Thermodynamics and phenomenology

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Abstract: The general way to derive constitutive equations for different phenomena, coupled or not, is briefly described with an emphasis on the choice of state variables and analytical expressions for the state potential and the (pseudo) potential of dissipation by qualitative experiments. Quantitative experiments are used to determine the numerical values of the materials dependent parameters. Applications are given for anisotropic damage coupled to stress-strain behaviour of materials, for aging coupled to plasticity and for magneto-mechanical couplings. A brief «state of the art» on localisation phenomena is given as a conclusion.

1. INTRODUCTION

1975! When «L.M.T.-Cachan» just started, it was clear that experiments should play a major role in the development of mechanics of materials, but how to bridge the gap between esoteric theories and pragmatic measurements? Up to that time, the worlds of theoreticians of thermodynamics and the world of experimentalists were completely disconnected, both writing for their own world without almost any mutual understanding. «Useless theories» against «empiricism»!

Fortunately we were the children of Paul Germain; he spent much time making basic thermodynamics clear in order to teach it in an understandable way to graduate students (Germain, 1973). It became clear that any material behaviour can be stated as a mathematical model, provided the second principle of thermodynamics is fulfilled and provided a proper choice of state variables, a proper choice of an analytical expression of a state potential Ψ, and a proper choice of another analytical expression of a dissipative (pseudo-)potential F are made (Lemaitre and Chaboche, 1994). In this sentence the word «choice» is used three times. This means that Thermodynamics gives the framework for building models, but not all the keys. Fortunately, human sensibility still plays a role:

The choice of variables depends upon the purpose of modelling: which phenomena should be modelled, under which conditions, and which predictions should

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be made. The observable variables and internal variables V_i define the state of a Representative Volume Element of any material. Each of them represents a mechanism to be identified by observations either at a meso-scale or at a micro-scale and by quantitative experiments. This is often a difficult task, since hidden variables correspond to hidden mechanisms: elastic and plastic strains ε_e and ε_p, isotropic and kinematic hardenings r and α, damage D, phase change a^*, anhysteretic and internal magnetic fields H_{an}, H_i.;

- The choice of an analytical expression for the state potential taken as the specific Helmholtz free energy Ψ, (or its dual the Gibbs specific enthalpy if more convenient), a function of all the state variables, is the second key to establishing the state laws. It gives the definition of the state couplings between phenomena and allows one to calculate the associated forces A_k such that the product $A_k \dot{V}_k$ is the dissipated power involved by the mechanism described by V_k within the hypothesis of small strains: $\Psi(\varepsilon_e, T, V_k)$

$$\begin{aligned} &\frac{\partial \rho\Psi}{\partial \varepsilon} = \sigma \rightarrow \text{law of elasticity} \\ &-\frac{\partial \Psi}{\partial T} = s \rightarrow \text{entropy density} \\ &\frac{\partial \rho\Psi}{\partial V_k} = A_k \rightarrow \text{associated variables} \end{aligned} \tag{1}$$

Only experiments can help to make this choice.

- The choice of an analytical expression for the dissipation potential is the third key to writing the kinetic laws of evolution of the dissipative phenomena: $F(A_k..;V_k)$

$$\dot{\lambda}\frac{\partial F}{\partial A_k} \text{ or } \frac{\partial F}{\partial A_k} = \dot{V}_k \rightarrow \text{kinetic laws} \tag{2}$$

Here again experiments are needed in order to define the function $F(A_k;V_k)$. Let us add that the intellectual dexterity of the « **modeller** » is also an important factor in obtaining models easy to use. The choice of the analytical form may also depend on the way the problem is numerically solved.

2. QUALITATIVE AND QUANTITATIVE IDENTIFICATION

A Physical model is meant to describe observed phenomena and to be predictive for situations that were not included in its construction and identification process. The few experiments needed for the identification must be carefully chosen to represent basic situations. The choice of the variables and the choice of the analytical expressions of the potentials require a sensitive analysis that indicates trends and do not require a good accuracy of the test results: this is qualitative identification. Conversely, the numerical values of the material parameters need precise results for a good quantitative identification.

For qualitative identification, the number of variables necessary to describe observed phenomena should be a minimum and consistent with the continuum description if one expects to use the models for design. Therefore, the route to the choice of these variables is the identification of all the physical mechanisms involved in the situation. This is based on experiments carried out on the Representative Volume Element but it may be complemented by physical observations and/or by homogenisation procedures whenever possible. Once all the mechanisms are exhibited, a variable is affected by each of them. The state potential will be then dependent on all these variables. The (pseudo-) dissipation potential will be dependent on forces A_k but also on the variables V_k. The first step in the construction of these potentials is the analysis of the interactions between all the defined mechanisms. Experiments can be designed to check if a given mechanism has or has not an influence on another, and what type of influence it is.

