JOURNAL OF APPLIED PHYSICS VOLUME 27, NUMBER 5 MAY, 1956

Theory of Deformation of a Porous Viscoelastic Anisotropic Solid

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(Received February 7, 1956)

Equations are established for the deformation of a viscoelastic porous solid containing a viscous fluid under the most general assumptions of anisotropy. The particular cases of transverse and complete isotropy are discussed. General solutions are also developed for the equations in the isotropic case. As an example the problem of the settlement of a loaded column is treated. The second-order effect of the change of permeability with deformation is also discussed.

1. INTRODUCTION

The theory of deformation of porous materials containing a viscous fluid was developed some years ago by Biot[1] for the case of an elastic solid. This constituted an extension of Terzaghi's original one-dimensional theory. Several analytical and numerical applications were subsequently developed in problems of consolidation and the settlement of foundations.[2–6] More recently the theory was extended to the case of an anisotropic solid.[7] The present paper constitutes a generalization to the case where the solid exhibits the most general properties of anisotropy and linear viscoelasticity. The word viscoelastic here must be understood in a very broad sense. It includes for instance such phenomena as the thermoelastic effect, i.e., the heat exchange between the various parts of the inhomogeneous material due to the additional local heating and cooling associated with the deformation. It also includes what might be called a sponginess effect. This will occur in case the walls of the main pores contain small cracks or micropores into which the fluid will seep in and out.

The present theory is applicable to a great variety of materials and, of course, contains the case of an elastic solid as a particular case. To mention only a few problems let us cite those of creep at high temperature in porous wall cooling, stresses in a dam, flow of oil or water in petroleum reservoirs, settlement and consolidation of clay in foundations. With reference to clay the property of viscoelasticity was introduced by Tan Tjong-Kie[8] who assumed a Maxwell-type solid and an experimental confirmation was given Geuze and Tan Tjong-Kie.[9]

The problem of seepage through a plastic porous medium has been treated by Krylov and Barenblatt.[10] They also refer to important work on porous materials by Gersevanov and Florin.

The literature on viscoelasticity of nonporous materials is quite extensive. The essentially nonreversible nature of the thermodynamics of viscoelastic phenomena was pointed out by Bridgman.[11] One-dimensional stress-strain relations were treated theoretically by many investigators, among others by Leaderman[12,13] and Gross.[14–16] A thermodymanic approach to viscoelasticity was initiated by Staverman,[17,18] Schwarzl,[17] and developed by Meixner[19,20] and Biot.[21,22] The latter's theory, which was originated independently, is more general in several respects. It introduces a generalized form of the free energy which applies to systems with nonuniform temperatures, thereby including thermoelastic effects in granular, porous, or multiphase systems, as well as couplings of thermochemical nature and others. The thermoelastic aspects of the theory were given more explicit development in a recent publication.[23] The general irreversible thermodynamic problem was also formulated for the case of arbitrary time-dependent perturbations applied to any system defined by its generalized free energy and a dissipation function. Variational principles of a very general nature were also developed[22] which lead to practical methods for the treatment of dynamical problems and stress analysis

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in viscoelastic materials with special reference to rods, plates, and shells.[24–26]

The present theory is derived along the lines developed by the writer.[21,22] The thermodynamic principles are outlined in Sec. 2 and used in Sec. 3 to derive the basic operational relations between stress, deformation, fluid content, and fluid pressure. Section 4 is a discussion of the field equations for various cases. Particular attention is given to transverse isotropy because it is the natural type of symmetry of rock which has consolidated under gravity. The physical significance of the operators is also discussed for the isotropic case with two decay constants.

An interesting sideline is considered in Sec. 5 where the second-order effect of the variation of permeability with deformation is discussed. This leads of course to a nonlinear theory.

General operational solutions of the field equations for an isotropic material are introduced in Sec. 6. They are a natural extension of the solutions developed in reference 2 for the elastic case. As an example the problem of settlement of a loaded column is treated in Sec. 7.

2. GENERAL THERMODYNAMIC RELATIONS FOR DISSIPATIVE SYSTEMS

We shall first recall some fundamental results on the thermodynamics of systems exhibiting viscosity and relaxation as established in references 21 and 22.

It was shown that the general equations for the time history of a thermodynamic system may be written

$$\frac{\partial D}{\partial \dot{q}_i}+\frac{\partial V}{\partial q_i}=Q_i \tag{2.1}$$

where the q_i are the state variables of the system describing its thermodynamic deviation from equilibrium. The equilibrium point is defined as $q_i=0$. The generalized forces Q_i perturbing the system are defined as conjugate variables to the coordinates q_i. The quadratic positive definite form

$$V=\tfrac{1}{2}\sum a_{ij}q_iq_j \tag{2.2}$$

is a generalized free energy of the system and D a generalized dissipation function which is proportional to the rate of entropy production. It may be written as a positive definite form in the time derivatives $\dot{q}_i$ of the variables.

