it is common experience that the stress at a particle depends on both the history of (localized) motion of the solid as well as the temperature history at the particle. Although a general theory of thermomechanical behaviour of materials has been developed by Coleman, among others, the application to engineering practice seems remote at the present time. However, if limitations on the generality of the theory are introduced, it is possible to develop more specialized methods of characterizing thermomechanical behaviour leading to computational techniques for boundary value problems. This paper is concerned with such a procedure. A thermomechanical constitutive equation appropriate to viscoelastic solids undergoing small, quasi-static deformations is utilized, along with field equations for this class of deformation. The heat conduction equation is assumed to be unaffected by the deformation and is therefore solved separately, but simultaneously, with the mechanical field problem.

We begin with a discussion of a mechanical constitutive equation for solids undergoing small deformations and subjected to temperature changes. A constitutive functional linear in deformation but non-linear in temperature is adopted and various representations are discussed. Specialization appropriate to thermorheologically simple solids is indicated.

On page 48, field equations are adjoined to the constitutive equation and the uncoupled, quasi-static boundary value problem for a class of viscoelastic solids is posed. A functional whose stationary value is equivalent to the indicated boundary value problem is stated on page 48.

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On pages 49 and 51, a computational algorithm based upon the finite element technique of discretizing the stationary value problem of a functional is described. As an application of the method a problem recently studied by Lockett and Morland[7] is examined. Agreement with their results, which depend on a method of more limited scope, is generally good.

CONSTITUTIVE EQUATIONS FOR A CLASS OF VISCOELASTIC SOLIDS

Consider a body undergoing small deformations from an unstressed reference state and simultaneously subjected to temperature changes relative to the same reference state. The theory of simple materials postulates that the stress at a particle of the body is determined by the histories of deformation and temperature at the particle. In the present context in symbolic form† this is expressed by the equation

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{F}_{s=-\infty}^{s=t} [\boldsymbol{\varepsilon}(\mathbf{x}, s), T(\mathbf{x}, s); \mathbf{x}, t] \tag{1}$$

where $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ are the stress and (small) strain tensors at the place $\mathbf{x}$ at time t; T is the temperature at $\mathbf{x}$ and $\mathbf{F}$ is the thermomechanical response functional of the material of the body, i.e. the functional that assigns to every small strain history and temperature history the value of the stress tensor at $\mathbf{x}$. Guided by experience with many engineering applications of viscoelastic solids, we introduce the assumption that the thermomechanical response functional is linear in strain and non-linear in temperature. With this in mind and with further restriction to homogeneous, non-ageing materials we replace equation (1) by the hereditary integral representation‡

$$\boldsymbol{\sigma}(t) = \int_{\tau=-\infty}^{\tau=t} \mathbf{C}_{s=0}^{s=t} [T(s); t-\tau] \frac{\partial}{\partial \tau} [\boldsymbol{\varepsilon}(\tau) - \boldsymbol{\theta}(\tau)] \, \mathrm{d}\tau \tag{2}$$

In equation (2) we have introduced the pseudo-temperature

$$\boldsymbol{\theta}(\tau) = \int_{T_0}^{T} \boldsymbol{\alpha}(T') \, \mathrm{d}T' \tag{3}$$

where $\boldsymbol{\alpha}$ is the temperature-dependent thermal coefficient of expansion tensor and the kernel

$$\mathbf{C} = \mathbf{C}_{s=0}^{s=t} [T(s); t-\tau] \tag{4}$$

is a fourth-rank relaxation modulus tensor whose value depends upon the temperature history of the material. For a *prescribed* temperature history the relaxation modulus reduces to

$$\mathbf{C} = \mathbf{C}(t; t-\tau) \tag{5}$$

This form resembles the kernel of an ageing linear viscoelastic solid and emphasizes the role of temperature history on viscoelastic material properties, i.e. temperature has an effect equivalent to 'ageing' of the material whose relaxation modulus is of the form of equation (4). Development of related computational algorithms will not be discussed further here in this connection (see Reference 2). Instead we first return to equation (4) and examine the case where the temperature is constant, but different from the reference temperature T_0.

We then adopt the postulate for thermorheologically simple (TS) materials, i.e.

$$\mathbf{C}(t; T) = \mathbf{C}[\xi(t); T_0] \tag{6}$$

† For the present direct notation will be employed, i.e. symbols in bold type are tensors of order indicated by the context. Further, for the class of small deformations considered, no distinction between 'particle' and 'place' need be made.