A state coupling is said to exist between mechanisms i and j if a change in state variable V_i induces a change in the associated force A_j, i.e. if: $\frac{\partial^2 \Psi}{\partial V_i \partial V_j} \neq 0$. If such a coupling does not exist, then the state potential contains two independent terms in V_i and V_j. Along the same lines, a kinetic coupling between two mechanisms i and j exists when a change in V_i implies a change in the rate of V_j (Lemaitre and Marquis, 1992).

For quantitative identification, since the determination of the parameters is in general a very difficult task (though many efficient optimisation techniques exist), the number of parameters entering a model should also be kept to a minimum. Here again, well-defined experiments on the Representative Volume Element (homogeneous behaviour) should be used in order to be able to measure all the variables properly, directly or indirectly. However, in many instances it is not possible to design experiments where a significant part of the specimen is in a homogeneous state; in this case, quantitative identification also requires the use of inverse procedures and corresponding algorithms.

3. APPLICATION TO DAMAGE MODELLING

Let us consider how to make the three choices introduced in Section 1 when modelling the progressive mechanical degradation of materials whatever their nature: metals, polymers, concrete, wood, ceramics.

3.1. Choice of the variable

Physically, damage corresponds to the creation of new micro free surfaces inside a Representative Volume Element at a meso-scale. A natural choice of the variable is the relative density of these defects: volume fraction of cavities for ductile damage (Gurson, 1977), crack density for brittle damage and fatigue (Kachanov, 1958). To link the two mechanisms in only one variable let us consider the maximum surface density $D_{(\vec{n})}$ of « defects » in any plane of normal $\vec{n}$ of the R.V.E.

If damage is isotropic, the intrinsic variable can be chosen as a scalar: D ; if not it is a tensor: a fourth order tensor in the most general case (Krajcinovic, 1985), a second order tensor if damage may be considered as orthotropic (Murakami, 1988). When damage of metals is observed under a microscope it is always, or almost always, associated with plasticity either at a meso-scale or at a micro-scale. Then, if damage is governed by plasticity described by a second order tensor, damage can also be represented by a second order tensor. Furthermore, the micro-mechanical analysis of an array of randomly distributed cracks shows that the elastic strain energy is represented by three terms that are dependent upon the crack pattern. The first term is a scalar; the second a second order tensor; and the third a fourth order tensor, negligible with respect to the other two terms (Kachanov, 1994). Thus, at least for two main reasons, the choice of a second order tensor D_{ij} is reasonable (but not rigorous!).

3.2. Choice of the state potential

When modelling linear isotropic elasticity, and isotropic and kinematic hardening plasticity, the free energy Ψ is a quadratic function of the elastic strain ε^e_{ij} and the sum of two non linear functions of accumulated plastic strain p and back strain α_{ij}.

Many experiments have shown a decrease of Young's modulus as damage progresses either in ductile creep or low-cycle fatigue phenomena. Coupling damage with elasticity is imposed by experiments, but how? One hypothesis, that is still a principle since it has not yet been demonstrated in the general case, solves the problem (Lemaitre, 1971; Lemaitre, 1978). The *principle of strain equivalence* states that « any strain constitutive equation of a damaged material is derived in the same way as for the virgin material if the stress is replaced by the effective stress ». This effective stress $\tilde{\sigma}$ is the stress acting on the unit surface diminished by the surface of defects. Due to the definition of the damage variable, in one dimension the effective stress is defined as: $\tilde{\sigma} = \sigma/(1-D)$.

In three dimensions with anisotropic damage it is more complicated because the simple extension of the above statement does not yield the existence of an elastic potential (Cordebois and Sidoroff, 1982). This difficulty is solved if the elastic energy is divided into two parts, distortion and hydrostatic. By using the Gibbs energy Ψ^* with the second order damage tensor D_{ij} of mean value $D_H = 1/3D_{kk}$ (Lemaitre et al, 1999), we find

$$\rho\Psi^* = \frac{1+\nu}{2E} H_{ij}\sigma^D_{jk}H_{kl}\sigma^D_{li} + \frac{3(1-2\nu)}{2E} \frac{\sigma_H^2}{1-\eta D_H} + \Psi_{\rho(r,\alpha)} \tag{3}$$

where,

$$(\delta_{ij} - D_{ij})^{1/2} \text{ and } \sigma^D_{ij} = \sigma_{ij} - \sigma_H\delta_{ij} \quad \sigma_H = \frac{1}{3}\sigma_{kk} \tag{4}$$

and where ρ denotes the mass density, E and ν the elastic constants of the virgin material and η a parameter that is required for a correct representation of experiments concerning the variation of the contraction ratio. For several metals $\eta \cong 3$, and $\eta = 1$ corresponds to isotropic damage. This is the price to pay for phenomenology; the parameter η has to be measured to evaluate the components of damage, but it does not appear in the damage kinetic law.