$$D=\tfrac{1}{2}\sum b_{ij}\dot{q}_i\dot{q}_j. \tag{2.3}$$

It was pointed out that in many dissipative systems the number of observed variables may be small compared to the total number of degrees of freedom. An element of viscoelastic material, for instance, may exhibit only its strain components as observed coordinates, while a great many hidden internal variables may contribute to its properties. In general, a system with n total variables may contain $n-k$ hidden degrees of freedom, and only k forces Q_i may be applied to the corresponding observed coordinates. In general, we are interested in the relationship between these applied forces and observed coordinates. We have shown that this relationship can be written in operational form as

$$Q_i=\sum^{j} T_{ij}q_j, \; j=1\cdots k \tag{2.4}$$

where

$$T_{ij}=\sum^{s}\frac{p}{p+r_s}D_{ij}^{(s)}+D_{ij}+pD_{ij}' \tag{2.5}$$

and

$$p=\frac{d}{dt}. \tag{2.6}$$

The tensor T_{ij} may be considered as an impedance matrix for the thermodynamic system. It is symmetric

$$T_{ij}=T_{ji}.$$

This is a consequence of Onsager's reciprocity relations in irreversible thermodynamics. It should be noted that $D_{ij}^{(s)}D_{ij}D_{ij}'$ and r_s are constants which characterize the system in the vicinity of a certain equilibrium state. The symbol p is the standard notation as used in electrical impedances. If the q_i's are harmonic functions of time $q_ie^{i\omega t}$, the forces Q_i are also harmonic functions of time given by expressions (2.4) where p is replaced by $i\omega$. If the variable q_i is any function of time, Eqs. (2.4) may be considered as relating the Laplace transforms of q_i and Q_i. For instance, consider a simplified relation reduced to one term

$$Q=\frac{p}{p+r}q. \tag{2.7}$$

If q is a unit step function of time

$$q=\mathbf{1}(t)$$

$$\mathbf{1}(t)=\begin{cases}1 & \text{for} \quad t>0\\ 0 & \text{for} \quad t<0,\end{cases} \tag{2.8}$$

then by Laplace transform or operational calculus we may write

$$Q=\frac{p}{p+r}\mathbf{1}(t)=e^{-rt}\mathbf{1}(t). \tag{2.9}$$

This corresponds to a relaxation of stress for a unit deformation. If the variable q is a given function of time, the time history of the force is given by Duhamel's

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integral

$$Q(t)=\int_0^t e^{-r(t-\tau)}dq(\tau). \quad (2.10)$$

For generality this must be interpreted as a Stieltjes integral.

3. GENERAL OPERATIONAL STRESS-STRAIN RELATIONS

We follow a procedure similar to that of reference 7 for the elastic case.

The porosity is defined as

$$f=V_p/V_b, \quad (3.1)$$

the ratio of the pore volume V_p to the volume V_b of the bulk material.

The stress system in the material is defined as

$$\begin{Bmatrix} \sigma_{xx}+\sigma & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}+\sigma & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}+\sigma \end{Bmatrix} \quad (3.2)$$

with the symmetry property $\sigma_{ij}=\sigma_{ji}$. The components σ_{ij} represent the forces applied to the solid portion of the material per unit area of bulk material while σ is the force applied to the fluid. It is related to the hydrostatic pressure p' by

$$\sigma=-fp'. \quad (3.3)$$

Strain components for the solid are defined as

$$e_{xx}=\frac{\partial u_x}{\partial x}\; e_{xy}=\left(\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)\text{etc.}, \quad (3.4)$$

where (u_x, u_y, u_z) represent the displacement field of the solid. The only significant strain component for the fluid is

$$\epsilon=\epsilon_{xx}+\epsilon_{yy}+\epsilon_{zz}, \quad (3.5)$$

where

$$\epsilon_{xx}=\frac{\partial U_x}{\partial x}\text{etc.},$$

and (U_x,U_y,U_z) represent the displacement field of the fluid.

Assuming the solid to be viscoelastic the stress components (3.2) in the solid may be considered as the seven generalized forces applied to a thermodynamic system in the vicinity of equilibrium with a great number of internal degrees of freedom. The seven stress components may therefore be identified with Q_i and the seven strain components e_{ij} and ϵ with the conjugate variables q_i. With $k=7$ we may apply the thermodynamic expression (2.4) for the relation between the stress and strain components. We note that because of the symmetry of the T_{ij} operators it is possible to introduce the operational invariant

$$I=\tfrac{1}{2}\sum^{ij} T_{ij}q_iq_j=\tfrac{1}{2}\sum^{i} Q_iq_i, \quad (3.6)$$

and write relations (2.4) as

$$Q_i=\frac{\partial I}{\partial q_i}. \quad (3.7)$$

Introducing the seven stress and strain components, we write

$$2I=\sigma_{xx}e_{xx}+\sigma_{yy}e_{yy}+\sigma_{zz}e_{zz}+\sigma_{yz}e_{yz} +\sigma_{zx}e_{zx}+\sigma_{xy}e_{xy}+\sigma\epsilon. \quad (3.8)$$

The invariant I is a quadratic form in the strain e_{ij} with operational coefficients T_{ij}. The stress-strain relations are given by

$$\sigma_{xx}=\frac{\partial I}{\partial e_{xx}}\; \sigma_{yy}=\frac{\partial I}{\partial e_{yy}}\text{etc.}$$

$$\sigma=\frac{\partial I}{\partial \epsilon}. \quad (3.9)$$

It is noted that formally all relations are identical with the purely elastic case if we replace the elastic potential energy by the operational invariant I and all elastic constants by the corresponding operational expressions. We have here an extension to the case of porous materials of a general *correspondence rule* between elastic problems and viscoelasticity as introduced in previous work.[24,25] The stress-strain relation may be written

