SHAPE HESSIAN BY THE VELOCITY METHOD: A LAGRANGIAN APPROACH*

M.C. DELFOUR
Centre de recherches mathématiques et
Département de mathématiques et de statistique,
Université de Montréal, C.P. 6128 Succ. A,
Montréal, Québec, Canada, H3C 3J7

J.P. ZOLÉSIO
Laboratoire de physique théorique,
Université des Sciences et Techniques du Languedoc,
Place Eugène Bataillon,
34060 - Montpellier, Cédex 02, France

ABSTRACT. In this paper we announce new results on the Shape Hessian of a shape functional by the Velocity (Speed) Method. We review and extend the Velocity Method and clarify its connections with methods using first or second order Perturbations of the Identity. We show that all these methods yield the same Shape gradient but a different and unequal Shape Hessian since each one depends on the choice of "connection". For autonomous velocity fields the Velocity Method yields a canonical bilinear Hessian. Expressions obtained by other methods can be recovered by adding to this canonical term the Shape gradient acting on the acceleration of the velocity field associated with the choice of perturbation of the identity. In the second part of the paper we give an application of the Lagrangian Method with Function Space Embedding to compute the Shape gradient and Hessian of a simple cost function associated with the non-homogeneous Dirichlet problem.

1. INTRODUCTION.

In this paper we announce new results on the *Shape Hessian* by the *Velocity (Speed) Method* (cf. J. CÉA [1,2,3] and J.P. ZOLÉSIO [1,2]) and apply the *Lagrangian Method* with *Function Space Embedding* to compute the Shape gradient and Hessian of a simple cost function associated with the non-homogeneous Dirichlet problem.

We describe a general method which applies to *differentiable semiconvex* cost functionals with applications to more general problems than the simple illustrative example we have chosen to consider. We also emphasize the use of the Function Space Embedding method (cf. DELFOUR-ZOLÉSIO [1,2,3,4,7]) combined with the implicit use of Lagrange multipliers. Therefore this paper complements our previous work where we have used a variational formulation for the Neumann problem (cf. ZOLÉSIO-DELFOUR [8]). In Shape Sensitivity Analysis the size of the computations can be quite large. Therefore it is extremely important to understand the fundamental structure of the Shape gradient and Hessian in order to simplify the computations and obtain mathematically meaningful expressions.

In the process we make a revision of the Velocity (Speed) Method and show how to associate with Methods of *Perturbation of the Identity* (first and second order) an appropriate non-autonomous family of velocity fields. For the Shape gradient, the different

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methods yield expressions which may look different but are all equal. However this is no longer true for the Shape Hessian. In fact we shall show in section 2.4 that different perturbations of the identity can yield final expressions which are not equal. It turns out that we can introduce an infinity of definitions based on perturbations of the identity. However we shall show that they always contain a *canonical bilinear term* plus the Shape gradient of the functional acting in the direction of an *acceleration* field which is characteristic of the chosen perturbation. The canonical bilinear term exactly coincides with the second order Shape derivative obtained by the Velocity (Speed) Method for autonomous velocity fields. Moreover each expression obtained by a method of perturbation of the identity can be strictly recovered by adding to the canonical term the Shape gradient acting in the direction of an appropriate acceleration field. Therefore we propose to refer to this canonical term as the *Shape Hessian.*

The above considerations clarify the fundamental concepts and reduce their complexity, but they do not eliminate all the associated computations. We need methods which provide both quick formal computations and appropriate mathematical justifications. We use Lagrangian methods combined with the use of theorems on the derivative of a MinMax with respect to a parameter. Such methods are well known and extensively used in Mechanical Sciences, Mathematical Programming and Optimal Control Theory. Their application to Shape Sensitivity Analysis is not completely straightforward since it leads to the "time-dependence" of the underlying function spaces appearing in the MinMax formulation. This phenomenon seems to be specific to that class of problems. Two techniques are available to get around this difficulty: the *Function Space Parametrization* and the *Function Space Embedding* methods. The first one has been used in DELFOUR-ZOLÉSIO [8,9], the second one will be used here.

It is fair to say that the use of Shape Hessians for discretized Finite Element Models and finitely parametrized shapes have been used in many places in the Engineering and Mechanics literatures. Some numerical expertise is available (cf. for instance, A. BERN [1], BERN-CHENOT-DEMAY-ZOLÉSIO [1]) and it is suspected that the really performing algorithms are not available in the open literature since they are marketable industrial products.

Few papers have dealt with the second variation of a Shape cost function for linear partial differential equations models. To our knowledge the first one by N. FUJII [1] used a second order perturbation of the identity along the normal to the boundary for second order linear elliptic problems. An extremely interesting paper by ARUMUGAN-PIRONNEAU [1,2] used the Shape second variation to solve the "ribblet problem". Finally J. SIMON [1] presented a computation of the second variation using a first order perturbation of the identity. The first general approach to the computation of Shape Hessians can be found in DELFOUR-ZOLÉSIO [8,9]. It uses the Velocity (Speed) Method and includes a simple illustrative example for the Neumann problem.

In conclusion, we would like to reiterate that the Velocity method and methods using first and second order perturbations of the identity lead to three different second order Shape derivatives which are not equal. The Velocity method with autonomous velocity fields provides the canonical bilinear Shape Hessian and all the other derivatives can be recovered by special choices of non-autonomous velocity fields.

2. SHAPE DERIVATIVES: DEFINITIONS AND PROPERTIES.

In this section we recall and extend the definitions of a Shape gradient and a Shape Hessian based on the Velocity Method (cf. J.P. ZOLÉSIO [1,2], DELFOUR-ZOLÉSIO [1,2]) and discuss their relationship to various methods based on perturbations of the identity operator.

2.1. Velocity (Speed) method and Perturbations of the Identity Operator. Let $V : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N$ be a given velocity field for some fixed $\tau > 0$. The map V can be viewed as a family $\{V(t)\}$ of non-autonomous velocity fields on $\mathbf{R}^N$ defined by

$$x \mapsto V(t)(x) \stackrel{\text{def}}{=} V(t,x) : \mathbf{R}^N \mapsto \mathbf{R}^N. \tag{1}$$

Assume that

$$(V) \quad \begin{cases} \forall x \in \mathbf{R}^N, \quad V(\cdot,x) \in C^0([0,\tau];\mathbf{R}^N) \\ \exists c > 0, \forall x,y \in \mathbf{R}^N, \quad \|V(\cdot,y) - V(\cdot,x)\|_{C^0([0,\tau];\mathbf{R}^N)} \leq c|y-x|, \end{cases}$$

where $V(.,x)$ is the function $t \mapsto V(t,.x)$. Associate with V the solution $x(t;X)$ of the ordinary differential equation

$$\frac{dx}{dt}(t) = V(t,x(t)), \quad t \in [0,\tau], \quad x(0) = X \in \mathbf{R}^N. \tag{2}$$

and introduce the homeomorphisms

$$X \mapsto T_t(V)(X) \stackrel{\text{def}}{=} x(t;X) : \mathbf{R}^N \to \mathbf{R}^N. \tag{3}$$

and the maps

$$(t,X) \mapsto \tau(t,X) \stackrel{\text{def}}{=} T_t(V)(X) : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N, \tag{4}$$

$$(t,x) \mapsto T_V^{-1}(t,x) \stackrel{\text{def}}{=} T_t^{-1}(V)(x) : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N. \tag{5}$$

NOTATION 2.1. In the sequel we shall drop the V in $\tau(t,X)$ and $T_t(V)$ whenever no confusion is possible.