The law of elasticity derives from the Gibbs potential, so that

$$\varepsilon^e_{ij} = \rho\frac{\partial\Psi^*}{\partial\sigma_{ij}} = \frac{1+\nu}{E}\tilde{\sigma}_{ij} - \frac{\nu}{E}\tilde{\sigma}_{kk}\delta_{ij} \tag{5}$$

and the effective stress is defined as

$$\tilde{\sigma}_{ij} = (H_{ik}\sigma^d_{kl}H_{li})^D + \frac{\sigma_H}{1-\eta D_H}\delta_{ij} \tag{6}$$

The force associated with damage variable is the strain energy density release rate tensor *Y:*

$$Y_{ij} = \rho\frac{\partial\Psi^*}{\partial D_{ij}} \tag{7}$$

Since it is easier to introduce the coupling between damage and plasticity through the dissipation potential, the plastic part Ψ^*_p of the state potential Ψ^* is not affected by damage.

3.3. Choice of the dissipation potential

There is much more freedom to choose a «good» dissipation potential. This is the reason why so many models of kinetic laws have been proposed!

It has been observed that:

- Ductile damage is governed by plasticity ;
- Fracture and ductile damage are influenced by stress triaxiality, which is related to the ratio of the hydrostatic elastic energy to the distortion energy;

- Damage measurements by means of elasticity changes show that ductile damage grows approximately proportionally to cumulative plastic strain, whereas fatigue damage evolution evolves non linearly with stress or elastic strain.

Hence, the (pseudo-)dissipation potential F has to be a function of the corresponding variables. Furthermore, since plastic constitutive equations must also be derived from this potential F, it must include the yield criterion f. The coupling with damage is introduced by means of the effective stress associated with the principle of strain equivalence, so that

$$F(\tilde{\sigma}, R, X, Y) = f(\tilde{\sigma}, R, X) + F_D(Y, \varepsilon^p) \tag{8}$$

Here f denotes the von Mises function giving the plastic multiplier $\dot{\lambda}$ by the consistency condition $\dot{f} = 0$: $\dot{\lambda} = \dot{r}, r$ is the variable associated with isotropic hardening R.

For isotropic damage the following choice of F_D has given a damage law of evolution extensively used with success for more than 15 years for ductile, quasi-brittle and fatigue damage (Lemaitre, 1983):

$$F_D = \frac{S}{s+1}\left(\frac{Y}{S}\right)^{s+1} \rightarrow \dot{D} = \dot{\lambda}\frac{\partial F}{\partial Y} = \left(\frac{Y}{S}\right)^{s} \dot{p} \tag{9}$$

where $\dot{p}$ denotes the cumulative plastic strain rate: $\dot{p} = \left(\frac{2}{3}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}\right)^{1/2}$.

Its extension to anisotropic damage imposes the choice: $F_D = \left(\frac{\overline{Y}(\varepsilon^e)}{S}\right)^{s} Y_{ij} \left|\frac{d\varepsilon^p}{dr}\right|_{ij}$,

where S and s denote material parameters, variable $\overline{Y}$ denotes the effective elastic strain energy such that: $\overline{Y} = \frac{1}{2}E_{ijkl}\varepsilon^e_{kl}\varepsilon^e_{ij}$, and operator $|\circ|$ applied to a tensor means the absolute value of the principal values. Finally, the damage evolution law appears as:

$$\dot{D}_{ij} = \frac{\partial F}{\partial Y_{ij}}\dot{\lambda} = \left(\frac{\overline{Y}}{S}\right)^{s} \left|\dot{\varepsilon}^p\right|_{ij} \tag{10}$$

Two other material-dependent parameters must be introduced in order to fit experimental observations:

- A damage threshold p_D is introduced such that $D_{ij} = 0$ if $p \leq p_D$;
- A critical value of the damage parameters is written in accordance with the principle of strain equivalence on the largest principal value of the damage tensor. This critical value is introduced such that

$$\underset{I}{Max}\ D_I = D_c \rightarrow \text{mesocrack intitiation.}$$

4. APPLICATION TO AGING AND PLASTICITY COUPLINGS

Among the many applications of this formalism to the modelling of the interactions between the mechanical behaviour and the chemical and physical transformations of materials, we present the modelling of an aluminium-copper alloy subjected to plastic deformation during the aging process. Aging of these alloys takes place after a dissolution heat treatment (1 hour at 500°C) followed by quenching in water at 20°C. After this heat treatment, the copper atoms are in solution in the aluminium matrix, and this configuration is not stable. Precipitates (Al_2Cu) appear, and lead to a strong hardening of the material. This aging process takes about 100 hours at 20°C. If the material experiences some plastic deformation during the aging process, it is assumed for sake of simplicity that the main interactions between the dislocations and the precipitates lead to a modification of the aging kinetics. The set of constitutive equations given hereafter is written for the isothermal case.