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \\ \sigma \end{Bmatrix} = \begin{bmatrix} \\ \\ \\ T_{ij} \\ \\ \\ \\ \end{bmatrix} \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \\ \epsilon \end{Bmatrix}. \quad (3.10)$$

The operators T_{ij} constitute a symmetric matrix of twenty-eight independent elements. Assuming a continuous spectrum of relaxation constants r of density $\gamma(r)$ the operators may be written

$$T_{ij}=p\int_0^\infty \frac{D_{ij}(r)\gamma(r)dr}{p+r}+D_{ij}+pD_{ij}'. \quad (3.11)$$

Because of the particular tensor nature of the elements the stress-strain relations (3.10) may be written

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \\ \sigma \end{Bmatrix} = \left[\begin{array}{cccccc|c} & & & & & & Q_{xx} \\ & & & & & & Q_{yy} \\ & & & P_{\mu\nu}{}^{ij} & & & Q_{zz} \\ & & & & & & Q_{yz} \\ & & & & & & Q_{zx} \\ \hline Q_{xx} & Q_{yy} & Q_{zz} & Q_{yz} & Q_{zx} & Q_{xy} & R \end{array}\right] \begin{Bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \\ \epsilon \end{Bmatrix} \quad (3.12)$$

or

$$\sigma_{\mu\nu}=P_{\mu\nu}{}^{ij}e_{ij}+Q_{\mu\nu}\epsilon$$
$$\sigma=Q_{ij}e_{ij}+R\epsilon. \quad (3.13)$$

The tensor $P_{\mu\nu}{}^{ij}$ is analogous to the twenty-one elastic moduli of an elastic solid, $Q_{\mu\nu}$ is a second-order symmetric tensor, and R is an invariant.

Some discussion is given in the next section of the significance of these operators in the simpler case of an isotropic material.

4. FIELD EQUATIONS

Let us now proceed to find the equations for the distribution of stress and deformation. There are six uuknowns, namely, the six components of displacement of the solid and the fluid. Additional equations are obtained by writing the condition of equilibrium of the stress field.

$$
\begin{aligned}
&\frac{\partial}{\partial x}(\sigma_{xx}+\sigma)+\frac{\partial\sigma_{xy}}{\partial y}+\frac{\partial\sigma_{xz}}{\partial z}+\rho X=0\\
&\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial}{\partial y}(\sigma_{yy}+\sigma)+\frac{\partial\sigma_{yz}}{\partial z}+\rho Y=0\\
&\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y}+\frac{\partial}{\partial z}(\sigma_{zz}+\sigma)+\rho Z=0
\end{aligned}
\tag{4.1}
$$

with ρ the mass density of the bulk material and X, Y, Z the body force per unit mass. As in reference 7, we write the generalized form of Darcy's law with a symmetric matrix, representing the anisotropic permeability properties of the material,

$$
\begin{bmatrix}\partial\sigma/\partial x+\rho_1 X\\ \partial\sigma/\partial y+\rho_1 Y\\ \partial\sigma/\partial z+\rho_1 Z\end{bmatrix}=\begin{bmatrix}b_{xx} & b_{xy} & b_{xz}\\ b_{yx} & b_{yy} & b_{yz}\\ b_{zx} & b_{zy} & b_{zz}\end{bmatrix}\begin{bmatrix}p(U_x-u_x)\\ p(U_y-u_y)\\ p(U_z-u_z)\end{bmatrix}
\tag{4.2}
$$

the velocities are pU_xpU_y, etc. We put $\rho_1=\rho_f f$ where ρ_f is the mass density of the fluid.

As a consequence of the existence of a dissipation function, the matrix b_{ij} is symmetric ($b_{ij}=b_{ji}$). We shall call it the *flow resistance matrix* in analogy with the case of electrical conductors. The matrix defines a positive definite quadratic form in the fluid velocities $\dot{U}_i$.

$$2D'=\sum^{ij} b_{ij}\dot{U}_i\dot{U}_j. \tag{4.3}$$

Substituting the values of the stress components (3.12) into the equilibrium equations (4.1) leads to three equations which along with the three equations (4.2) for the flow, determine the six components u_i and U_i of the deformation and flow fields. Various cases can be considered as in the elastic case, the operational invariant I now playing the role of the elastic potential energy. For instance, the case of transverse isotropy with the z-axis as axis of symmetry leads to operational stress-strain relations with eight distinct operators.

$$
\begin{aligned}
\sigma_{xx}&=2Ne_{xx}+A(e_{xx}+e_{yy})+Fe_{zz}+M\epsilon\\
\sigma_{yy}&=2Ne_{yy}+A(e_{xx}+e_{yy})+Fe_{zz}+M\epsilon\\
\sigma_{zz}&=Ce_{zz}+F(e_{xx}+e_{yy})+Q\epsilon\\
\sigma_{yz}&=Le_{yz}\\
\sigma_{zx}&=Le_{zx}\\
\sigma_{xy}&=Ne_{xy}\\
\sigma\ &=M(e_{xx}+e_{yy})+Qe_{zz}+R\epsilon.
\end{aligned}
\tag{4.4}
$$