THEOREM 2.1. *(i) Under hypothesis (V) the maps T and T^{-1} have the following properties*

$$(T1) \quad \begin{cases} \forall X \in \mathbf{R}^N, \quad T(\cdot,X) \in C^1([0,\tau];\mathbf{R}^N) \\ \exists c > 0, \forall X,Y \in \mathbf{R}^N, \quad \|T(\cdot,Y) - T(\cdot,X)\|_{C^1([0,\tau];\mathbf{R}^N)} \leq c|Y-X|, \end{cases}$$

$$(T2) \quad \forall t \in [0,\tau], \quad X \mapsto T_t(X) = T(t,X) : \mathbf{R}^N \to \mathbf{R}^N \quad \text{is bijective,}$$

$$(T3) \quad \begin{cases} \forall x \in \mathbf{R}^N, \quad T^{-1}(\cdot,x) \in C^0([0,\tau];\mathbf{R}^N) \\ \exists c > 0, \forall x,y \in \mathbf{R}^N, \quad \|T^{-1}(\cdot,y) - T^{-1}(\cdot,x)\|_{C^0([0,\tau];\mathbf{R}^N)} \leq c|y-x|. \end{cases}$$

(ii) If there exists a real $\tau > 0$ and a map $T : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N$ verifying hypotheses (T1), (T2) and (T3), then the map

$$(t,x) \mapsto V(t,x) = \frac{\partial T}{\partial t}(t, T_t^{-1}(x)) : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N, \tag{6}$$

verifies hypothesis (V), where T_t^{-1} is the inverse of $X \mapsto T_t(X)$. □

This first theorem is an equivalence result which says that we can either start from a family of velocity fields $\{V(t)\}$ on $\mathbf{R}^N$ or a family of transformations $\{T_t\}$ of $\mathbf{R}^N$ provided that the map $V, V(t,x) = V(t)(x)$, verifies (V) or the map $T, T(t,X) = T_t(X)$, verifies (T1), (T2) and (T3).

When we start from V, we obtain the velocity method. Given an initial domain Ω, the family of homeomorphisms $T_t(V)$defines a family of transformed domains

$$\Omega_t = T_t(V)(\Omega) = \{T_t(V)(X) : X \in \Omega\}. \tag{7}$$

In examples where we start from T, it is usually possible to verify hypotheses (T1),(T2) and (T3) and construct the corresponding velocity field V defined in (6). For instance perturbations of the identity to the first or second order fall in that category:

$$T_t(X) = X + tU(X) + \frac{t^2}{2}A(X) \quad (A = 0 \text{ for the first order }),\ t \geq 0, X \in \mathbf{R}^N, \tag{8}$$

where U and A are sufficiently smooth transformations of $\mathbf{R}^N$. It turns out that for Lipschitz transformations U and A, hypotheses (T1), (T2) and (T3) are verified.

THEOREM 2.2. *Let U and A be two uniform Lipschitz transformations of $\mathbf{R}^N$:*

$$\exists c > 0,\ \forall X,\ Y \in \mathbf{R}^N,\ |U(Y) - U(X)| \leq c|Y - X|,\ |A(Y) - A(X)| \leq c|Y - X|.$$

(i) Let $\tau = \min\{1, 1/4c\}$ and T be given by (8). Then the velocity

$$(t,x) \mapsto V(t,x) = U(T_t^{-1}(x)) + tA(T_t^{-1}(x)) : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N, \tag{9}$$

verifies hypotheses (V). □

REMARK 2.1. Observe that from (8) and (9)

$$V(0) = U,\ \overset{\bullet}{V}(0)(x) = \frac{\partial V}{\partial t}(t,x)|_{t=0} = A - [DU]U. \tag{10}$$

where DU is the Jacobian matrix of U. The term $\overset{\bullet}{V}(0)$ is an "acceleration" at $t = 0$ which will always be present even when $A = 0$. □

2.2. Shape gradient. In general a *shape functional* will be a map

$$\Omega \mapsto J(\Omega) : \mathcal{A} \subset \mathcal{P}(\mathbf{R}^N) \to \mathbf{R}. \tag{11}$$

defined on a subset $\mathcal{A}$ of the set $\mathcal{P}(\mathbf{R}^N)$ of all subsets of $\mathbf{R}^N$. Under the action of a velocity V verifying (V), the domain Ω is transformed into a new domain $\Omega_t(V) = T_t(V)(\Omega)$.

DEFINITION 2.1. *Given a velocity field V verifying (V), J is said to have an Eulerian semiderivative at Ω in the direction V if the following limit exists and is finite*

$$\lim_{t \searrow 0} [J(\Omega_t(V)) - J(\Omega)]/t. \tag{12}$$

When it exists, it is denoted $dJ(\Omega; V)$. □

This definition is quite general and may include situations where $dJ(\Omega; V)$ is not only a funtion of $V(0)$ but also of $V(t)$ in a neighbourhood of $t = 0$. This will not occur under some appropriate continuity hypothesis on the map $V \mapsto dJ(\Omega, V)$. To be more precise we introduce some notation. For any integers $k \geq 0$ and $m \geq 0$, and any compact subset K of $\mathbf{R}^N$

$$\mathcal{V}_K^{m,k} = C^m([0,\tau]; \mathcal{D}^k(K, \mathbf{R}^N)) \cap \mathcal{L}, \tag{13}$$

where $\mathcal{D}^k(K, \mathbf{R}^N)$ is the space of all k-time continuously differentiable maps from $\mathbf{R}^N$ to $\mathbf{R}^N$ with compact support in K and

$$\mathcal{L} = \left\{ V : [0,\tau] \times \mathbf{R}^N \to \mathbf{R}^N : V \text{verifies } \ (\mathrm{V}) \right\}. \tag{14}$$

With the above definitions we introduce the space

$$\vec{\mathcal{V}}^{m,k} \stackrel{\text{def}}{=} \varinjlim_{K} \left\{ \mathcal{V}_K^{m,k} : \forall K \ \text{ compact in } \mathbf{R}^N \right\} \tag{15}$$

where $\varinjlim$ denotes the inductive limit endowed with its natural inductive limit topology. For autonomous fields, the above constructions reduce to

$$\mathcal{V}^k = \left\{ \begin{array}{ll} \mathcal{D}^0(\mathbf{R}^N, \mathbf{R}^N) \cap \mathrm{Lip}(\mathbf{R}^N, \mathbf{R}^N), & \text{if } k = 0 \\ \mathcal{D}^k(\mathbf{R}^N, \mathbf{R}^N) & \text{if } k \geq 1 \end{array} \right\} \tag{16}$$

where Lip $(\mathbf{R}^N, \mathbf{R}^N)$ denotes the space of transformations of $\mathbf{R}^N$ which are uniformly Lipschitzian. In all cases (V) will be verified.

THEOREM 2.3. *Let Ω be a domain in $\mathbf{R}^N$ and $m \geq 0$ and $k \geq 0$ be integers. Assume that for all V in $\vec{\mathcal{V}}^{m,k}$, $dJ(\Omega; V)$ exists and that the map*

$$V \mapsto dJ(\Omega; V) : \vec{\mathcal{V}}^{m,k} \to \mathbf{R} \tag{17}$$

is continuous. Then

$$\forall V \in \vec{\mathcal{V}}^{m,k}, \quad dJ(\Omega; V) = dJ(\Omega; V(0)). \quad \square$$

In the above analysis we have chosen to follow the classical framework of the Theory of distributions (cf. L. SCHWARTZ [1]) and perturb the domain Ω by velocity fields V with compact support.

DEFINITION 2.2. *Let Ω be a domain in $\mathbf{R}^N$.*

*(i) The shape funtional J is said to be **shape differentiable** at Ω if the Eulerian semiderivative $dJ(\Omega; V)$ exists for all V in $\mathcal{D}(\mathbf{R}^N, \mathbf{R}^N)$ and the map*

$$V \mapsto dJ(\Omega; V): \ \mathcal{D}(\mathbf{R}^N, \mathbf{R}^N) \to \mathbf{R} \tag{19}$$

is linear and continuous.

*(ii) The map (19) defines a vector distribution $G(\Omega)$ which will be called the **shape gradient** of J at Ω.*

(iii) When $G(\Omega)$ is continuous on $\mathcal{D}^k(\mathbf{R}^N, \mathbf{R}^N)$ for some finite $k \geq 0$, we say that $G(\Omega)$ is of order k. □

The next theorem gives additional properties of shape differentiable functionals.