The state variables are the classical state variables of elasto-plasticity, and an aging variable denoted by a^*. The limiting values for a^* are 0 (absence of precipitates) and 1 (complete precipitation). To determine the state potential, the state couplings or uncouplings are analysed. The only state coupling is the coupling between aging and isotropic hardening due to the dislocations and precipitates interactions. The following expression of the free energy is deduced:

$$\psi(\varepsilon^e,\alpha,r,a)=\frac{1}{2}A:\varepsilon^e:\varepsilon^e+\frac{1}{2}Btr(\alpha^2)+W(r)+a^*(Cr-L)+L \tag{11}$$

The last terms (where C and L are material dependent parameters) lead to the state coupling:

$$R=W'(r)+Ca^* \tag{12}$$

This expression shows that the higher the aging value, the higher the isotropic hardening (or the yield stress).

The kinetic coupling of the plastic strain with the ageing mechanism is deduced from the interaction between the diffusion rate of copper atoms and the density of dislocations. The following expression of the potential of dissipation is deduced from specific experiments (Marquis and Costa Mattos 1991)

$$\varphi=I_f(\sigma,X,R)+\frac{1}{\tau}<a^*_\infty-a^*>Z \tag{13}$$

where τ and a^*_∞ denote functions of r and ε^p, and $\langle x\rangle=xH(x)$ where H is the Heaviside function.

From these two potentials, the following set of constitutive equations is obtained:

$$\varepsilon = \varepsilon^e + \varepsilon^p ; \varepsilon^e = \frac{1+\nu}{E}\sigma - \frac{\nu}{E} tr(\sigma) I ; \dot{\varepsilon}^p = \dot{\lambda} \frac{\partial f}{\partial \sigma'} \tag{14}$$

$$\dot{X} = B^{\infty} \varepsilon^p ; \dot{R} = W''(r)\dot{r} + C\dot{a}^* \tag{15}$$

$$\dot{a}^* = \frac{1}{\tau} < a^*_{\infty} - a^* > \tag{16}$$

where f denotes the von Mises yield function with kinematic and isotropic hardenings such that

$$f = \left[\frac{3}{2} tr(\sigma^D - X)^2\right]^{1/2} - R - \sigma_y \tag{17}$$

where σ_y denotes the initial yield stress of the material, and σ^D the stress deviator.

The plastic multiplier $\dot{\lambda}$ is calculated from the consistency condition: $\dot{f} = 0$, and the internal variable r associated with the isotropic hardening is the cumulative plastic strain.

From specific experiments on the material at different levels of plastic strain and aging, the following expressions for the functions τ and a^* are identified:

$$\frac{1}{\tau} = \left[\alpha - \beta \exp(-\gamma' r)\right] \tag{18}$$

$$a^*_{\infty} = A + (1-A)\exp\left[-k(\varepsilon^p_{II})M\right] \tag{19}$$

where $\alpha, \beta, \gamma', A$ and k denote material parameters, and $(\varepsilon^p_{II})_M$ denotes the maximum equivalent plastic strain over the time interval, so that $(\varepsilon^p_{II})_M = \max\left[\frac{2}{3} tr(\varepsilon^p)^2\right]^{1/2}$.

The expression of the strain hardening function $W''(r)$ is chosen to ensure the limitation of the size of the elastic domain:

$$W''(r) = b\gamma \exp(-\gamma r) \tag{20}$$

where b and γ denote material parameters.

The set of all numerical values of the material parameters is given below for the 2024 aluminium alloy considered in the application:

$E = 72500$ Mpa, $\nu = 0.33$, $B = 30000$ Mpa, $\sigma_y = 75 Mpa$,

b = 40.5 Mpa, $\gamma = 5$, $\alpha = 0.00167 s^{-1}$,

$\beta = 0.00153 s^{-1}$, $\gamma' = 1$, $A = 0.5$, $k' = 28$, $C = 130$Mpa.