The eight operators are:

$$
\begin{aligned}
N&=p\int_0^\infty\frac{N(r)}{p+r}dr+N+N'p\\
A&=p\int_0^\infty\frac{A(r)}{p+r}dr+A+A'p,\ \text{etc.}
\end{aligned}
\tag{4.5}
$$

The flow resistance equations are:

$$
\begin{bmatrix}\partial\sigma/\partial x+\rho_1 X\\ \partial\sigma/\partial y+\rho_1 Y\\ \partial\sigma/\partial z+\rho_1 Z\end{bmatrix}=\begin{bmatrix}b_{xx} & 0 & 0\\ 0 & b_{xx} & 0\\ 0 & 0 & b_{zz}\end{bmatrix}\begin{bmatrix}p(U_x-u_x)\\ p(U_y-u_y)\\ p(U_z-u_z)\end{bmatrix}.
\tag{4.6}
$$

In this case there are two coefficients b_{xx} and b_{zz} of flow resistance.

Similarly for the isotropic case, there are four distinct operators and the stress-strain relations are:

$$
\begin{aligned}
\sigma_{ii}&=2Ne_{ii}+Ae+Q\epsilon\\
\sigma_{ij}&=Ne_{ij}(i\neq j)\\
\sigma&=Qe+R\epsilon
\end{aligned}
\tag{4.7}
$$

with

$$e=e_{xx}+e_{yy}+e_{zz}.$$

Note that the appearance of the same operator Q in the expressions for σ_{ii} and σ is a direct consequence of Onsager's principle.

The flow resistance equations are:

$$
\begin{bmatrix}\partial\sigma/\partial x+\rho_1 X\\ \partial\sigma/\partial y+\rho_1 Y\\ \partial\sigma/\partial z+\rho_1 Z\end{bmatrix}=\begin{bmatrix}b & 0 & 0\\ 0 & b & 0\\ 0 & 0 & b\end{bmatrix}\begin{bmatrix}p(U_x-u_x)\\ p(U_y-u_y)\\ p(U_z-u_z)\end{bmatrix}.
\tag{4.8}
$$

Let us consider the isotropic case in more detail. Assume that there are only two relaxation constants, r and s, and that the solid is elastic for rapid deformations while it contains a viscous term for slow deformations. The operators are:

$$
\begin{aligned}
N&=p\frac{N_1}{p+s}+N_2\\
A&=p\frac{A_1}{p+r}+A_2\\
Q&=p\frac{Q_1}{p+r}+Q_2\\
R&=p\frac{R_1}{p+r}+R_2.
\end{aligned}
\tag{4.9}
$$

On the right-hand side are constants characteristic of the material appearing as coefficients N_1, N_2, etc., in the operators and the two time constants r and s.

Consider the case of hydrostatic stress

$$\begin{aligned}\sigma_{xx}&=\sigma_{yy}=\sigma_{zz}\\ e_{xx}&=e_{yy}=e_{zz}=e/3.\end{aligned}\tag{4.10}$$

Relations (4.7) become

$$\begin{aligned}\sigma_{xx}&=\left[\tfrac{2}{3}p\frac{N_1}{p+s}+p\frac{A_1}{p+r}+\tfrac{2}{3}N_2+A_2\right]e+\left(\frac{pQ_1}{p+r}+Q_2\right)\epsilon\\ \sigma&=\left(\frac{pQ_1}{p+r}+Q_2\right)e+\left(\frac{pR_1}{p+r}+R_2\right)\epsilon.\end{aligned}\tag{4.11}$$

If the volume of the bulk material is maintained constant, $e=0$ and

$$\sigma=\left(\frac{pR_1}{p+r}+R_2\right)\epsilon.\tag{4.12}$$

The operator represents the apparent compressibility of the fluid in the pores. It should be noted that this operator leads to a property of apparent compressional viscosity. Even if the porous solid is purely elastic, such an effect can very well occur owing to thermoelastic heat transfer between solid and fluid. It can also occur, for purely mechanical reasons, in the following way. Assume, for instance, that the solid contains two types of pores. One type of a certain size enables the fluid to percolate while another type, micropores, of a much smaller size, may be considered as some sort of sponginess of the walls of the main pores. In this case, the viscous effect, for hydrostatic pressure, is due to the time lag required for the fluid to penetrate into the micropores. In this sense, therefore, the term viscoelasticity embodies properties of a more general nature than those of the solid alone.

Suppose, for example, that we suddenly force the fluid into the main pores by establishing a sudden increase of fluid content of the material at $t=0$. This is expressed by putting

$$\epsilon=-1(t)=\begin{cases}-1 & t>0\\ 0 & t<0.\end{cases}\tag{4.13}$$

By operational rules or Laplace transform we find from (4.12)

$$-\sigma=R_1e^{-rt}+R_2.\tag{4.14}$$

The pressure in the fluid is relaxed exponentially from the instantaneous value at $t=0$ to a smaller value corresponding to the penetration of the fluid in the micropores.