‡ Dependence of field variables on $\mathbf{x}$ is understood.

where the reduced time ξ is defined by

$$\xi(t) = t\varphi(T) \tag{7}$$

and the temperature shift function $\varphi(T)$ is assumed to be intrinsic material property normalized by the condition $\varphi(T_0) = 1$. Using equation (7) it is possible to compare the mechanical behaviour of a material at different *constant* temperatures. To extend the idea of a temperature shift function to non-isothermal applications a further postulate is required. In the past, more from lack of contrary evidence than from experimental confirmation, it has been assumed that equation (4) could be replaced by

$$\mathbf{C}_{s=0}^{s=t}[t-\tau;\, T(s)] = \mathbf{C}[\xi(t)-\xi(\tau);\, T_0] \tag{8}$$

where the reduced time is now defined by

$$\xi(\tau) = \int_0^\tau \varphi[T(s)]\,\mathrm{d}s \tag{9}$$

For constant temperature, equation (9) clearly reduces to equation (6). Equation (9) is only one of many postulates that might be used to extend the notion of a TS material to non-isothermal cases. For example, we might assume that the reduced time depends on both temperature and temperature rate histories, i.e.

$$\xi^*(\tau) = \int_0^\tau f\left[T(s), \frac{\partial T}{\partial s}\right] \mathrm{d}s \tag{10}$$

or possibly on time derivatives of arbitrary order. In the case of equation (10), assuming that the temperature history shift function is separable,

$$f\left(T, \frac{\partial T}{\partial s}\right) = \varphi(T)\,\psi\left(\frac{\partial T}{\partial s}\right) \tag{11}$$

we have for the case of a linear (in time) temperature history $T(s) = ks$, the following result

$$\xi^*(\tau) = \int_0^\tau \varphi[T(s)]\,\psi(k)\,\mathrm{d}s = \psi(k)\,\xi(\tau) \tag{12}$$

from which it follows that

$$\log \xi^* = \log \psi + \log \xi \tag{13}$$

that is, the presence of a temperature rate function ψ admits the possibility of two time shifts, one for temperature and one for temperature rate. In the case of constant temperature, $k = 0$ and by taking $\psi(0) = 1$ we have $\xi^* = \xi$, i.e. the TS postulate is recovered. We will not pursue this matter further here but remark once again that determination of the response functional for non-isothermal histories requires a further postulate for TS materials. In the sequel any definition of reduced time of the form (10) or generalizations thereof will be admissible.

Returning to equation (2) and using equation (6) we adopt as our response functional

$$\boldsymbol{\sigma}(t) = \int_{-\infty}^{t} \mathbf{C}[\xi(t)-\xi(\tau);\, T_0]\frac{\partial}{\partial \tau}[\boldsymbol{\varepsilon}(\tau)-\boldsymbol{\theta}(\tau)]\,\mathrm{d}\tau \tag{14}$$

Anticipating later applications of equation (14) we introduced the equivalent abbreviated form

$$\boldsymbol{\sigma}(t) = \mathbf{C} \otimes [\boldsymbol{\varepsilon}^{(1)} - \boldsymbol{\theta}^{(1)}] \tag{15}$$

where

$$\boldsymbol{\varepsilon}^{(1)} \equiv \frac{\partial \boldsymbol{\varepsilon}}{\partial \tau}, \quad \text{etc.,}$$

and the symbol ($\otimes$) denotes the integral of the composition of the adjoined tensors. In equation (14) since $\mathbf{C}$ is of rank four and $\boldsymbol{\varepsilon}, \boldsymbol{\theta}$ of rank two, the symbol in equation (15) denotes the integral of a doubly contracted composition whose value is a second-rank stress tensor. In component form, equation (15) is

$$\sigma_{ij}(t) = \int_{-\infty}^{t} C_{ij}^{kl}[\xi(t)-\xi(\tau);\, T_0]\frac{\partial}{\partial\tau}[\varepsilon_{kl}(\tau)-\theta_{kl}(\tau)]\,\mathrm{d}\tau$$

We assume in the sequel that $C_{ijkl} = C_{klij}$.

FORMULATION OF THE BOUNDARY VALUE PROBLEM

By a thermomechanical boundary value problem for a viscoelastic solid we understand the following: a mechanically linear, TS material undergoing quasistatic deformation and subjected to an independently determined temperature field satisfies the equilibrium equations

$$\nabla.\boldsymbol{\sigma}+\mathbf{f} = \mathbf{0} \tag{16}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

where $\mathbf{f}$ is a prescribed body force vector, the strain-displacement equations

$$2\boldsymbol{\varepsilon} = [\nabla\mathbf{u}+(\nabla\mathbf{u})^T] \tag{17}$$

where $\mathbf{u}$ is the displacement vector, and constitutive equations

$$\boldsymbol{\sigma} = \mathbf{C}\otimes[\boldsymbol{\varepsilon}^{(1)}-\boldsymbol{\theta}^{(1)}] \tag{18}$$

in a region of space R occupied by the body. To these equations are adjoined the following boundary conditions:

$$\mathbf{T}(\mathbf{x},t) = \mathbf{n}.\boldsymbol{\sigma} = \bar{\mathbf{T}} \quad \text{on } S_\sigma \tag{19}$$

$$\mathbf{u}(\mathbf{x},t) = \bar{\mathbf{u}} \quad \text{on } S_u$$

In equation (19) $\mathbf{n}$ is the outward unit vector normal to the boundary surface of the body, and $\bar{\mathbf{T}}, \bar{\mathbf{u}}$ are prescribed values of the surface traction vector and displacement vector on complementary parts of the boundary of the body, S_σ and S_u, respectively. The temperature of the body is assumed to be a prescribed function of position and time.

A direct computational method for attacking the boundary value problem follows in the next section.