THEOREM 2.4. *(Generalized Hadamard's structure theorem). Let Ω be a domain in $\mathbf{R}^N$ with boundary Γ and assume that J is shape differentiable.*
(i) The support of $G(\Omega)$ is contained in Γ. Moreover when $G(\Omega)$ is of finite order its support is compact.
(ii) If $G(\Omega)$ is of finite order k and Ω is an open domain in $\mathbf{R}^N$ with boundary Γ in C^{k+1}, then there exists a scalar distribution $g(\Omega)$ in $\mathcal{D}^k(\Gamma)'$ such that

$$dJ(\Omega; V) = < g(\Omega), \ V \bullet n >_{\mathcal{D}^k(\Gamma)} \tag{20}$$

where n is the unit outward normal to Ω on Γ and $V \bullet n$ denotes the scalar product of V and n in $\mathbf{R}^N$. □

REMARK 2.2. When Γ is compact $\mathcal{D}^k(\Gamma)$ coincides with $C^k(\Gamma)$. □

2.3. Shape Hessian. We first study the second order Eulerian semiderivative $d^2J(\Omega; V; W)$ of a functional $J(\Omega)$ for two non-autonomous vector fields V and W. A first theorem shows that under some natural continuity hypotheses, $d^2J(\Omega; V; W)$ is the sum of two terms: the "canonical term $d^2J(\Omega; V(0); W(0))$" plus the first order Eulerian semiderivative $dJ(\Omega; \dot{V}(0))$ at Ω in the direction $\dot{V}(0)$ of the time-partial derivative $\partial_t V(t, x)$ at $t = 0$.

As for first order Eulerian semiderivatives, this first theorem reduces the study of second order Eulerian semiderivatives to the autonomous case. So we shall specialize to fields V and W in $\mathcal{D}^k(\mathbf{R}^N, \mathbf{R}^N)$ and give the equivalent of Hadamard's structure theorem for the "canonical term".

2.3.1. Non-autonomous case. The basic framework introduced in sections 2.1 and 2.2 has reduced the computation of the Eulerian semiderivative of $J(\Omega)$ to the computation of the derivative

$$j'(0) = dJ(\Omega; V(0)) \tag{21}$$

of the function

$$j(t) = J(\Omega_t(V)). \tag{22}$$

For $t \geq 0$, we naturally obtain

$$j'(t) = dJ(\Omega_t(V); V(t)). \tag{23}$$

This suggests the following definition.

DEFINITION 2.3. *Let V and W belong to $\mathcal{L}$ and assume that for all $t \in [0, \tau]$, $dJ(\Omega_t(W); V(t))$ exists for $\Omega_t(W) = T_t(W)(\Omega)$. The functional J is said to have a* ***second order Eulerian semiderivative*** *at Ω in the directions (V, W) if the following limit exists*

$$\lim_{t \searrow 0} [dJ(\Omega_t(W); V(t)) - dJ(\Omega; V(0))]/t. \tag{24}$$

When it exists, it is denoted $d^2 J(\Omega; V; W)$. □

REMARK 2.3. This last definition is compatible with the second order expansion of $j(t)$ with respect to t around $t = 0$:

$$j(t) \cong j(0) + tj'(0) + \frac{t^2}{2} j''(0), \tag{25}$$

where

$$j''(0) = d^2 J(\Omega; V; V). \quad \Box \tag{26}$$

REMARK 2.4. It is easy to construct simple examples with time-invariant fields V and W showing that $d^2 J(\Omega; V; W) \neq d^2 J(\Omega; W; V)$ (cf. DELFOUR-ZOLÉSIO [8]). □

The next theorem is the analogue of Theorem 2.3 and provides the canonical structure of the second order Eulerian semiderivative.

THEOREM 2.5. *Let Ω be a domain in $\mathbf{R}^N$ and $m \geq 0$ and $l \geq 0$ be integers. Assume that*

(i) $\forall V \in \vec{\mathcal{V}}^{m+1,l}$, $\forall W \in \vec{\mathcal{V}}^{m,l}$, *$d^2 J(\Omega; V; W)$ exists,*

(ii) $\forall W \in \vec{\mathcal{V}}^{m,l}$, $\forall t \in [0, \tau]$, J has a shape gradient at $\Omega_t(W)$ of order l,

(iii) $\forall\, U \in \mathcal{V}^l$, the map

$$W \mapsto d^2 J(\Omega; U; W) : \vec{\mathcal{V}}^{m,l} \to \mathbf{R} \tag{27}$$

is continuous. Then for all V in $\vec{\mathcal{V}}^{m+1,l}$ and all W in $\vec{\mathcal{V}}^{m,l}$

$$d^2 J(\Omega; V; W) = d^2 J(\Omega; V(0); W(0)) + dJ(\Omega; \dot{V}(0)), \tag{28}$$

where

$$\dot{V}(0)(x) = \lim_{t \searrow 0} [V(t, x) - V(0, x)]/t. \Box \tag{29}$$

2.3.2. Autonomous case.

DEFINITION 2.4. *Let Ω be a domain in $\mathbb{R}^N$.*
(i) The funtional $J(\Omega)$ is said to be shape differentiable at Ω if

$$\forall\, V,\ \forall\, W \text{ in } \mathcal{D}(\mathbb{R}^N,\mathbb{R}^N),\ d^2J\,(\Omega;V;W) \text{ exists} \tag{30}$$

and the map

$$(V,W)\mapsto d^2J(\Omega;V;W):\mathcal{D}(\mathbb{R}^N,\mathbb{R}^N)\times\mathcal{D}(\mathbb{R}^N,\mathbb{R}^N)\to\mathbb{R} \tag{31}$$

is bilinear and continuous. We denote by h the bilinear and continuous map (31).
(ii) Denote by $H(\Omega)$ the continous linear map on the tensor product $\mathcal{D}(\mathbb{R}^N,\mathbb{R}^N)\otimes\mathcal{D}(\mathbb{R}^N,\mathbb{R}^N)$, associated with h:

$$d^2J(\Omega;V;W)=\langle H(\Omega),V\otimes W\rangle=h(V,W), \tag{32}$$

where $V\otimes W$ is the tensor product of V and W defined as

$$(V\otimes W)_{ij}(x,y)=V_i(x)W_j(y),\quad 1\le i,\ j\le N, \tag{33}$$

and $V_i(x)$ (resp. $W_j(y)$) is the i-th (resp. j-th) component of the vector V(resp. W) (cf. L. SCHWARTZ [2]'s kernel theorem and GELFAND-VILENKIN [1]. $H(\Omega)$ will be called the Shape Hessian of J at Ω.
(iii) When there exists an integer $\ell\ \ge\ 0$ such that $H(\Omega)$ is continuous on $\mathcal{D}^\ell(\mathbb{R}^N,\mathbb{R}^N)\otimes\mathcal{D}^\ell(\mathbb{R}^N,\mathbb{R}^N)$ we say that $H(\Omega)$ is of order ℓ. □

THEOREM 2.6. *Let Ω be a domain in $\mathbb{R}^N$ with boundary Γ and assume that J is twice shape differentiable at Ω.*
(i) $H(\Omega)$ has support in $\Gamma\times\Gamma$. Moreover the support of $H(\Omega)$ is compact when its order is finite.
(ii) If $H(\Omega)$ is of finite order ℓ, $\ell\ \ge\ 0$, Ω is an open domain in $\mathbb{R}^N$ with boundary Γ in $C^{\ell+1}$, then there exists a continuous linear map on the tensor product $\mathcal{D}^\ell(\Gamma,\mathbb{R}^N)\otimes\mathcal{D}^\ell(\Gamma)$ such that

$$d^2J(\Omega);V;W)=\langle h(\Omega),(\gamma_\Gamma V)\otimes((\gamma_\Gamma W)\bullet n)\rangle \tag{34}$$

where $(\gamma_\Gamma V)\otimes((\gamma_\Gamma W)\bullet n)$ is defined as the tensor product

$$((\gamma_\Gamma V)\otimes(\gamma_\Gamma W)\bullet n))_i(x,y)=(\gamma_\Gamma V_i)(x)((\gamma_\Gamma W)\bullet n)(y),\ x,y\in\Gamma, \tag{35}$$

$V_i(x)$ is the i-th component of $V(x)$ and

$$(\gamma_\Gamma(W)\bullet n)(y)=(\gamma_\Gamma W)(y)\bullet n(y),\ y\in\Gamma.\quad \square \tag{36}$$

REMARK 2.5. Finally under the hypotheses of Theorem 5 and 6

$$\begin{aligned} d^2J(\Omega;V;W)=&\langle h(\Omega),(\gamma_\Gamma V(0))\otimes((\gamma_\Gamma W(0))\bullet n)\rangle\\ &+\langle g(\Omega),(\gamma_\Gamma \dot{V}(0))\bullet n\rangle \end{aligned} \tag{37}$$

for all V in $\vec{\mathcal{V}}_D^{m+1,l}$ and W in $\vec{\mathcal{V}}_D^{m,l}$. □

2.4. Comparison with Methods of Perturbation of the Identity. At this juncture it is instructive to compare first and second order Eulerian semiderivatives obtained by the Velocity (Speed) Method with those obtained by first and second order perturbations of the identity: that is, when the transformations T_t are specified a priori by

$$T_t(X) = X + tU(X) + \frac{t^2}{2}A(X),\ X \in \mathbf{R}^N, \tag{38}$$

where U and A are transformations of $\mathbf{R}^N$ verifying the hypotheses of Theorem 2.2. The transformation T_t in (38) is a ***second order*** perturbation when $A \neq 0$ and a ***first order*** perturbation when $A = 0$.