The operator N represents the shear properties of the porous solid. For pure shear the fluid does not enter into the picture. We have

$$\sigma_{xy}=\left(\frac{pN_1}{p+s}+N_2\right)e_{xy}.\tag{4.15}$$

A unit step shearing strain

$$e_{xy}=1(t)\tag{4.16}$$

produces a shearing stress

$$\sigma_{xy}=N_1e^{-st}+N_2.\tag{4.17}$$

The decay constant s is related to the viscoelastic properties of the porous solid independently of the fluid. The case of a solid which is purely viscous is given by the limiting case

$$N_1=s=\infty\quad \frac{N_1}{s}=\mu\quad N_2=0$$

hence (4.18)

$$\sigma_{xy}=p\mu e_{xy}=\mu\frac{\partial e_{xy}}{\partial t}.$$

If we seal a sample of material so that no fluid enters or leaves the sample, we may put

$$e=\epsilon.$$

In this case

$$\begin{aligned}\sigma_{xx}+\sigma&=[\tfrac{2}{3}N+A+2Q+R]e\\ \sigma&=(Q+R)e.\end{aligned}\tag{4.19}$$

Observation of the total stress $\sigma_{xx}+\sigma$ and the pore pressure $-\sigma/f$ will yield additional information on the physical constants appearing in the operators. It is noticed that relaxation of the total stress will involve both time constants r and s. It should be possible to determine experimentally all the physical constants by devising experiments corresponding to the above simplified loading conditions.

The six equations for the field of deformation contain the six unknown components of the displacement vectors $\bar{U}$ and $\bar{u}$ of fluid and solid. They are obtained by eliminating the stress component between Eqs. (4.1), (4.7), and (4.8). If we assume for simplicity that the body force is zero, we find

$$\begin{aligned}N\nabla^2\bar{u}+(N+A+Q)\,\mathrm{grad}e+(Q+R)\,\mathrm{grad}\epsilon&=0\\ \mathrm{grad}(Qe+R\epsilon)&=bp(\bar{U}-\bar{u}).\end{aligned}\tag{4.20}$$

These equations are formally the same as for the elastic case but the coefficients are now functions of the time operator p.

5. NONLINEAR CASE FOR VARIABLE PERMEABILITY

We shall consider the case of a material which is initially isotropic. In this case the flow resistance matrix is initially

$$\begin{bmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{bmatrix}. \tag{5.1}$$

A deformation of the material will modify the permeability and change the flow resistance matrix to

$$\begin{bmatrix} b+\Delta b_{xx} & \Delta b_{xy} & \Delta b_{xz} \\ \Delta b_{yx} & b+\Delta b_{yy} & \Delta b_{yz} \\ \Delta b_{zx} & \Delta b_{zy} & b+\Delta b_{zz} \end{bmatrix}. \tag{5.2}$$

Because of Onsager's principle, we have the reciprocity relations

$$\Delta b_{ij}=\Delta b_{ji}. \tag{5.3}$$

If we consider the first-order perturbation of the deformation on the porosity, we may assume that the incremental resistance matrix Δb_{ij} is a linear function of the strain components e_{ij}. Because of the assumed initial isotropy of the material, the relationship between the tensors Δb_{ij} and the components e_{ij} involves two coefficients in analogy with the stress-strain relations in an isotropic elastic solid. They may be written

$$\begin{aligned} \Delta b_{xx}&=2\alpha e_{xx}+\beta e \\ \Delta b_{yy}&=2\alpha e_{yy}+\beta e \\ \Delta b_{zz}&=2\alpha e_{zz}+\beta e \\ \Delta b_{yz}&=\alpha e_{yz} \\ \Delta b_{zx}&=\alpha e_{zx} \\ \Delta b_{xy}&=\alpha e_{xy}. \end{aligned} \tag{5.4}$$

The coefficient β represents the effect of volume change on the permeability, while α represents the effect of the shearing deformations. It is the latter which introduces the anisotropy in the flow resistance tensor.

We have assumed here that the coefficients α and β are constants, i.e., that the flow resistance returns to its original value when the deformation vanishes. This, of course, is not necessarily true. This hysteresis effect could be introduced as a refinement by making α and β functions of the operator p.

A similar discussion could be made for a material which is initially anisotropic. For instance, in the case of transverse isotropy, the incremental resistance matrix can be written

$$\begin{aligned} \Delta b_{xx}&=2\alpha e_{xx}+\beta(e_{xx}+e_{yy})+\gamma e_{zz} \\ \Delta b_{yy}&=2\alpha e_{yy}+\beta(e_{xx}+e_{yy})+\gamma e_{zz} \\ \Delta b_{zz}&=\delta e_{zz}+\gamma_1(e_{xx}+e_{yy}) \\ \Delta b_{yz}&=\mu e_{yz} \\ \Delta b_{zx}&=\mu e_{zx} \\ \Delta b_{xy}&=\alpha e_{xy}. \end{aligned} \tag{5.5}$$

There are six coefficients in these relations in contrast to five for the elastic moduli. The reason is that the matrix in this case is not in general symmetric.

The field equations (4.8) become nonlinear when the change of permeability is taken into account. They contain products of the field velocity by the strain components. In this sense, the effect of the porosity change is of the second order. Care must be exercised in handling nonlinear expressions containing operators by insuring that the latter are applied only to the proper quantities.