A VARIATIONAL THEOREM

For computational purposes it is expedient to recast the boundary value problem posed in the previous section as a stationary value problem for a functional. Accordingly, we define a thermomechanical state functional $V\{\mathbf{u}\}$ through the equation

$$V\{\mathbf{u}\} = \int_R [\tfrac{1}{2}\mathbf{C}\otimes\boldsymbol{\varepsilon}*\boldsymbol{\varepsilon}-\mathbf{C}\otimes\boldsymbol{\theta}*\boldsymbol{\varepsilon}-h*\mathbf{f}*\mathbf{u}]\,\mathrm{d}v-\int_{S_\sigma} h*\bar{T}*\mathbf{u}\,\mathrm{d}s \tag{20}$$

In equation (20) h is the Heaviside step function defined as unity for $t>0$ and zero for $t<0$; the asterisk ($*$) denotes the convolution of two functions in the sense

$$\mathbf{f}*\mathbf{g} = \int_{-\infty}^{t} \mathbf{f}(t-\tau)\,\mathbf{g}(\tau)\,\mathrm{d}\tau \tag{21}$$

We assume that the body is undisturbed over the interval $-\infty<\tau<0^+$; consequently an explicit statement of initial conditions is not required.

We define an admissible thermomechanical state associated with the functional $V\{\mathbf{u}\}$ as follows:

1. The (symmetric) stress tensor is determined by the constitutive equation (18).
2. The strain-displacement equations (17) are satisfied.
3. The displacement vector satisfies equation (19).
4. The pseudo-temperature $\boldsymbol{\theta}$ is a prescribed function of place and time associated with a solution of the heat conduction boundary value problem for the body.

We now state the variational theorem: Among all admissible thermomechanical states, that which satisfies the equilibrium equations (16) and stress boundary conditions (19) is given by

$$\delta V = 0 \tag{22}$$

Executing the variation of equation (20), using the Divergence Theorem and equation (18), leads to

$$\delta V = -\int_R \{h * [\nabla.(\mathbf{C} \otimes \boldsymbol{\varepsilon}^{(1)} - \mathbf{C} \otimes \boldsymbol{\theta}^{(1)}) + \mathbf{f}] * \delta\mathbf{u}\}\, \mathrm{d}v + \int_{S_\sigma} [h * (\mathbf{T} - \bar{\mathbf{T}}) * \delta\mathbf{u}]\, \mathrm{d}s = 0 \tag{23}$$

Application of a corollary of Titchmarsh's Theorem[12] (i.e. $f * g = 0$ implies either $f = 0$ or $g = 0$) and using the constitutive equation (18) in the volume integral yields the equilibrium equation (16) and stress boundary condition, equation (19).

In the sequel, along with equation (23) we will adopt a variational theorem, presented by Wilson and Nickell[13] for obtaining solutions of the heat conduction equation. The finite element computer algorithm developed therein, along with the algorithm for the thermomechanical problem to be developed on this page and on page 51 form the basis of the computational work reported here.

SPECIALIZATION FOR AXISYMMETRIC DEFORMATION OF ISOTROPIC VISCOELASTIC SOLIDS—FINITE ELEMENT SOLUTION

In this section the previous results are specialized for a particular class of problems for isotropic solids. For axisymmetric solids subjected to axisymmetric loads (both mechanical and thermal) response occurs in the r, z plane (r, θ, z-co-ordinates) only, hence u_θ is zero and u_r, u_z are functions of r and z only. From equation (17) the non-zero strains are (in terms of physical components)

$$\left.\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\ \varepsilon_{\theta\theta} &= \frac{u_r}{r} \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\ \varepsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \end{aligned}\right\} \tag{24}$$

For isotropic thermomechanical response both $\mathbf{C}$ and $\boldsymbol{\theta}$ appearing in equation (18) are isotropic functions and the following constitutive equations result:

$$\left.\begin{aligned} \sigma_{rr} &= (3K - 2G) \otimes \varepsilon^{(1)} + 2G \otimes \varepsilon_{rr}^{(1)} - K \otimes \theta^{(1)} \\ \sigma_{\theta\theta} &= (3K - 2G) \otimes \varepsilon^{(1)} + 2G \otimes \varepsilon_{\theta\theta}^{(1)} - K \otimes \theta^{(1)} \\ \sigma_{zz} &= (3K - 2G) \otimes \varepsilon^{(1)} + 2G \otimes \varepsilon_{zz}^{(1)} - K \otimes \theta^{(1)} \\ \sigma_{rz} &= 2G \otimes \varepsilon_{rz}^{(1)} \end{aligned}\right\} \tag{25}$$

where

$$3\varepsilon = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} \tag{26}$$

and

$$\theta = \int_{T_0}^{T} \alpha(T')\,\mathrm{d}t'$$

and the material functions K, G have the the form, equation (8). Equation (20) can now be more conveniently written†

$$V = \int_R [\tfrac{1}{2}A_{ij} \otimes S_i * S_j - \theta_i * S_i - h * f_\alpha * u_\alpha]\, r\,\mathrm{d}r\,\mathrm{d}z - \int_{S_r} h * \bar{T}_\alpha * u_\alpha\, r\,\mathrm{d}r\,\mathrm{d}z \tag{20a}$$

where

$$S_i = \left(\frac{u_r}{r}; \frac{\partial u_r}{\partial r}; \frac{\partial u_z}{\partial z}; \frac{\partial u_z}{\partial r}; \frac{\partial u_r}{\partial z}\right) \tag{27}$$