5. APPLICATION TO MAGNETO-ELASTIC COUPLINGS

This section is devoted to some aspects of the complex couplings between the different phenomena that are involved in real thermo-magneto-elastic behaviour of soft ferromagnetic materials. The global (structural) couplings [Germain et al 1983, Maugin, 1980-1988] that involve the structural conservation laws viz. the heat equation, the mechanical conservation equation and the Maxwell equations, are not discussed here.

The phenomena that are discussed here refer to the fact that the magnetic behaviour is affected by stresses and conversely, to the fact that changes in the magnetisation generate so-called magnetostrictive strains. One of the practical consequences of these phenomena is the following: the magnetic behaviour of electrical machines is dependent not only on the behaviour of the raw materials they are made of, but also on the stresses induced in these machines by their manufacturing process and the in-service loadings.

A possible approach to developing coupled magneto-elastic constitutive equations consists in using so-called homogenisation techniques [Buiron et al, 1999]. More phenomenological models have also been proposed [see for instance: Maugin, 1991; Jiles, 1995; Sabir, 1995]. The approach used herein is also phenomenological though based on continuum thermodynamics.

5.1. Phenomenology at the mascroscopic scale

When subjected to a cyclic magnetic field, industrial materials exhibit a hysteretic behaviour. However, by superimposing a slowly decaying alternative field on a magnetic field of given amplitude, it is possible to reach the state of lowest internal energy that corresponds to one point of the so-called anhysteretic curve. This experimental procedure has been applied to an industrial non-oriented Fe-3%Si alloy commonly used in electrical power engineering. During the tests, the 0.5mm thick laminations were subjected to collinear uniaxial mechanical and magnetic excitations. The experimental results presented in Fig.1 show the influence of stresses on both the magnetostriction strain measured in the direction of the applied field, and on the magnetic response.

5.2. Microscopic behaviour

The magnetisation process can be sketched as follows. The material comprises numerous microscopic magnetic domains. In each domain, the magnetisation is uniform and oriented according to the local orientation of the crystal axes. The boundaries between the domains are so-called domain walls and correspond to a few atomic distances. At the macroscopic level, if no external field is applied, the average magnetisation tends to zero. This demagnetised state is often taken as a reference for the strain state of the material.

When an external magnetic field is applied, the domain configuration evolves: domains oriented in the direction of the external field grow, whereas other domains tend to shrink, so that both the average magnetisation and the average strain state change. Moreover, after a certain level of mean magnetisation has been reached, the magnetisation in each domain tends to rotate from the crystallographic axes to the direction of the applied magnetic field. The final stage of this evolution corresponds to magnetic saturation: in Fig.1a this microscopically approximately uniform state has been chosen as a reference for the strain state. During the magnetisation process, the domain walls motions are restrained by all kinds of defects in the material (impurities, grain boundaries, inclusions, local stresses, etc); these interactions lead to repeated pinning, bulging and unpinning of the walls; this explains the hysteretic character of the magnetic behaviour.

The anhysteretic behaviour corresponds to the behaviour that would be observed for a material with no defect. Real materials may exhibit this ideal behaviour, which corresponds to the state of minimum energy, if the domain walls are «shaken» for instance by an alternating magnetic field so that an additional energy overcomes the local energy barriers created by the various obstacles.

5.3. A magneto-elastic model (Hirsinger, 1994; Gourdin, 1998)

5.3.1 State variables

To model these mechanisms, and as sketched on Fig.2, it is proposed to make a partition of the magnetic field into two contributions: the anhysteretic «effective» field $\mathbf{H}_{an}$ corresponds to the reversible behaviour and the internal variable $\mathbf{H}_i$ corresponds to the dissipative phenomena. Further partitioning of the internal field enables us to take account of the reversible bending of the pinned walls and irreversible jump to the next obstacle that are respectively described by fields $\mathbf{H}_{rev}$ and $\mathbf{H}_{irr}$:

$$\mathbf{H}_{an} = \mathbf{H} - \mathbf{H}_i \tag{21}$$

$$\mathbf{H}_i = \mathbf{H}_{rev} + \mathbf{H}_{irr} \tag{22}$$

$$\mathbf{M}(\mathbf{H}) = \mathbf{M}_{an}(\mathbf{H}_{an}) = \mathbf{M}_{an}(\mathbf{H} - \mathbf{H}_i) \tag{23}$$

Besides, it is assumed that total strains ε are the sum of pure mechanical strains ε^m, pure magnetostriction strains ε^μ and thermal expansion strains ε^{th}, so that when small strain and linear elasticity assumptions are made, the following strain decomposition can be used:

$$\varepsilon = \varepsilon^m + \varepsilon^\mu + \varepsilon^{th} = \mathbf{E}^{-1} : \sigma + \varepsilon^\mu(\sigma, \mathbf{H}, \mathbf{H}_{rev}, \mathbf{H}_{irr}, T) + \varepsilon^{th}(T) \tag{24}$$

where $\mathbf{E}$, σ and T respectively denote the elasticity moduli tensor, the stress tensor and temperature. This relation is based on the following uncoupling hypothesis: the elastic and thermal expansion moduli are unaffected by the magnetic field.