According to Theorem 2.2, first and second order Eulerian semiderivatives associated with (38) can be equivalently obtained by applying the Velocity (Speed) Method to the time-varying velocity fields V_{UA} given by (9)

$$dJ(\Omega; V_{UA}) = dJ(\Omega; V_{UA}(0)) = dJ(\Omega; U) \tag{39}$$

where we have used Remark 2.1 which says that

$$V_{UA}(0) = U \text{ and } \dot{V}_{UA}(0) = A - [DU]U. \tag{40}$$

Similarly if V_{WB} is another velocity field corresponding to

$$T_t(X) = X + tW(X) + \frac{t^2}{2}B(X),\ X \in \mathbf{R}^N, \tag{41}$$

where W and B verify the hypotheses of Theorem 2.2, then

$$d^2J(\Omega; V_{UA}; V_{WB}) = d^2J(\Omega; V_{UA}(0); V_{WB}(0)) + dJ(\Omega; \dot{V}_{UA}(0)) \tag{42}$$

and

$$d^2J(\Omega; V_{UA}; V_{WB}) = d^2J(\Omega; U; W) + dJ(\Omega; A - [DU]U). \tag{43}$$

Expressions (39) and (43) are to be compared with the following expressions obtained by the Velocity (Speed) Method for two time-invariant vector fields V and W

$$dJ(\Omega; V) \text{ and } d^2J(\Omega; V; W). \tag{44}$$

For the Shape gradient the two expressions coincide; for the Shape Hessian we recognize the bilinear term in (43) and (44) but the two expressions differ by the term

$$dJ(\Omega; A - [DU]U). \tag{45}$$

Even for a first order perturbation ($A = 0$), we have a quadratic term in U.

This situation is analogous to the classical problem of defining second order derivatives on a manifold. The term (45) would correspond to the connexion while the bilinear term $d^2J(\Omega; V; W)$ would be the candidate for the ***canonical*** second order shape derivative. In this context we shall refer to the corresponding distribution $H(\Omega)$ as the ***canonical Shape***

Hessian. All other second order shape derivatives will be obtained from $H(\Omega)$ by adding the gradient term $g(\Omega)$ acting as the appropriate acceleration field (connexion). □

REMARK 2.6. The method of perturbation of the identity can be made "more canonical" by using the following family of transformations

$$T_t(X) = X + tU(X) + \frac{t^2}{2}(A + [DU]U) \tag{46}$$

which yields

$$dJ(\Omega; U) \text{ for the gradient} \tag{47}$$

and

$$d^2J(\Omega; U; W) + dJ(\Omega; A) \text{ for the Hessian,} \tag{48}$$

where for a first order perturbation ($A = 0$) the second term disappears. □

REMARK 2.7. When Ω^* is an appropriately smooth domain which minimizes a twice Shape differentiable functional $J(\Omega)$ without constraints on Ω, the classical necessary conditions would be (at least formally)

$$dJ(\Omega^*; V) = 0, \quad \forall V, \tag{49}$$

$$d^2J(\Omega^*; W; W) \geq 0, \quad \forall W, \tag{50}$$

or equivalently for "smooth velocity fields V and W"

$$dJ(\Omega^*; V(0)) = 0, \quad \forall V \tag{51}$$

$$d^2J(\Omega^*; W(0); W(0)) + dJ(\Omega^*; \dot{V}(0)) \geq 0, \quad \forall W. \tag{52}$$

But in view of (51), condition (52) reduces to the following condition on the "canonical Shape Hessian"

$$d^2J(\Omega^*; W(0); W(0)) \geq 0, \quad \forall W. \quad \square \tag{53}$$

3. A SADDLE POINT FORMULATION OF THE DIRICHLET PROBLEM.

Let Ω be a bounded open domain in $\mathbf{R}^N$ with a sufficiently smooth boundary Γ. Let f and g be two fixed functions in $H^{\frac{1}{2}+\epsilon}(\mathbf{R}^N)$ and $H^{2+\epsilon}(\mathbf{R}^N)$, respectively, for some arbitrary small $\epsilon > 0$. Consider the solution y in $H^2(\Omega)$ to the non-homogeneous Dirichlet boundary value problem.

$$-\Delta y = f \text{ in } \Omega, y = g \text{ on } \Gamma. \tag{1}$$

We can also say that y is the solution of the weak equation

$$\int_\Omega (\Delta y + f)\psi \, dx + \int_\Gamma (y - g)\mu \, d\Gamma = 0 \tag{2}$$

for all ψ in $H^2(\Omega)$ and μ in $H^{\frac{1}{2}}(\Gamma)$, since the corresponding functional

$$L(\phi,\ \psi,\ \mu) = \int_\Omega (\Delta\phi + f)\psi\ dx + \int_\Gamma (\phi - g)\mu\ d\Gamma. \tag{3}$$

It has a unique saddle point $(\hat{\phi}, \hat{\psi}, \hat{\mu})$ which is completely characterized by the equations

$$\Delta\hat{\phi} + f = 0 \text{ in } \Omega, \tag{4}$$

$$\hat{\phi} - g = 0 \text{ in } \Gamma, \tag{5}$$

$$\int_\Omega \Delta\phi\ \hat{\psi}\ dx + \int_\Gamma \phi\hat{\mu}\ d\Gamma = 0,\ \forall\phi \in H^2(\Omega), \tag{6}$$

where the last equation yields

$$\Delta\hat{\psi} = 0 \text{ in } \Omega,\ \hat{\psi} = 0 \text{ on } \Gamma \textit{ and } \hat{\mu} = \frac{\partial\hat{\psi}}{\partial n} \text{ on } \Gamma. \tag{7}$$

Of course, this implies that the saddle point is unique and given by

$$(\hat{\phi}, \hat{\psi}, \hat{\mu}) = (y, 0, 0). \tag{8}$$

The purpose of the above computation was to find out the form of the multiplier $\hat{\mu}$

$$\hat{\mu} = \frac{\partial\hat{\psi}}{\partial n} \text{ on } \Gamma, \tag{9}$$

in order to rewrite the previous functional as a function of two variables instead of three:

$$L(\phi, \psi) = \int_\Omega (\Delta\phi + f)\psi\ dx + \int_\Gamma (\phi - g)\frac{\partial\psi}{\partial n}\ d\Gamma, \tag{10}$$

for (ϕ, ψ) in $H^2(\Omega) \times H^2(\Omega)$. It is also advantageous for shape problems to get rid of boundary integrals whenever it is possible. So noting that

$$\int_\Gamma (\phi - g)\frac{\partial\psi}{\partial n}\ d\Gamma = \int_\Omega \operatorname{div}[(\phi - g)\nabla\psi]\ dx, \tag{11}$$

we finally use the functional

$$L(\phi, \psi) = \int_\Omega \{(\Delta\phi + f)\psi + (\phi - g)\Delta\psi + \nabla(\phi - g) \bullet \nabla\psi\}\ dx \tag{12}$$

on $H^2(\Omega) \times H^2(\Omega)$. It is readily seen that it has a unique saddle point $(\hat{\phi}, \hat{\psi})$ in $H^2(\Omega) \times H^2(\Omega)$ which is completely characterized by the saddle point equations:

$$\Delta\hat{\phi} + f = 0 \textit{ in } \Omega, \quad \hat{\psi} = g \textit{ on } \Gamma, \quad \Delta\hat{\psi} = 0 \text{ in } \Omega, \quad \hat{\psi} = 0 \text{ on } \Gamma. \tag{13}$$

4. SHAPE GRADIENT FOR THE DIRICHLET PROBLEM .

4.1. Formulation and formal computations. Consider the cost function

$$J(\Omega) = \frac{1}{2}\int_{\Omega} |y(\Omega) - y_d|^2 \, dx \tag{1}$$

associated with the solution $y = y(\Omega)$ of the Dirichlet problem (3.1) and the fixed function y_d in $H^{\frac{1}{2}+\epsilon}(\mathbf{R}^N)$ for some arbitrary fixed $\epsilon > 0$.