A_{ij} is a 5×5 symmetric array whose non-zero components are

$$\left.\begin{aligned} A_{11} &= A_{22} = A_{33} = \tfrac{1}{3}(3K+4G) \\ A_{12} &= A_{13} = A_{23} = \tfrac{1}{3}(3K-2G) \\ A_{44} &= A_{55} = A_{45} = G \end{aligned}\right\} \tag{28}$$

θ_i denotes thermal terms whose non-zero components are

$$\theta_1 = \theta_2 = \theta_3 = K \otimes \theta$$

$$u_\alpha = (u_r, u_z)$$

$$f_\alpha = (f_r, f_z)$$

and

$$\bar{T}_\alpha = (\bar{T}_r, \bar{T}_z)$$

A Ritz-type solution to equation (20a) may be obtained by a finite element method. To this end, the volume and surface integrals are expressed as a sum of integrals over a set of subregions (finite elements) defining R. Assumed solutions are taken for each element in such a way that displacement continuity is maintained between contiguous elements. In the present developments triangular elements are used together with a linear expansion of the displacement field in each element.

Use of standard finite element procedures for spatial discretization[15] and application of the first variation of V yields a set of $2N$ linear integral equations in terms of the $2N$ nodal displacements (N equals the number of nodes). These may be expressed as

$$\int_{-\infty}^{t} K_{mn}(\xi - \xi')\frac{\partial u_n}{\partial t'}\,\mathrm{d}t' = R_m(t) \tag{29}$$

where $\xi' = \xi(t')$, K_{mn} is an assemblage of element stiffness relaxation functions for the body, and R_m is an assemblage of surface and body loads. The solution of equation (29) yields the nodal point displacement history. The strain and stress histories can then be computed from equations (25) and (26).

† Latin indices range from 1 to 5 while Greek indices range over 1, 2. Summation convention is implied.

SOLUTION OF SIMULTANEOUS INTEGRAL EQUATIONS

The approximate spatial reduction by a finite element method leads to a set of simultaneous integral equations. In the absence of variable temperature history equation (29) reduces to a set of Volterra integral equations of the second kind which theoretically can be solved by integral transform methods. However, with variable temperature history these equations are no longer tractable by transform techniques. Therefore, in order to solve equation (29) direct numerical methods will be employed: A standard numerical technique for solving this class of equations is a step-forward integration procedure. In connection with viscoelastic analyses a finite difference technique has been used to solve the convolution inter-relationship between creep and relaxation.[3,5,10] Stress analyses have also been performed utilizing a finite difference numerical step-forward integration procedure.[1,11,14] The numerical integration procedure consists in expansions into a series of time increments where integrations are performed over each increment according to some difference approximation. The great disadvantage of this method (in connection with computer applications) is that *all* past solutions are required. Thus, in the case of equation (29) extensive amounts of information are required to obtain solutions over extended time periods. Also, considerable computer time is required (it is to be anticipated that a viscoelastic analysis will require a considerable increase in computer time over that of a similar elastic analysis).

Recently some modifications to the above procedure have been proposed. One is based upon the premise of a finite memory in the material, hence, the solution at any time involves only knowledge of a limited history of the past deformations.[9] Thus it is necessary to retain only a finite number of past solutions to obtain a solution at any time. This modification achieves a considerable saving in computation time. However, for large numbers of nodes, the finite element reduction to equation (29) results in excessive computation times.

To obtain a computationally efficient compromise between representation of material response by a finite number of parameters and time required to solve equation (29), an alternative scheme is used here. We represent the kernel functions of the integral equation (29) by the series

$$K(\xi-\xi') = \sum_{i=1}^{I} K_i f_i(\xi) g_i(\xi') \tag{30}$$

If the functions f_i, g_i are elements of a complete set, as I approaches infinity the series uniformly approximates the kernel.[18] As a practical matter I cannot be taken too large and care must be taken in the choice of approximating functions. Furthermore, since a theoretically complete characterization of response cannot be obtained when I is finite, care must be taken to evaluate the free parameters in a given approximation using histories that are representative of the problem under consideration. Experience has shown that generalized Maxwell models (with related negative exponential functions) often constitute good approximations to the response of viscoelastic materials. Furthermore, from a theoretical viewpoint it is known that negative exponentials are complete on the range $0 < t < \infty$.[19] Finally, it should be noted that for an ageing viscoelastic material in the sense of Arutyunyan,[20] a generalization of equation (30) can still be used. When the stress–strain equations are expressed in integral form the kernel function then has the series representation

$$G(\xi) = \sum_{i=1}^{I} G_i \exp(-\xi/\lambda_i) + G_0 \tag{31}$$

where G_i are constants associated with instantaneous response, λ_i are constants associated with a discrete relaxation spectrum (each λ_i can be called a relaxation time) and I is the number of

Maxwell elements used to approximate the materials' relaxation modulus. From equation (31) it follows that

$$G(\xi-\xi') = \sum_{i=1}^{I} G_i \exp(-\xi/\lambda_i)\exp(\xi'/\lambda_i)+G_0 \tag{32}$$