5.3.2 State potential

The specific free enthalpy Ψ is used as a state potential. This global state potential is divided into several uncoupled contributions:

$$\rho\Psi = \rho\Psi(\sigma, \mathbf{H}, \mathbf{H}_{rev}, \mathbf{H}_{irr}, T) \tag{25}$$

$$\rho\Psi = \rho\Psi^{\mu m}(\sigma, \mathbf{H}, \mathbf{H}_{rev}, \mathbf{H}_{irr}, T) + \rho\Psi^{\mu an}(\mathbf{H} - \mathbf{H}_{rev} - \mathbf{H}_{irr}, T) + \rho\Psi^{\mu i}(\mathbf{H}, \mathrm{H}_{rev}, \mathbf{H}_{irr}, T) \tag{26}$$

where ρ denotes the mass density.

The first term $\Psi^{\mu m}$ represents the state coupling via the magnetostriction strain; its expression is postulated as

$$\rho\Psi^{\mu m}(\sigma,\mathbf{H},\mathbf{H}_{rev},\mathbf{H}_{irr},T)=\frac{1}{2}\sigma:\mathbf{E}^{-1}:\sigma+\int_0^{\sigma}\varepsilon^{\mu}(\sigma,\mathbf{H},\mathbf{H}_{rev},\mathbf{H}_{irr},T):d\sigma+\varepsilon^{th}(T):\sigma \quad (27)$$

The second term $\rho\Psi^{\mu an}$ is linked to the anhysteretic behaviour of the material, its expression is postulated as

$$\rho\Psi^{\mu an}(\mathbf{H}-\mathbf{H}_{rev}-\mathbf{H}_{irr},T)=\int_0^{\mathbf{H}-\mathbf{H}_{irr}-\mathbf{H}_{rev}}\mu_0\mathbf{M}_{an0}(\mathbf{h},T)\mathbf{dh} \quad (28)$$

where subscript 0 refers to $\sigma=\mathbf{0}$ and μ_0 denotes the vacuum permeability.

The last term $\rho\Psi^{\mu i}$ accounts for the hysteretic part of magnetic behaviour.

5.3.2 State laws

The mechanical and magnetic state laws are derived from the state potential:

$$\varepsilon=\frac{\partial(\rho\Psi^{\mu m})}{\partial\sigma}=\mathbf{E}^{-1}:\sigma+\varepsilon^{\mu}(\sigma,\mathbf{H},\mathbf{H}_{rev},\mathbf{H}_{irr},T)+\varepsilon^{th}(T) \quad (29)$$

$$\mu_0\mathbf{M}_l=\frac{\partial(\rho\Psi^{\mu an}+\rho\Psi^{\mu i})}{\partial\mathbf{H}}+\int_0^{\sigma}\frac{\partial\varepsilon^{\mu}}{\partial\mathbf{H}}(\sigma,\mathbf{H},\mathbf{H}_{rev},\mathbf{H}_{irr},\mathbf{T}):d\sigma \quad (30)$$

$$\mu_0\mathbf{M}_{rev}=\frac{\partial(\rho\Psi)}{\partial\mathbf{H}_{rev}} \quad (31)$$

$$\mu_0\mathbf{M}_{irr}=\frac{\partial(\rho\Psi)}{\partial\mathbf{H}_{irr}} \quad (32)$$

where $\mathbf{M}_l$, $\mathbf{M}_{rev}$ and $\mathbf{M}_{irr}$ respectively denote the forces associated to $\mathbf{H}$, $\mathbf{H}_{rev}$ and $\mathbf{H}_{irr}$.