As in section 3, we reformulate this problem as the saddle point of a functional by introducting the Lagrangian

$$\begin{aligned} G(\Omega, \phi, \psi) = & \frac{1}{2}\int_{\Omega} |\phi - y_d|^2 dx \\ & + \int_{\Omega} \{(\Delta\phi + f)\psi + (\phi - g)\Delta\psi + \nabla(\phi - g) \bullet \nabla\psi\} \, dx \end{aligned} \tag{2}$$

on $H^2(\Omega) \times H^2(\Omega)$. It is readily seen that $G(\Omega, \cdot, \cdot)$ has a unique saddle point$(\hat{\phi}, \hat{\psi})$ which is completely characterized by the following saddle point equations:

$$\Delta\hat{\phi} + f = 0 \text{ in } \Omega, \quad \hat{\phi} = g \text{ on } \Gamma \tag{3}$$

$$\int_{\Omega} \{(\hat{\phi} - y_d)\phi + \Delta\phi\hat{\psi} + \phi\Delta\hat{\psi} + \nabla\phi \bullet \nabla\hat{\psi}\} \, dx = 0, \quad \forall\, \phi \in H^2(\Omega). \tag{4}$$

But the last equation is equivalent to

$$\int_{\Omega} [(\hat{\phi} - y_d) + \Delta\hat{\psi}]\phi \, dx + \int_{\Gamma} \frac{\partial\phi}{\partial n}\hat{\psi} \, d\Gamma = 0, \quad \forall\phi \in H^2(\Omega) \tag{5}$$

or

$$\Delta\hat{\psi} + (\hat{\phi} - y_d) = 0 \text{ in } \Omega, \quad \hat{\psi} = 0 \text{ on } \Gamma, \tag{6}$$

by using the theorem on the surjectivity of the trace. In the sequel, we shall use the notation (y, p) for the saddle point $(\hat{\phi}, \hat{\psi})$. As a result, we have

$$J(\Omega) = \underset{\phi\in H^2(\Omega)}{\mathrm{Min}} \quad \underset{\psi\in H^2(\Omega)}{\mathrm{Max}} \quad G(\Omega, \phi, \psi). \tag{7}$$

We shall now use the above Lagrangian formulation combined with the Velocity method (cf. J.CÉA [1,2,3], J.P.ZOLÉSIO[1,2], DELFOUR-ZOLÉSIO [1,2,3,4,7]) to compute the Shape gradient of $J(\Omega)$. Recall that the domain Ω is perturbed by a velocity vector field V which defines a homeomorphism (cf. section 2.1)

$$T_t : \mathbf{R}^N \to \mathbf{R}^N, \; T_t(X) = x(t), \tag{8}$$

and a new domain

$$\Omega_t = T_t(\Omega). \tag{9}$$

The Shape semiderivative is defined as(cf. section 2.2)

$$dJ(\Omega; V) = \lim_{t \searrow 0} [J(\Omega_t) - J(\Omega)]/t \tag{10}$$

whenever the limit exists. It is easy to check that

$$J(\Omega_t) = \underset{\phi \in H^2(\Omega_t)}{\text{Min}} \quad \underset{\psi \in H^2(\Omega_t)}{\text{Max}} \quad G(\Omega_t, \phi, \psi). \tag{11}$$

There are two ways to get rid of the time dependence in the underlying function spaces (cf. DELFOUR-ZOLÉSIO [1,2]):
- the *Function Space Parametrization Method*
- the *Function Space Embedding Method.*

In the first case, we parametrize the functions in $H^2(\Omega_t)$ by elements of $H^2(\Omega)$ through the transformation

$$\phi \mapsto \phi \circ T_t^{-1} = H^2(\Omega) \to H^2(\Omega_t), \tag{12}$$

where "o" denotes the composition of the two maps and we introduce the *Parametrized Lagrangian*,

$$\tilde{G}(t, \phi, \psi) = G(T_t(\Omega), \phi \circ T_t^{-1}, \psi \circ T_t^{-1}) \tag{13}$$

on $H^2(\Omega) \times H^2(\Omega)$. In the Function Space Embedding Method, we introduce a large enough domain, D which contains all the transformations $\{\Omega_t : 0 \leq t \leq \bar{t}\ \}$ of Ω for some small $\bar{t} > 0$.

In this paper, we shall use the Function Space Embedding Method with $D = \mathbb{R}^N$

$$J(\Omega_t) = \underset{\Phi \in H^2(\mathbb{R}^N)}{\text{Min}} \quad \underset{\Psi \in H^2(\mathbb{R}^N)}{\text{Max}} \quad G(\Omega_t, \Phi, \Psi). \tag{14}$$

As can be expected the price to pay for the use of this method, is the fact that the set of saddle points

$$S(t) = X(t) \times Y(t) \ \subset H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N) \tag{15}$$

is not a singleton anymore since

$$X(t) = \{\Phi \in H^2(\mathbb{R}^N) : \Phi|_{\Omega_t} = y_t\} \tag{16}$$

$$Y(t) = \{\Psi \in H^2(\mathbb{R}^N) : \Psi|_{\Omega_t} = p_t\} \tag{17}$$

where (y_t, p_t) is the unique solution in $H^2(\Omega_t) \times H^2(\Omega_t)$ to the previous saddle point equations on Ω_t

$$\Delta y_t + f = 0 \ \text{ in } \ \Omega_t, \quad y_t = g \ \text{ on } \ \Gamma_t, \tag{18}$$

$$\Delta p_t + (y_t - y_d) = 0 \ \text{ in } \ \Omega_t, \quad p_t = 0 \ \text{ on } \ \Gamma_t. \tag{19}$$

We are now ready to apply the theorem of CORREA-SEEGER [1] which says that under appropriate hypotheses (to be checked in the next section)

$$dJ(\Omega; V) = \underset{\Phi \in X(0)}{\text{Min}} \quad \underset{\Psi \in Y(0)}{\text{Max}} \quad \partial_t G(\Omega_t, \Phi, \Psi). \tag{20}$$

Since we have already characterized $X(0)$ and $Y(0)$, we only need to compute the partial derivative of

$$G(\Omega_t, \Phi, \Psi) = \int_{\Omega_t} \{\frac{1}{2}|\Phi - y_d|^2 + (\Delta\Phi + f)\Psi + (\Phi - g)\Delta\Psi + \nabla(\Phi - g) \bullet \nabla\Psi\}dx. \quad (21)$$

If we assume that Ω_t is sufficiently smooth, then

$$f, y_d \in H^{\frac{1}{2}+\epsilon}(\mathbf{R}^N) \text{ and } g \in H^{2+\epsilon}(\mathbf{R}^N) \Rightarrow y \text{ and } p \in H^{\frac{5}{2}+\epsilon}(\Omega) \quad (22)$$

and we can choose to consider our saddle points $S(t)$ in $H^{\frac{5}{2}+\epsilon}(\mathbf{R}^N) \times H^{\frac{5}{2}+\epsilon}(\mathbf{R}^N)$ rather than $H^2(\mathbf{R}^N) \times H^2(\mathbf{R}^N)$. If Φ and Ψ belong to $H^{\frac{5}{2}+\epsilon}(\mathbf{R}^N)$, then

$$\partial_t G(\Omega_t, \Phi, \Psi) = \int_{\Gamma_t} \{\frac{1}{2}(\Phi - y_d)^2 + (\Delta\Phi + f)\Psi + (\Phi - g)\Delta\Psi + \nabla(\Phi - g) \bullet \nabla\Psi\} V \bullet n_t \, d\Gamma_t. \quad (23)$$