The above decomposition has previously been used in connection with a finite difference integration technique for approximate solution of viscoelasticity problems.[1,14,17] The method is used herein as a finite element concept for the approximate solution of viscoelastic problems which include temperature effects through the thermorheologically simple postulate. The solution technique is similar to that used in References 8 and 13. In order to discuss the solution technique we consider the single integral equation

$$G(\xi)\,u(0)+\int_0^t G(\xi-\xi')\frac{\partial u}{\partial t'}\,\mathrm{d}t' = r(t) \tag{33}$$

Equation (33) may be considered as a typical term in equation (29), consequently, any conclusions obtained from equation (33) are directly applicable to the solution of equation (29). In equation (33) $G(\xi)$ represents a particular relaxation modulus function (i.e. shear or bulk), $r(t)$ is a known forcing function and $u(t)$ is the sought solution. Substituting into equation (33) the material property representation given by equation (32) we obtain

$$G_0\int_0^t \frac{\partial u}{\partial t'}\,\mathrm{d}t' + \sum_{i=1}^{I} G_i \exp(-\xi/\lambda_i)\int_0^t \exp(\xi'/\lambda_i)\frac{\partial u}{\partial t'}\,\mathrm{d}t' = r(t)-u(0)\left[G_0+\sum_{i=1}^{I} G_i\exp(-\xi/\lambda_i)\right] \tag{34}$$

As in the spatial reduction by a finite element method, equation (34) may be discretized by piecewise expansions in time of the dependent variable $u(t)$. A continuous time response may be obtained by assuming a polynomial time expansion and matching nodal displacements between each succeeding time expansion. The simplest expansion is given by the linear Lagrangian interpolation function

$$u(t) = \frac{1}{\Delta t_n}[(t_n-t)\,u_{n-1}+(t-t_{n-1})\,u_n], \quad t_{n-1}\leqslant t\leqslant t_n \tag{35}$$

where

$$u_n = u(t_n)$$
$$\Delta t_n = t_n - t_{n-1}$$

Consequently the time derivative of u during each time increment is constant and is expressed by

$$\frac{\partial u}{\partial t} = \frac{u_n-u_{n-1}}{\Delta t_n} = \frac{\Delta u_n}{\Delta t_n}, \quad t_{n-1}\leqslant t\leqslant t_n \tag{36}$$

If we introduce the notation†

$$h_i(\Delta t_j) = \frac{1}{\Delta t_j}\int_{t_{j-1}}^{t_j} \exp[-(\xi_j-\xi')/\lambda_i]\,\mathrm{d}t' = \frac{1}{\Delta t_j}\int_0^{\Delta t_j}\exp(-\xi'/\lambda_i)\,\mathrm{d}t' \tag{37}$$

then at time t_n an approximate solution to equation (33) is given by

$$\sum_{j=1}^{n}\left\{G_0+\sum_{i=1}^{I} G_i\exp[-(\xi_n-\xi_j)/\lambda_i]\,h_i(\Delta t_j)\right\}\Delta u_j = r(t_n)-\left[G_0+\sum_{i=1}^{I} G_i\exp(-\xi_n/\lambda_i)\right]u_0 \tag{38}$$

where $t_0 = 0$; all other t_j represent previous or present discrete solution points and ξ_n is the value

† It should be noted that $h_i(0) = 1$; hence, instantaneous loading and unloading may be considered by setting Δt to zero.

of the reduced time at the present real time t_n. It is possible to rewrite equation (38) such that each new solution may be computed directly from the previous solution. To this end we let

$$g_i(t_n) = G_i\left\{\exp(-\xi_n/\lambda_i)\,u_0 + \sum_{j=1}^{n-1} \exp[-(\xi_n-\xi_j)/\lambda_i]\,h_i(\Delta t_j)\,\Delta u_j\right\} \tag{39}$$

and note that a recursion formula may be deduced as

$$g_i(t_n) = \exp(-\Delta\xi_n/\lambda_i)\,[g_i(t_{n-1}) + G_i\,h_i(\Delta t_{n-1})\,\Delta u_{n-1}], \quad n \geqslant 1 \tag{40}$$

where $g_i(t_0) = 0$, $\Delta u_0 = u_0$ and $\Delta\xi_n = \xi_n - \xi_{n-1}$. Equation (38) may now be written as

$$\left[G_0 + \sum_{i=1}^{I} G_i\,h_i(\Delta t_n)\right]\Delta u_n = r(t_n) - G_0\,u_{n-1} - \sum_{i=1}^{I} g_i(t_n) \tag{41}$$

For a single integral equation (41) is an efficient solution algorithm for both short and long duration loads. The solution effort at each discrete time is proportional to the number of Maxwell elements used in the material characterization, whereas, in previous developments (see Reference 5), the solution effort was proportional to the number of previous solution points. In connection with a finite element method of spatial discretization equation (41) may be applied to each term of equation (29) separately and the resulting simultaneous linear algebraic equations may be solved by standard techniques, e.g. Gauss elimination is used in a computer program, called THVISC, listed in Reference 16.