5.4 Experimental validation of the state (un)couplings

To complete the model, evolution laws for the internal variables must be postulated and experimentally identified. These laws must satisfy the Clausius-Duhem inequality:

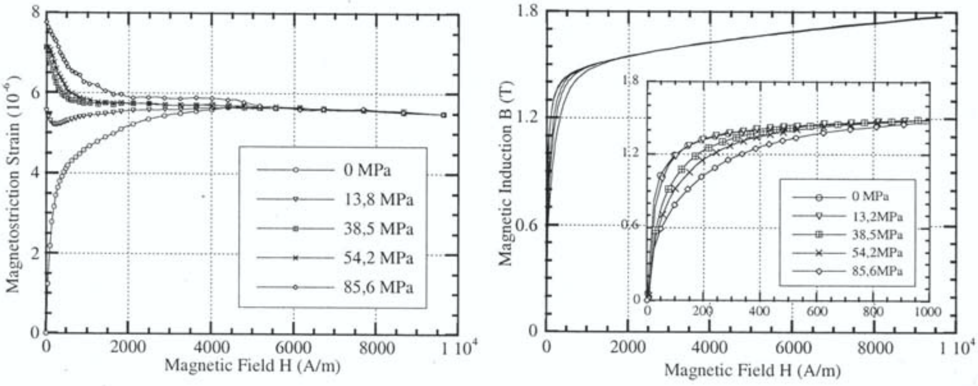
$$D=\varepsilon:\dot{\sigma}+\mu_0\mathbf{M}.\dot{\mathbf{H}}-\rho(\dot{\Psi}+s\dot{T})\geq 0 \quad (33)$$

where s denotes the entropy.

However, before proceeding towards this last step it is worthwhile -when possible- to validate the choices made when writing the state potential.

When written for anhysteretic evolutions, the Clausius-Duhem inequality shows that

$$\mu_0\mathbf{M}_l=\mu_0\mathbf{M}_{an0}(\mathbf{H}-\mathbf{H}_{rev}-\mathbf{H}_{irr},T)+\int_0^{\sigma}\frac{\partial\varepsilon^{\mu}}{\partial\mathbf{H}}(\sigma,\mathbf{H}-\mathbf{H}_{rev}-\mathbf{H}_{irr},\mathbf{T}):d\sigma$$



a. **b.**

Figure 1. Effect of an elastic tensile stress on the anhysteretic magnetostriction strain measured in the direction of applied field (**a**) and on the magnetic anhysteretic curve (**b**) for 0.5 mm thick laminations made of N.O. Fe-3%Si.

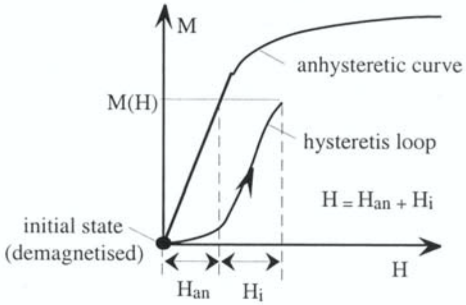
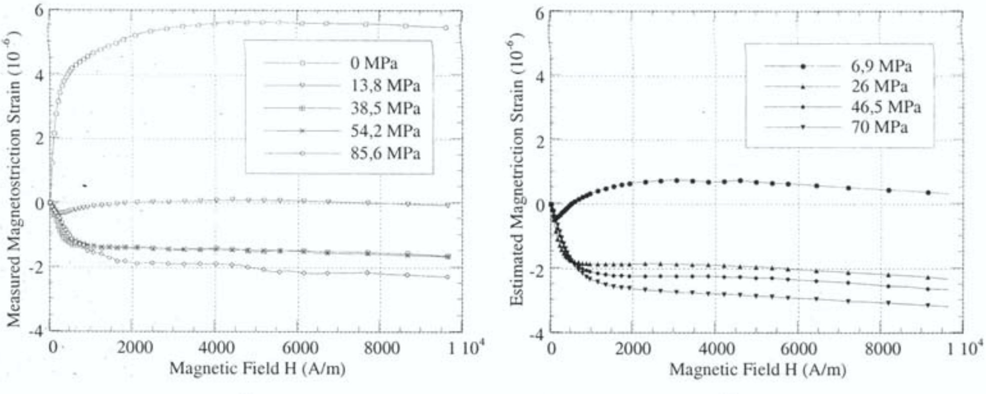


Figure 2. Definition of the state variables



a. **b.**

Figure 3. Validation of the magnetoelastic state couplings: comparison between the measured (**a**) and the predicted (**b**) relative anhysteretic curves ($\varepsilon^{\mu an}_{//}(\mathbf{H},\sigma) - \varepsilon^{\mu an}_{//}(\mathbf{H} = \mathbf{0}, \sigma)$).