This expression is an integral over the boundary Γ which will not depend on Φ and Ψ outside of $\bar{\Omega}$. As a result the Min and the Max can be dropped in expression (20) which reduces to

$$dJ(\Omega; V) = \int_{\Gamma} \{\frac{1}{2}(y - y_d)^2 + (\Delta y + f)p + (y - g)\Delta p + \nabla(y - g) \bullet \nabla p\} V \bullet n \, d\Gamma. \quad (24)$$

But

$$p = 0 \text{ and } y - g = 0 \quad \Rightarrow \quad \nabla p = \frac{\partial p}{\partial n}\, n, \quad \nabla(y - g) = \frac{\partial}{\partial n}(y - g)\, n \quad \text{on } \Gamma \quad (25)$$

and finally

$$dJ(\Omega; V) = \int_{\Gamma} \{\frac{1}{2}(g - y_d)^2 + \frac{\partial}{\partial n}(y - g)\frac{\partial p}{\partial n}\} V \bullet n \, d\Gamma. \quad (26)$$

4.2. Verification of the Hypotheses. As we have seen the computations of the Shape gradient is both quick and easy. We now turn to the step by step verification of the hypotheses of the underlying theorem. Many of the constructions given below are "canonical" and can be repeated for different problems in different contexts.

THEOREM 4.1. *(CORREA AND SEEGER [1]). Let $\tau > 0$, the sets X and Y and the functional $L : [0, \tau] \times X \times Y \to \mathbf{R}$ be given. Denote by*

$$S(t) = X(t) \times Y(t) \subset X \times Y \quad (27)$$

the set of saddle points of the functional $L(t, \cdot\,, \cdot)$ on $X \times Y$. Assume that
(H1) $\quad \forall t \in [0, \tau], \; S(t) \neq \varnothing$.
and that
(H2) $\quad \forall (x, y) \in [X(0) \times \bigcup_{0 \le t \le \tau} Y(t)] \cup [\bigcup_{0 \le t \le \tau} X(t) \times Y(0)], \; \partial_t L(t, x, y)$ exists on $[0, \tau]$.

Moreover, assume that there exist topologies $\mathcal{T}_X$ on X and $\mathcal{T}_Y$ on Y such that

(H3) *for all sequences $t_n \to 0$ as $n \to \infty$, $0 \le t_n \le \tau$, there exists $(x_0, y_0) \in S(0)$ and a subsequence of $\{t_n\}$, still denoted $\{t_n\}$, such that $\forall n$, $\exists (x_n, y_n) \in S(t_n)$ and $(x_n, y_n) \to (x_0, y_0)$ in $\mathcal{T}_X \times \mathcal{T}_Y$,*

(H4) *for all y in $Y(0)$ (resp. x in $X(0)$)*
$\liminf\limits_{\substack{t\searrow 0\\ n\to\infty}} \partial_t L(t, x_n, y) \ge \partial_t L(0, x_0, y)$ (resp. $\limsup\limits_{\substack{t\searrow 0\\ n\to\infty}} \partial_t L(t, x, y_n) \le \partial_t L(0, x, y_0)$).

Then the function

$$g(t) = \min_{x\in X} \max_{y\in Y} L(t, x, y)$$

on $[0, \tau]$ has a derivative at $t = 0$ given by

$$dg(0) = \lim_{t\searrow 0}[g(t) - g(0)]/t = \operatorname*{Inf}_{x\in X(0)} \operatorname*{Sup}_{y\in B(0)} \partial_t L(0, x, y)$$
$$= \operatorname*{Sup}_{y\in B(0)} \operatorname*{Inf}_{x\in X(0)} \partial_t L(0, x, y). \quad \square$$

Let y_d and $f \in H^1(\mathbf{R}^N)$ and $g \in H^{\frac{5}{2}}(\mathbf{R}^N)$ so that

$$X = Y = H^3(\mathbf{R}^N). \tag{28}$$

The saddle points $S(t) = X(t) \times Y(t)$ are given by

$$X(t) = \{\Phi \in X : \Phi|_{\Omega_t} = y_t\} \tag{29}$$

$$Y(t) = \{\Psi \in Y : \Psi|_{\Omega_t} = p_t\} \tag{30}$$

The sets $X(t)$ and $Y(t)$ are not empty since it is always possible to construct a continuous linear extension

$$\Pi^m : H^m(\Omega) \to H^m(\mathbf{R}^N) \tag{31}$$

for each $m \ge 1$. For instance with $m = 1$ and a boundary Γ which is $W^{1,\infty}$, see AGMON-DOUGLIS-NIRENBERG [1,2] and V.M.BABIĆ [1] for $m > 1$ (see also J.NEČAS [1]). Using this Π^m, then we define the following extension

$$\Pi_t^m : H^m(\Omega_t) \to H^m(\mathbf{R}^N) \tag{32}$$

$$\Pi_t^m(\phi) = [\Pi^m(\phi \circ T_t)] \circ T_t^{-1}. \tag{33}$$

In the sequel m is fixed and equal to 3, so we shall drop the superscript m and define the extensions

$$Y_t = \Pi_t y_t \quad P_t = \Pi_t p_t \tag{34}$$

of y_t and p_t, respectively. Hence,

$$Y_t \in X(t) \text{ and } P_t \in Y(t) \Rightarrow S(t) \ne \emptyset. \tag{35}$$

So hypothesis $(H1)$ is verified. Hypothesis $(H2)$ follows by hypotheses on f, y_d and g. To check hypothesis $(H3)$ and $(H4)$, we need two general theorems which can be used in various contexts and problems.

THEOREM 4.2. *For $V \in \mathcal{D}^1(\mathbb{R}^N, \mathbb{R}^N)$ and $\Phi \in L^2(\mathbb{R}^N)$*

$$\lim_{t \searrow 0} \Phi \circ T_t = \Phi \quad \textit{and} \quad \lim_{t \searrow 0} \Phi \circ T_t^{-1} = \Phi \text{ in } L^2(\mathbb{R}^N). \tag{36}$$

PROOF. (i) The space $\mathcal{D}(\mathbb{R}^N)$ of continuous functions with compact support in $\mathbb{R}^N$ is dense in $L^2(\mathbb{R}^N)$. So given $\epsilon > 0$, there exists Φ_ϵ in $\mathcal{D}(\mathbb{R}^N)$ such that

$$\|\Phi - \Phi_\epsilon\|_{L^2}^2 < \epsilon^2 / \max\{J_t^{-1} : 0 \le t \le \tau\}.$$

Hence,

$$\|\Phi \circ T_t - \Phi\| \le \|\Phi_\epsilon \circ T_t - \Phi_\epsilon\| + \|\Phi \circ T_t - \Phi_\epsilon \circ T_t\| + \|\Phi - \Phi_\epsilon\|. \tag{37}$$

But, $\forall t \in [0, \tau]$

$$\int_{\mathbb{R}^N} |\Phi \circ T_t - \Phi_\epsilon \circ T_t|^2 dx = \int_{\mathbb{R}^N} |\Phi - \Phi_\epsilon|^2 J_t^{-1} dx \le \epsilon^2$$

So the last two terms in (37) are less than 2ϵ. It remains to evaluate the first term for a fixed function Φ_ϵ. Φ_ϵ has a compact support K in $\mathbb{R}^N$. Now, choose a bounded open domain D which contains $T_t(K)$ for all t in $[0, \tau]$. Since Φ_ϵ is uniformly continuous on $\mathbb{R}^N$

$$\exists \delta > 0, \ |x - y| < \delta \Rightarrow |\Phi_\epsilon(y) - \Phi_\epsilon(x)| < \epsilon / m(D)^{\frac{1}{2}}.$$

But, T_t is also uniformly continuous and

$$\exists \eta > 0, \ \forall 0 \le t < \eta, \ \forall x \in D, \ |T_t x - x| < \delta.$$

By construction

$$\text{supp } (\Phi_\epsilon \circ T_t) = T_t \ (\text{supp } \Phi_\epsilon) \subset D$$

and

$$\Phi_\epsilon = 0 \ \text{ and } \ \Phi_\epsilon \circ T_t = 0 \text{ outside of } D.$$