The discretization errors involved in the above process are related to the order of the time interpolation polynomials. It is possible to increase the discretization error by using poor approximations to equation (37). The evaluation of equation (37) in closed form is in general not possible for non-uniform temperature states. An approximation may be obtained by some numerical integration or other approximation which will allow a closed form evaluation. If it is assumed, as in Reference 6, that ξ is linear in time between t_{j-1} and t_j, which corresponds to the assumption of constant temperature in the time interval, the integrals, equation (37), may be evaluated, yielding

$$h_i(\Delta t_j) \doteq \lambda_i[1 - \exp(-\Delta\xi_j/\lambda_i)]/\Delta\xi_j \tag{42}$$

An alternative to equation (42) is the simple trapezoidal integration procedure. This method has previously been used with a finite difference approximation to effect solutions of integral equations.[1,14] In this procedure each integral is approximated by

$$h_i(\Delta t_j) \doteq \tfrac{1}{2}[1 + \exp(-\Delta\xi_j/\lambda_i)] \tag{43}$$

For isothermal problems equation (42) represents an exact evaluation to equation (37) while equation (43) is in all cases an approximation (see Figure 1). By considering the solution to a single integral equation we can illustrate that for a given time discretization careful evaluation of equation (37) is much more crucial in controlling the numerical error than the approximation of the dependent variable. It should be noted at the outset that our argument is based upon the fact that material characterization uses relaxation modulus functions together with a displacement method. Thus, in equation (33) if we consider a unit step forcing function, $u(t)$ is the creep compliance. For arbitrary inputs $u(t)$ will have time variations that are related to the time characteristics of the creep compliance. Consequently the numerical determination of the creep compliance serves as a check on the accuracy of equation (41). From the inter-relationship between the relaxation modulus, $G(t)$, and the creep compliance, $J(t)$, it may be shown that the *retardation time*, τ, is always greater than the *relaxation time*, λ, by the amount

$$\tau = \frac{G_0}{G_\infty}\lambda = \frac{J_\infty}{J_0}\lambda \tag{44}$$

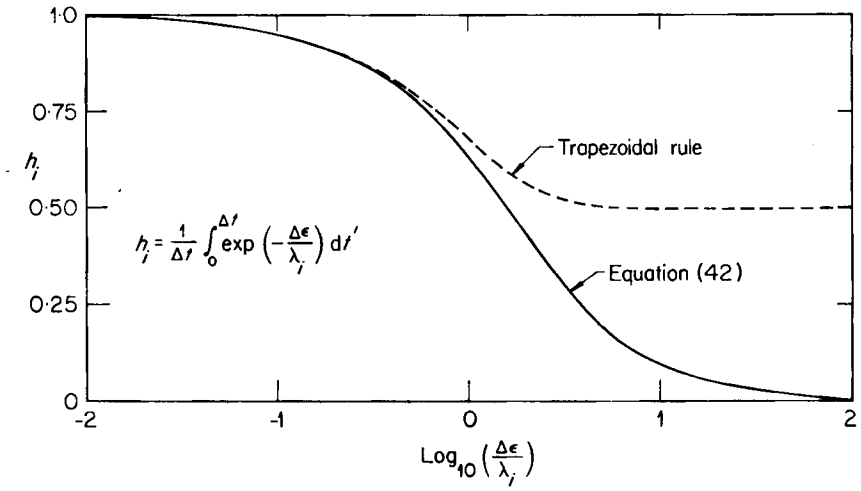

Figure 1. Evaluation of integrals, equation (42), versus trapezoidal rule

where G_0, J_0 are initial values and G_∞, J_∞ are final values of the relaxation and creep functions, respectively (see Figure 2). Consequently for relaxation moduli with low relative equilibrium

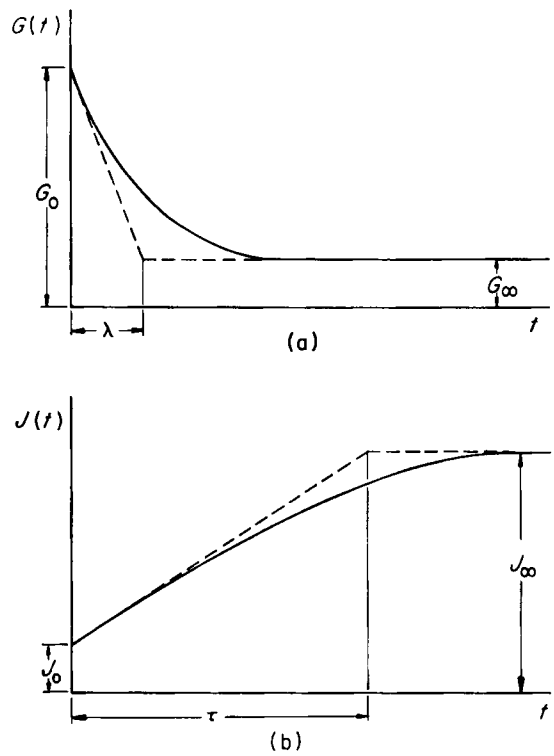

Figure 2. Comparison of relaxation and retardation times. (a) Relaxation modulus, λ = relaxation time; (b) creep compliance, τ = retardation time

values the creep compliance and consequently $u(t)$ will have significant time effects long after the relaxation modulus has reached a near equilibrium value. To illustrate this effect we consider the relaxation modulus used in Reference 7.