6. GENERAL SOLUTION OF THE FIELD EQUATION

General solutions of Eqs. (4.20) were developed for the elastic case.[2] We will show here how these solutions may be extended to include the most general case of isotropic viscoelasticity. Let us perform the following substitution in Eqs. (4.20)

$$\begin{aligned} \bar{u}&=\bar{u}_1-\frac{R+Q}{H}\operatorname{grad}\varphi \\ \bar{U}&=\bar{u}_1+\frac{P+Q}{H}\operatorname{grad}\varphi, \end{aligned} \tag{6.1}$$

with the operators

$$\begin{aligned} H&=P+R+2Q \\ P&=2N+A. \end{aligned} \tag{6.2}$$

We find

$$\begin{aligned} &N\nabla^2\bar{u}_1+(H-N)\operatorname{grad}\operatorname{div}\bar{u}_1=0 \\ &(Q+R)\operatorname{div}\bar{u}_1+\frac{PR-Q^2}{H}\nabla^2\varphi=bp\varphi. \end{aligned} \tag{6.3}$$

This reduces the number of unknowns to four, the vector $\bar{u}$ and the scalar φ.

Moreover φ does not appear in the first three equations. These equations are formally identical to the Lamé equations for elastic deformations, except for the replacement of the coefficients by time operators. These equations allow general solutions which are operational generalizations of the Boussinesq-Papkovitch solutions for the theory of elasticity.

A general expression for $\bar{u}_1$ is therefore

$$\bar{u}_1=\operatorname{grad}(\psi_0+\bar{r}\cdot\bar{\psi})-\frac{2H}{H-N}\bar{\psi}. \tag{6.4}$$

These introduce a scalar ψ_0 and a vector $\bar{\psi}$ function of the coordinates and the operator p.

$$\begin{aligned} \psi_0&=\psi_0(x,y,z,p) \\ \bar{\psi}&=\bar{\psi}(x,y,z,p). \end{aligned} \tag{6.5}$$

The coordinates x, y, z are the components of the vector $\bar{r}$. It can be verified by substitution in the first three equations (6.3) that expression (6.4) for $\bar{u}_1$ is a solution of these equations if

$$\nabla^2\psi_0=\nabla^2\bar{\psi}_0=0. \tag{6.6}$$

Although the Laplace operator does not contain the time operator p the functions ψ_0 and $\bar{\psi}$ may be functions of p through the boundary conditions. For instance ψ_0 may be a function of the type

$$\psi_0=[C_1(p)e^{-\mu z}+C_2(p)e^{\mu z}]\sin\mu x. \quad (6.7)$$

This satisfies Laplace's equation, but the customary constants C_1 and C_2 are replaced by operators. The example of Sec. 7 will further clarify this point. We still have to consider the fourth equation (6.3). We may write

$$\operatorname{div}\bar{u}_1=-\frac{2N}{H-N}\operatorname{div}\bar{\psi}. \quad (6.8)$$

A particular solution of the fourth equation (6.3) is therefore

$$\varphi=-\frac{2(Q+R)N}{b(H-N)p}\operatorname{div}\bar{\psi}, \quad (6.9)$$

and the general solution may be written

$$\varphi=-\frac{2(Q+R)N}{b(H-N)p}\operatorname{div}\bar{\psi}+\phi, \quad (6.10)$$

where ϕ is a solution of the equation

$$\frac{PR-Q^2}{H}\nabla^2\phi=bp\phi. \quad (6.11)$$

Since $(PR-Q^2)/H$ is a function of the time operator p, it is interesting to note that this equation may be considered as a generalization of the diffusion equation.

Another point worth mentioning is that if we disregard operationally dependent boundary conditions we may solve the eigenvalue problem

$$\nabla^2\phi+\kappa^2\phi=0, \quad (6.12)$$

leading to characteristic values. Each characteristic value κ_n and solution ϕ_n is then associated with a certain number of relaxation constants which are roots of the equation in the unknown p

$$\frac{PR-Q^2}{H}\kappa_n^2+bp=0. \quad (6.13)$$

From a general theorem proved in reference 21 all roots of this equation must be negative, say

$$p=-\lambda_s. \quad (6.14)$$

A characteristic solution of (6.11) may thus be written

$$\phi=\phi_n\sum^s C_se^{-\lambda_s t}. \quad (6.15)$$

The constants C_s depend on the past history of the system at the instant $t=0$. The results above lead to a complete solution of Eq. (4.20) for $\bar{u}$ and U. We may write

$$\bar{u}=\operatorname{grad}(\psi_0+\bar{r}\cdot\bar{\psi})-\frac{2H}{H-N}\bar{\psi}+\frac{2N(R+Q)^2}{bH(H-N)p}\operatorname{grad}\operatorname{div}\bar{\psi}-\frac{R+Q}{H}\operatorname{grad}\phi. \quad (6.16)$$

$$\bar{U}=\operatorname{grad}(\psi_0+\bar{r}\cdot\bar{\psi})-\frac{2H}{H-N}\bar{\psi}-2N\frac{(P+Q)(Q+R)}{bH(H-N)p}\operatorname{grad}\operatorname{div}\bar{\psi}+\frac{P+Q}{H}\operatorname{grad}\phi. \quad (6.17)$$