$$=\mu_0\, \mathbf{M}_{an}\, (\mathbf{H}_{an}, \sigma, T) \qquad (34)$$

Differentiation of this relation with respect to stress σ, followed by integration with respect to **H** leads to

$$\mu_0 \frac{\partial \mathbf{M}_{an}}{\partial \sigma}(\mathbf{H},\sigma) = \frac{\partial \varepsilon^{\mu an}}{\partial \mathbf{H}}(\mathbf{H},\sigma) \qquad (35)$$

$$\varepsilon^{\mu an}(\mathbf{H}, \sigma) - \varepsilon^{\mu an}(\mathbf{H}=0, \sigma) = \mu_0 \int_0^{\mathbf{H}} \frac{\partial \mathbf{M}_{an}}{\partial \sigma}(\mathbf{h},\sigma).\mathbf{dh} = \int_0^{\mathbf{H}} \frac{\partial \varepsilon^{\mu an}}{\partial \mathbf{H}}(\mathbf{h},\sigma).\mathbf{dh} \qquad (36)$$

For isotropic materials, it is reasonable to assume that the magnetostriction strain tensor takes the following form

$$\varepsilon^{\mu}(\mathbf{H}, \sigma) = \begin{pmatrix} \varepsilon^{\mu}_{//}(\mathbf{H},\sigma) & 0 & 0 \\ 0 & \varepsilon^{\mu}_{\perp}(\mathbf{H},\sigma) & 0 \\ 0 & 0 & \varepsilon^{\mu}_{\perp}(\mathbf{H},\sigma) \end{pmatrix}_{(\mathbf{u}_{//},\mathbf{v}_{\perp},\mathbf{w}_{\perp})} \qquad (37)$$

where $\mathbf{u}_{//}$, $\varepsilon^{\mu}_{//}(\mathbf{H},\sigma)$ and $\varepsilon^{\mu}_{\perp}(\mathbf{H},\sigma)$ respectively denote a unit vector parallel to the magnetisation $\mathbf{M}$, magnetostriction strain in the direction of magnetisation, and magnetostriction strain in perpendicular directions.

The results that are presented in Fig.3.a., correspond to the identification of $(\varepsilon^{\mu an}_{//}(\mathbf{H},\sigma) - \varepsilon^{\mu an}_{//}(\mathbf{H} = 0, \sigma))$ through a straightforward treatment of the experimental results presented in Fig.1.a. The results that are presented in Fig.3.b are directly derived from the experimental results presented in Fig. 1 .b; a finite difference scheme has been used to compute the term $\frac{\partial \mathbf{M}_{an}}{\partial \sigma}(\mathbf{H},\sigma)$. It is worth noticing that both plots 3.a and 3.b are directly derived from experiments and the expression of the state potential. In other words, this procedure -comparison between the direct measurements of magnetostrictive strains and the prediction of these strains from the magnetic response of the material under different stress states– provides an example of the choices made in writing the state potential.

6. LOCALISATION PHENOMENON AND ITS CONSEQUENCE

If, as mentioned above, Paul Germain was at the beginning of the story about the use of thermodynamics in the modelling of the mechanical behaviour, we find him also at the end. Indeed, the boundary value problems arising in non-linear solid mechanics (with the type of constitutive equations sketched above, damage mechanics and non-associated plasticity) may change their type locally during a loading process. It is well established and understood now that various types of constitutive equations lead to a rate-boundary value problem that may become hyperbolic at a certain stage of a loading process, while it was elliptic at the beginning. This loss of ellipticity suggests the appearance of discontinuity surfaces, and happens for example in the presence of softening or for certain non-associative behaviours. Note however that none of these features is necessary or sufficient. It is this type of change, this loss of ellipticity that is used to describe localisation phenomena for rate-independent solids, phenomena that are precursors to rupture. This sets the transition from a diffuse deformation behaviour to a very localized one.

Most of the time this boundary value problem becomes actually mixed (elliptic and hyperbolic at the same time). There are both an elliptic zone and a hyperbolic zone in the structure just like in fluid mechanics and in the context of transonic compressible flows where a subsonic zone coexists with a supersonic one, a field to which Paul Germain has significantly contributed.

It is also known that this loss of ellipticity, when it occurs, leads to some kind of ill-posedness in the sense that the boundary value problems may have an infinite number of linearly independent solutions, and that these solutions may not depend continuously on the data. This has led these last years to the reappraisal of generalised continuum theories such as higher gradient theories, Cosserat continua and nonlocal models. Paul Germain was also concerned by these theories. This reappraisal is meant to solve some physical problems that the classical local continuum framework fails to describe. All these theories bring in a way or another an internal length in the continuum that allows for instance one to fix the wavelength of certain bifurcation modes and a thickness to shear bands. The loss of ellipticity manifests itself in many ways: loss of ellipticity of the field equations but also through a compatibility between these field equations and boundary conditions or interfacial conditions (when the material is heterogeneous). See [Benallal et al, 1993] for a detailed discussion.

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