$$G(t) = 0{\cdot}75 \times 10^7 + 8{\cdot}2925 \times 10^9 \exp(-t/2) \tag{45}$$

where the known solution is

$$u(t) = J(t) = \tfrac{4}{3} \times 10^{-7} \left[1 - \frac{8{\cdot}2925}{8{\cdot}3} \exp\left(-\frac{3t}{6640}\right)\right]$$

The retardation time is 6640/3 and the relaxation time is 2. The numerical solution for $J(t)$ using exact and trapezoidal integrals to represent h_i is shown in Table I. The necessity of accurate

Table I. Creep compliance computation comparison for exact and trapezoidal integration of equation (37)

Creep compliance: $J(t) \times 10^9$

t	Δt: 2 Trapezoidal	Δt: 2 Exact	Δt: 200 Trapezoidal	Δt: 200 Exact	Δt: 1000 Trapezoidal	Δt: 1000 Exact
0	0·12	0·12	0·12	0·12	0·12	0·12
200	10·79	11·63	0·36	11·17		
400	20·61	22·14	0·60	21·30		
600	29·64	31·75	0·84	30·59		
800	37·95	40·53	1·08	39·12		
1000	45·59	48·54	1·32	46·93	0·36	41·60
1200	52·62	55·87	1·56	54·10		
1400	59·09	62·56	1·80	60·67		
1600	65·04	68·68	2·03	66·70		
1800	70·51	74·26	2·27	72·22		
2000	75·54	79·37	2·51	77·29	0·60	70·17
Error at 2000 (%)	5	~0	97	2·5	99+	11·5

evaluation of equation (37) is clearly illustrated by the results in Table I. The trapezoidal integration scheme depends upon accurate estimates of the relaxation modulus integrals; these are obviously related to the λ_i and Δt as seen in Figure 1. If equation (42) is used the solution increment is primarily related to the retardation times, τ_i and these are always larger than λ_i. Consequently extensive reduction in computational effort is possible by using approximations to equation (37) which are accurate for large increments of time. This is especially important for problems involving change in temperature since the reduced time increment may be several orders of magnitude greater than the real time increment. A consequence of poor approximation to equation (37) is illustrated in the next section.

NUMERICAL EXAMPLE

As an application of the present development the thermal stress analysis of a thin-walled cylinder under conditions of plane strain with a time-dependent boundary temperature was investigated. This problem was considered by Lockett and Morland.[7] Previous closed-form solutions of thermorheologically simple problems by integral transform methods have been limited to one-dimensional slabs and spheres where symmetry was used to uncouple the single integral law in reduced time from the remaining field equations in real time. More general axisymmetric

geometries are not amenable to this method of solution. Lockett and Morland have shown that for thin-walled cylinders, a perturbation scheme in the thinness parameter permits a similar uncoupling at each stage of the solution.

The numerical example presented in Reference 7 shows only the first-order solution, which the authors state should be valid for a sufficiently thin cylinder. In general, this problem admits four characteristic times, that of the applied boundary temperature, the diffusion time of heat transfer, the relaxation time of the viscoelastic material and the time of the solution (retardation time). Since the primary purpose of the example was to show the effect of temperature-dependent material properties, the diffusion time was assumed to be negligible. As illustrated in the last section, the retardation time is even more significant than the relaxation time for numerical applications. In general the time steps must be selected small enough so that the piecewise linear approximation in time of temperature, displacements and applied loads sufficiently capture the transient phenomenon.

For a time-dependent inner boundary temperature and a prescribed zero outer boundary temperature, the first-order steady-state temperature solution is linear and assumed to be achieved instantaneously. Using the notation of Reference 7 with $\bar{\theta}$, ρ and x normalized temperature, time and distance respectively, the applied boundary temperature is taken as .

$$\bar{\theta}(0, \rho) = 1 - \exp(-2\rho) \tag{46}$$

Consequently, the instantaneous steady-state temperature is given by

$$\bar{\theta}(x, \rho) = (1 - x)[1 - \exp(-2\rho)] \tag{47}$$

The shift function, based on data for polymethylmethacrylate, is taken as

$$\varphi(x, \rho) = 3981{\cdot}1 \exp[-6{\cdot}2172(1 - \bar{\theta})(1{\cdot}333 + \bar{\theta} + 1{\cdot}095\bar{\theta}^2)] \tag{48}$$

where the strong dependence on temperature is seen from the following table:

Table II. Tabulated shift function

$\bar{\theta}$	ϕ	$\rho\ (x = 0)$
0·0	1·0	0
0·4	3·22	0·25
0·6	12·2	0·45
0·8	118	0·8
0·9	572	1·15
1·0	3981	2·5

The approximate times at which the temperature is attained at the inner boundary are indicated in the third column.

The material moduli are characterized by an elastic bulk modulus and a standard solid shear relaxation modulus:

$$\left.\begin{aligned} K &= 2{\cdot}50 \times 10^{10} \\ G(\xi) &= 0{\cdot}75 \times 10^7 + 829{\cdot}25 \times 10^7 \exp(-\xi) \end{aligned}\right\} \tag{49}$$

In Reference 7, the numerical solutions were obtained by using finite difference methods for both the spatial and time variables. The authors state that it was necessary to use 40 spatial points

and 100 time points to attain a solution with an overall estimated error of 2 per cent. A finite element solution based upon the present development was obtained using only 12 nodal points (11 elements as shown at the top of Figure 3) and 40 time points. The results of both analyses are reproduced in Figures 3 and 4. In Figure 3 a plot of the normalized hoop stress is shown for 3 times. Both analyses compare very favourably at $\rho = 0{\cdot}5$. This corresponds to a normalized temperature of about 0·6 and a reduced time increment of about 25 times the real time increment.