These solutions are expressed in terms of harmonic functions ψ_0, $\bar{\psi}$, and solutions ϕ of the generalized diffusion equation (6.11). In many problems we are interested in the deformation of the solid and the fluid pressure. It may therefore be more convenient to consider a different set of variables and operators. We introduce a variable ζ which represents the increment of fluid content of the material expressed in volume of fluid per unit volume. We write

$$\zeta=-f(\epsilon-e)=\frac{1}{M}p'+\alpha e, \quad (6.18)$$

with $p'=-\sigma/f$ the fluid pressure, and the operators

$$M=R/f^2 \qquad \alpha=\frac{Q+R}{R}f. \quad (6.19)$$

There is a direct relation between ζ and φ,

$$\zeta=-f\nabla^2\varphi. \quad (6.20)$$

We may also write the stress-strain relations (4.7) in the form

$$\sigma_{\mu\mu}+\sigma=2Ne_{\mu\mu}+Se-\alpha p' \qquad \sigma_{\mu\nu}=Ne_{\mu\nu}(\mu\neq\nu) \quad (6.21)$$

with the operators N and

$$S=A-\frac{Q^2}{R}. \quad (6.22)$$

With these variables, Eqs. (4.20) are replaced by

$$N\nabla^2\bar{u}+(N+S+\alpha^2M)\operatorname{grad}e-\alpha M\operatorname{grad}\zeta=0 \qquad \kappa\nabla^2\zeta=bp\zeta \quad (6.23)$$

where

$$\frac{\kappa}{b}=\frac{k(2N+S)M}{2N+S+\alpha^2M}=\frac{PR-Q^2}{Hb}. \quad (6.24)$$

The constant $k=f^2/b$ is a permeability defined in terms of the fluid pressure by

$$-k\,\mathrm{grad}p'=f\frac{\partial}{\partial t}(\bar{U}-\bar{u}). \tag{6.25}$$

The second equation (6.23) expresses the remarkable property that the fluid content satisfies the generalized diffusion equation (6.11). If we assume that we know a general solution ζ of this equation we find the field $\bar{u}$ by putting

$$\bar{u}=\bar{u}'+\mathrm{grad}\psi_1, \tag{6.26}$$

where ψ_1 is any particular solution of the equation

$$(2N+S+\alpha^2M)\nabla^2\psi_1=\alpha M\zeta. \tag{6.27}$$

Note that ψ_1 must satisfy

$$\nabla^2(\kappa\nabla^2-bp)\psi_1=0. \tag{6.28}$$

Substituting $\bar{u}$ in Eq. (6.23) we find

$$N\nabla^2\bar{u}'+(N+S+\alpha^2M)\,\mathrm{grad}\,\mathrm{div}\bar{u}'=0. \tag{6.29}$$

The general solution of these equations is

$$\bar{u}'=\mathrm{grad}(\psi_0+\bar{r}\cdot\bar{\psi})-\frac{2(2N+S+\alpha^2M)}{N+S+\alpha^2M}\bar{\psi} \tag{6.30}$$

with

$$\nabla^2\psi_0=\nabla^2\bar{\psi}=0. \tag{6.31}$$

Hence the final expression for $\bar{u}$

$$\bar{u}=\mathrm{grad}(\psi_0+\psi_1+\bar{r}\cdot\bar{\psi})-\frac{2(2N+S+\alpha^2M)}{N+S+\alpha^2M}\bar{\psi}. \tag{6.32}$$

This form of the solution is simpler than (6.16).

7. SETTLEMENT OF A LOADED COLUMN

As an illustrative example we shall solve the following simple problem. Consider a column of height h resting on an impervious base and supported laterally by a rigid impervious sheath. A load is applied to the top through a perfectly pervious slab. We wish to calculate the settlement as a function of time. There is only one coordinate z in this problem. We put the origin $z=0$ at the base, and the top at $z=h$. Solutions are even functions of z. The solution is expressed by means of a vector $\bar{\psi}$ and a scalar ϕ. We put

$$\begin{aligned}\bar{\psi}&=(0,0,\psi_2)\\ \psi_2&=C_1z.\end{aligned} \tag{7.1}$$

The scalar satisfies (6.11). Hence we put

$$\phi=C_2\cos\lambda z$$

with

$$\lambda=\left(\frac{-pbH}{PR-Q^2}\right)^{\frac{1}{2}}. \tag{7.2}$$

C_1 and C_2 are unknown operators to be determined. Applying expressions (6.16) and (6.17), we find for the solid and fluid displacements

$$\begin{aligned}u&=-\frac{2N}{H-N}C_1z-\frac{Q+R}{H}C_2\lambda\sin\lambda z\\ U&=-\frac{2N}{H-N}C_1z+\frac{P+Q}{H}C_2\lambda\sin\lambda z.\end{aligned} \tag{7.3}$$

We note that the boundary condition $u=U=0$ of zero displacement at the bottom ($z=0$) is satisfied. The stress components are:

$$\begin{aligned}\sigma_z&=-\frac{2N}{H-N}(P+Q)C_1-\frac{PR-Q^2}{H}C_2\lambda^2\cos\lambda z\\ \sigma&=-\frac{2N}{H-N}(Q+R)C_1+\frac{PR-Q^2}{H}C_2\lambda^2\cos\lambda z.\end{aligned} \tag{7.4}$$