Finally,

$$\int_{\mathbb{R}^N} |\Phi_\epsilon(T_t x) - \Phi_\epsilon(x)|^2 \, dx = \int_D |\Phi_\epsilon(T_t x) - \Phi_\epsilon(x)|^2 \, dx \le \epsilon^2$$

and this implies that

$$\forall \epsilon > 0, \ \exists \eta > 0, \ \forall 0 \le t \le \eta, \ \|\Phi \circ T_t - \Phi\|_{L^2(\mathbb{R}^N)} \le 3\epsilon.$$

(ii) For the second part of (36) we make a change of variable and use the result of part (i)

$$\int_{\mathbb{R}^N} |\Phi \circ T_t^{-1} - \Phi|^2 dx = \int_{\mathbb{R}^N} |\Phi - \Phi \circ T_t|^2 J_t dx \le \epsilon^2$$

This completes the proof. $\square$

COROLLARY. *Under the hypotheses of Theorem 4.2 for $m \geq 1$, V in $\mathcal{D}^m(\mathbf{R}^N, \mathbf{R}^N)$ and $\Phi \in H^m(\mathbf{R}^N)$,*

$$\lim_{t>0\to 0} \Phi \circ T_t = \Phi \quad and \quad \lim_{t>0\to 0} \Phi \circ T_t^{-1} = \Phi \ in \ H^m(\mathbf{R}^N). \quad \square \tag{38}$$

REMARK 4.1. In fact for $m \geq 1$ and $V \in \mathcal{D}^m(\mathbf{R}^N, \mathbf{R}^N)$ the transformation

$$S(t)\Phi = \Phi \circ T_t, \quad \forall \Phi \in H^m(\mathbf{R}^N), \ \forall t, \ 0 \leq t \leq \tau, \tag{39}$$

defines a strongly continuous semigroup of class C_0 on $H^m(\mathbf{R}^N)$ with infinitesimal generator

$$\mathcal{A}\Phi = \nabla\Phi \bullet V, \ D(\mathcal{A}) = \{\Phi \in H^m(\mathbf{R}^N) : \nabla\Phi \bullet V \in H^m(\mathbf{R}^N)\}. \quad \square$$

THEOREM 4.3. *Under the hypotheses of Theorem 4.2,*

$$y^t \to y^0 \ \text{ in } \ H^m(\Omega) - \text{strong (resp. weak)} \tag{40}$$

implies that

$$Y_t \to Y_0 \ \text{ in } \ H^m(\mathbf{R}^N) - \text{strong (resp. weak)}.$$

PROOF. The strong case is obvious. We prove the weak case for $m = 0$. By definition,

$$Y_t = (\Pi y^t) \circ T_t^{-1}$$

and for all Φ in $L^2(\mathbf{R}^N)$, we consider

$$\int_{\mathbf{R}^N} Y_t \Phi \, dx = \int_{\mathbf{R}^N} (\Pi y^t) \circ T_t^{-1} \Phi \, dx = \int_{\mathbf{R}^N} \Pi y^t \Phi \circ T_t \ J_t \, dx$$

We have shown in Theorem 4.2 that

$$\Phi \circ T_t \to \Phi \ \text{ in } \ L^2(\mathbf{R}^N) \text{ strong }.$$

In addition, $J_t \to 1$ and by linearity and continuity of Π

$$\Pi y^t \to \Pi y \ \text{ in } \ L^2(\mathbf{R}^N) - \text{weak}.$$

Hence,

$$\forall \Phi \in L^2(\mathbf{R}^N), \ \int_{\mathbf{R}^N} Y_t \Phi \, dx \to \int_{\mathbf{R}^N} \Pi y \Phi \, dx = \int_{\mathbf{R}^N} Y_0 \Phi \, dx.$$

This proves the weak convergence. $\square$

To verify hypothesis $H3$, we transform (y_t, p_t) on Ω_t to $(y^t, p^t) = (y_t \circ T_t, p_t \circ T_t)$ on Ω. The pair (y^t, p^t) is the transported pair of solutions from Ω_t to Ω. It is the unique solution in $H^1(\Omega) \times H^1(\Omega)$ of the system

$$-\text{div}[A(t)\nabla y^t] = J_t \ f \circ T_t \ \text{ in } \ \Omega, \ \ y^t = g \circ T_t \ \text{ on } \ \Gamma, \tag{41}$$

$$-\text{div}[A(t)\nabla p^t] = J_t(y^t - y_d \circ T_t) \text{ in } \Omega, \quad p^t = 0 \text{ on } \Gamma, \tag{42}$$

where

$$A(t) = J_t(DT_t)^{-1*}(DT_t)^{-1}, \quad J_t = |\text{det}DT_t|, \tag{43}$$

and DT_t is the Jacobian matrix of T_t.

For sufficiently smooth domains Ω and vector fields V, the pair $\{y^t, p^t\}$ is bounded in $H^1(\Omega) \times H^1(\Omega)$ as t goes to zero . Since $H^1(\Omega)$ is a Hilbert space, we can extract weakly convergent subsequences to some $(\bar{y}, \bar{p})$ in $H^1(\Omega) \times H^1(\Omega)$. However, by linearity of the equation with respect to (y^t, p^t) and continuity of the coefficients with respect to t, the limit point $(\bar{y}, \bar{p})$ will coincide with (y^0, p^0), since the system has a unique solution at $t = 0$. Then we go back to the equation for y^t and y and show that the convergence is strong in $H^1(\Omega)$. Finally by using the regularity of the data and the classical regularity theorems we show that $(y^t, p^t) \to (y, p)$ in $H^3(\Omega) \times H^3(\Omega)$.

For the verification of hypothesis H4, we go back to expression (3.25) which can be rewritten as a volume integral

$$\partial_t G(\Omega_t, \Phi, \Psi) = \int_{\Omega_t} \text{div}\{[\frac{1}{2}(\Phi - y_d)^2 + (\Delta\Phi + f)\Psi + (\Phi - g)\Delta\Psi + \nabla(\Phi - g) \bullet \nabla\Psi]V\} \, dx \tag{44}$$

for $(\Phi, \Psi) \in H^3(\mathbf{R}^N) \times H^3(\mathbf{R}^N)$. Now introduce the map

$$\begin{cases} (\Phi, \Psi) \mapsto F(\Phi, \Psi) = [\frac{1}{2}(\Phi - y_d)^2 + (\Delta\Phi + f)\Psi + (\Phi - g)\Delta\Psi + \nabla(\Phi - g) \bullet \nabla\Psi]V \\ : H^3(\mathbf{R}^N) \times H^3(\mathbf{R}^N) \to (H^1(\mathbf{R}^N))^N. \end{cases}$$

It is bilinear and continuous. Finally the map

$$(t, F) \mapsto \int_{\Gamma_t} F \circ n_t \, d\Gamma \; = \; \int_{\Omega_t} F \, dx \; = \; \int_{\Omega} (divF) \circ T_t \, J_t^{-1} \, dx \quad : [0, \tau] \times H^1(\mathbf{R}^N) \to \mathbf{R} \tag{45}$$

is continuous. Then

$$(t, \Phi, \Psi) \mapsto \partial_t G(\Omega_t, \Phi, \Psi) \; = \; \int_{\Gamma_t} F(\Psi, \Psi) \bullet n_t \, d\Gamma_t \tag{46}$$

is continuous and hypothesis (H4) is verified. This completes the verification of the hypotheses.