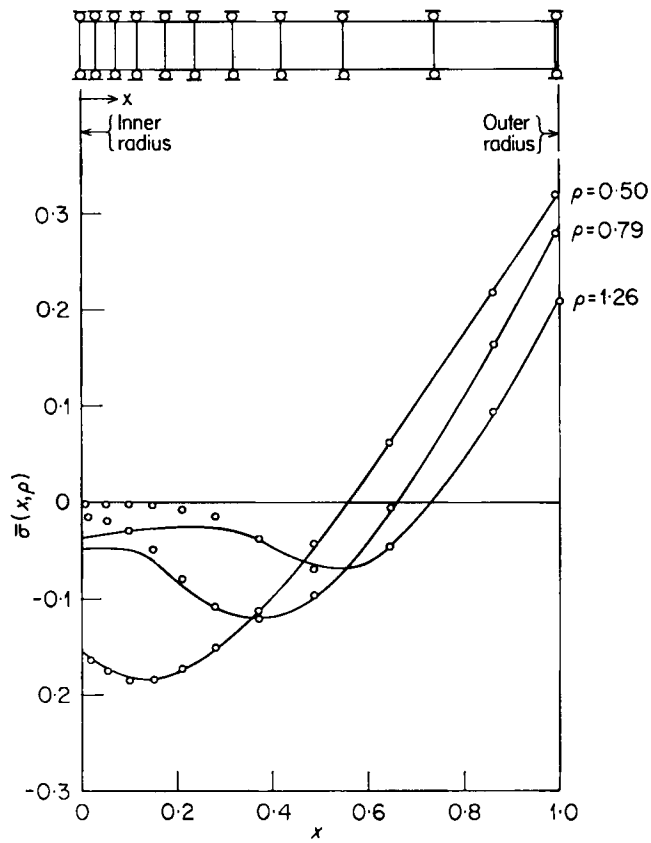

Figure 3. Hoop stress versus time, temperature-dependent properties; ○, per THVISC, Reference 16; —, per Reference 7

Based upon results from the previous section and the time increment used, we estimated that trapezoidal integration should be adequate. On the other hand, for $\rho = 0{\cdot}79$ and 1·26 the reduced time increments are about 110 and 600 times the real time increment respectively. For these time increments estimates show that the trapezoidal integration should lag behind the expected solution. The estimates are confirmed by the results obtained in Reference 7. It is physically impossible to obtain their response as shown near the inner radius (i.e. for small x). The stresses at the high temperature will decrease faster than at lower temperatures; consequently, due to the discretization error the results of Reference 7 are apparently in error. To verify this conclusion the solution was obtained for temperature-independent properties and compared to the results of Reference 7. Here excellent agreement has been obtained as seen in Figure 4. The results for the temperature-dependent solution are also replotted in Figure 4.

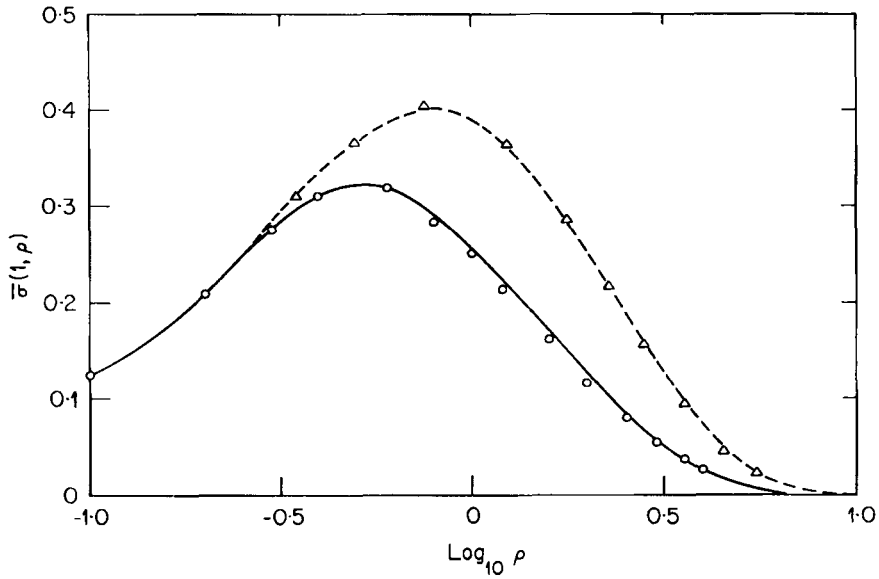

Figure 4. Hoop stress at outer radius versus log time. Temperature-independent properties: △, per THVISC, Reference 16; - - -, per Reference 7. Temperature-dependent properties: ○, per THVISC, Reference 16; —, per Reference 7

The above example serves to illustrate further the superiority of the time discretization presented herein since accurate estimates to the response time can be estimated beforehand and used for the selection of the time solution points.

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