Denoting by σ_{z0} the load at the top ($z=h$) and satisfying the condition that the fluid pressure is zero ($\sigma=0$) at the top, we find

$$\begin{aligned}C_1&=-\frac{H-N}{2NH}\sigma_{z0}\\ C_2&=\frac{Q+R}{pbH\cos\lambda h}\sigma_{z0}.\end{aligned} \tag{7.5}$$

The settlement at the top is

$$u=\sigma_{z0}\frac{h}{H}-\sigma_{z0}\frac{(R+Q)^2}{bpH^2}\lambda\tan\lambda h. \tag{7.6}$$

If a unit load is suddenly applied at $t=0$ we may write

$$\sigma_{z0}=1(t) \tag{7.7}$$

and expand the operators of expression (7.6) by using the identity

$$\lambda\tan\lambda h=\frac{2}{h}\sum_0^\infty\frac{\lambda^2h^2}{\kappa_n^2-\lambda^2h^2} \tag{7.8}$$

with

$$\kappa_n=(n+\tfrac{1}{2})\pi.$$

Hence

$$u=\frac{h}{H}1(t)+\frac{2h}{H}(R+Q)^2\sum_0^\infty\frac{1}{\kappa_n^2(PR-Q^2)+pbHh^2}1(t). \tag{7.9}$$

This is an operational expression operating on the unit step function $1(t)$. If the solid is elastic P, Q, R, and H are constants.

In this case putting

$$\beta_n=\frac{(PR-Q^2)}{bHh^2}\kappa_n^2, \quad (7.10)$$

we may write

$$u=\frac{h}{H}1(t)+\frac{2(R+Q)^2}{bH^2h}\sum_0^\infty\frac{1}{\beta_n}(1-e^{-\beta_n t}) \quad (7.11)$$

or

$$u=\frac{h}{H}1(t)+\frac{2h}{H}\frac{(R+Q)^2}{(PR-Q^2)}\sum_0^\infty\frac{1}{\kappa_n^2}(1-e^{-\beta_n t}). \quad (7.12)$$

The first term represents the instantaneous elastic deformations upon application of the load and the series represents the time history of the settlement.

Consider now the case when the solid is viscoelastic with properties represented by the operators (4.9) discussed above. The first term represents the viscoelastic settlement without seepage and the second term the additional effect of seepage. We may represent the time history by means of exponentials. To do this we must expand the operators in expression (7.9). We put

$$\Delta_1=[p(P_1p+P')+P_2(p+r)(p+s)](pR'+R_2r)-(pQ'+Q_2r)^2(p+s)$$

$$\Delta_2=p(P_1p+P')+P_2(p+r)(p+s)+(pR'+R_2r)(p+s)+2(pQ'+Q_2r)(p+s) \quad (7.13)$$

with coefficients

$$\begin{aligned}P_1&=2N_1+A_1\\ P_2&=2N_2+A_2\\ P'&=2N_1r+A_1s\\ Q'&=Q_1+Q_2\\ R'&=R_1+R_2.\end{aligned} \quad (7.14)$$

Also, we put

$$\Delta=\kappa_n^2\Delta_1+h^2bp(p+r)\Delta_2. \quad (7.15)$$

The operators are then

$$\frac{1}{H}=(p+s)(p+r)/\Delta_2$$

$$\frac{1}{H_n'}=\frac{(R+Q)^2}{H[\kappa_n^2(PR-Q^2)+pbHh^2]}=\frac{[p(R'+Q')+(R_2+Q_2)r]^2(p+s)^2(p+r)}{\Delta_2\Delta}. \quad (7.16)$$

The polynomials Δ_2 and Δ are of second and fourth degree in p, respectively. Denote by $-\lambda_1$, $-\lambda_2$ the roots of $\Delta_2=0$ and by $-\lambda_{3n}\cdots-\lambda_{6n}$ the four roots of $\Delta=0$. These roots are real and negative in view of a general property derived in reference 21 from thermodynamics. We expand the operators in partial fractions and write

$$\frac{1}{H}=\frac{\Gamma_1}{p+\lambda_1}+\frac{\Gamma_2}{p+\lambda_2}$$

$$\frac{1}{H_n'}=\frac{\Gamma_{1n}}{p+\lambda_1}+\frac{\Gamma_{2n}}{p+\lambda_2}+\sum_{s=3}^{6}\frac{\Gamma_{sn}}{p+\lambda_{sn}}. \quad (7.17)$$

These operators lead to the following expressions for the settlement as a function of time

$$u=\frac{h}{\lambda_1}(\Gamma_1+2\sum^n\Gamma_{1n})(1-e^{-\lambda_1 t})+\frac{h}{\lambda_2}(\Gamma_2+2\sum^n\Gamma_{2n})(1-e^{-\lambda_2 t})+2h\sum_{s=3}^{6}\sum^{n}\frac{\Gamma_{sn}}{\lambda_{sn}}(1-e^{-\lambda_{sn}t}). \quad (7.18)$$

The expansion of $1/H$ may contain a constant term, i.e., some Γ and the corresponding λ may be infinite. In that case the deflection takes the general form

$$u=C_1+\sum C_{2n}(1-e^{-\mu_n t})+\sum C_{3n}(1-e^{-\nu_n t}). \quad (7.19)$$

The first term is the instantaneous elastic deflection.