5. SHAPE HESSIAN FOR THE DIRICHLET PROBLEM.

5.1. Formulation and formal computations. We proceed as in sections 3 and 4 and provide the mathematical justification in section 5.2. For the second derivative, we need two time invariant vector fields V and W on $\mathbf{R}^N$ and the expression of the first derivative $dJ(\Omega_t(W); V)$ where $\Omega_t(W)$ is the perturbation of the domain Ω by the vector field W:

$$dJ(\Omega_t(W); V) = \int_{\Omega_t(W)} \text{div}\{[\frac{1}{2}(g - y_d)^2 + \nabla(y_t - g) \bullet \nabla p_t]V\} \, dx, \tag{1}$$

where (y_t, p_t) are the unique solutions in $H^3(\Omega_t(W)) \times H^3(\Omega_t(W))$ to the equations

$$\Delta y_t + f = 0 \ \text{ in } \Omega_t(W), \quad y_t = g \ \text{ on } \ \Gamma_t(W), \tag{2}$$

$$\Delta p_t + (y_t - y_d) = 0 \ \text{ in } \Omega_t(W), \quad p_t = 0 \text{ on } \Gamma_t(W). \tag{3}$$

Then, we express (1) as a MinMax over a new Lagrangian:

$$dJ(\Omega_t(W); V) = \operatorname*{Min}_{\Phi,\Psi \in H^3(\mathbb{R}^N)} \ \operatorname*{Max}_{P,\Sigma \in H^2(\mathbb{R}^N)} G(\Omega_t, \Phi, \Psi, P, \Sigma), \tag{4}$$

where $G = G(\Omega_t, \Phi, \Psi, P, \Sigma)$ is given by

$$\begin{aligned} G = \int_{\Omega_t} & \{\operatorname{div}[\frac{1}{2}(g - y_d)^2 + \nabla(\Phi - g) \bullet \nabla\Psi]V\} \\ & + [\Delta\Phi + f]P + (\Phi - g)\Delta P + \nabla(\Phi - g) \bullet \nabla P \\ & + [\Delta\Psi + \Phi - y_d]\Sigma + \Psi\Delta\Sigma + \nabla\Psi \bullet \nabla\Sigma\} \ dx. \end{aligned} \tag{5}$$

This new Lagrangian is affine in (P, Σ), but is not necessarly convex in (Φ, Ψ). However, it is semiconvex in (Φ, Ψ) and we shall see that CORREA-SEEGER [1] will still apply to our special Lagrangian where the sets $X(t) \times Y(t)$,

$$X(t) \subset H^3(\mathbb{R}^N) \times H^3(\mathbb{R}^N) \tag{6}$$

$$Y(t) \subset H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N) \tag{7}$$

will be given by the usual "saddle point equations":

$$\int_{\Omega_t} [\Delta\hat{\Phi} + f]P + (\hat{\Phi} - g)\Delta P + \nabla(\hat{\Phi} - g) \bullet \nabla P \ dx = 0, \ \forall P, \tag{8}$$

$$\int_{\Omega_t} [\Delta\hat{\Psi} + \hat{\Phi} - y_d]\Sigma + \hat{\Psi}\Delta\Sigma + \nabla\hat{\Psi} \bullet \nabla\Sigma \ dx = 0, \ \ \forall\Sigma, \tag{9}$$

$$\int_{\Omega_t} \operatorname{div}\{[\nabla(\hat{\Phi} - g) \bullet \nabla\Psi]V\} + \Delta\Psi\hat{\Sigma} + \Psi\Delta\hat{\Sigma} + \nabla\Psi \bullet \nabla\hat{\Sigma} \ dx = 0, \ \ \forall\Psi, \tag{10}$$

$$\int_{\Omega_t} \operatorname{div}\{[\nabla\Phi \bullet \nabla\hat{\Psi}]V\} + \Delta\Phi\hat{P} + \Phi\Delta\hat{P} + \nabla\Phi \bullet \nabla\hat{P} + \Phi\hat{\Sigma} \ dx = 0, \forall\Phi. \tag{11}$$

It is obvious that (8) and (9) yield

$$\hat{\Phi}|_{\Omega_t} = y_t \text{ and } \ \hat{\Psi}|_{\Omega_t} = p_t. \tag{12}$$

Similarly, equations (10) and (11) have solutions $(\hat{\Sigma}, \hat{P})$ in $H^2(\mathbb{R}^N) \times H^2(\mathbb{R}^N)$ such that

$$Y_t' = \hat{\Sigma}|_{\Omega_t}, \quad P_t' = \hat{P}|_{\Omega_t} \tag{13}$$

are unique in $H^2(\Omega_t)\times H^2(\Omega_t)$ and solution of

$$\Delta Y_t' = 0 \text{ in } \Omega_t(W), \quad Y_t' = -\frac{\partial}{\partial n_t}(y_t - g)V \bullet n_t \quad \text{on } \Gamma_t(W), \tag{14}$$

$$\Delta P_t' = 0 \text{ in } \Omega_t(W), \quad P_t' = -\frac{\partial p_t}{\partial n_t}V \bullet n_t \quad \text{on } \Gamma_t(W). \tag{15}$$

It can be shown that Y_t' and P_t' coincide with the "partial derivative" with respect to t of appropriate extensions of y_t and p_t from $\Omega_t(W)$ to $\mathbb{R}^N$.

Finally, the partial derivative of the Lagrangian G with respect to t is given by

$$\begin{aligned}\partial_t G = \int_{\Gamma_t} &\{\operatorname{div}\{[\frac{1}{2}(g-y_d)^2 + \nabla(\Phi - g)\bullet\nabla\Psi]V\} \\ &+ [\Delta\Phi + f]P + (\Phi - g)\Delta P + \nabla(\Phi-g)\bullet\nabla P \\ &+ [\Delta\Psi + \Phi - y_d]\Sigma + \Psi\Delta\Sigma + \nabla\Psi\bullet\nabla\Sigma\}W\bullet n_t\, d\Gamma_t\end{aligned} \tag{16}$$

for Φ,Ψ,P,Σ in $H^3(\mathbb{R}^N)$, y_d and f in $H^2(\mathbb{R}^N)$ and g in $H^{\frac{7}{2}}(\mathbb{R}^N)$. The immediate consequence of this computation is that y_t, p_t, Y_t', P_t' all belong to $H^3(\Omega_t)$. But, Y_t', P_t' in $H^3(\Omega_t)$ require that y_t and p_t belong to $H^4(\Omega_t)$. This is precisely why we chose the appropriate smoothness of y_d, f and g.

Therefore, we must choose our saddle points $X(t)\times Y(t)$ in $(H^4(\mathbb{R}^N)\times H^4(\mathbb{R}^N))\times(H^3(\mathbb{R}^N)\times H^3(\mathbb{R}^N))$,

$$X(t) = \{(\Phi,\Psi)\in H^4(\mathbb{R}^N)\times H^4(\mathbb{R}^N) : \Phi|_{\Omega_t} = y_t, \Psi|_{\Omega_t} = p_t\} \tag{17}$$

$$Y(t) = \{(P,\Sigma)\in H^3(\mathbb{R}^N)\times H^3(\mathbb{R}^N) : P|_{\Omega_t} = P_t', \Sigma|_{\Omega_t} = Y_t'\}. \tag{18}$$

Finally, since $\partial_t G$ is a functional on Ω_t, it will only use the restriction to Ω_t of the various functions in $X(t)\times Y(t)$. Therefore, the Min and the Max can be removed and

$$\begin{aligned}d^2J(\Omega;V;W) = \int_\Gamma &\{\operatorname{div}[\frac{1}{2}(y-y_d)^2 + \frac{\partial}{\partial n}(y-g)\frac{\partial p}{\partial n}]V\} \\ &+ [\Delta y + f]P_V' + (y-g)\Delta P_V' + \nabla(y-g)\bullet\nabla P_V' \\ &+ [\Delta p + y - y_d]Y_V' + p\Delta Y_V' + \nabla p\bullet\nabla Y_V'\}W\bullet n\, d\Gamma.\end{aligned} \tag{19}$$

But,

$$\Delta y + f = 0,\ y = g,\ \Delta p + y - y_d = 0 \ \text{ and } \ p = 0 \text{ on } \Gamma, \tag{20}$$

and

$$d^2J(\Omega;V;W) = \int_\Gamma \{\operatorname{div}\ \{[\frac{1}{2}(g-y_d)^2 + \frac{\partial}{\partial n}(y-g)\frac{\partial p}{\partial n}]V\} + \frac{\partial}{\partial n}(y-g)\frac{\partial P_V'}{\partial n} + \frac{\partial p}{\partial n}\frac{\partial Y_V'}{\partial n}\}W\bullet n\, d\Gamma, \tag{21}$$

where we have added the subscript V to P' and Y' to emphasize the fact that they both depend on V.

The last step consists in the elimination of P'_V which will introduce Y'_W. To do that, we set $\psi = \hat{\Psi}_V|_{\Omega_t} = P'_V$ in equation (10) with $V = W$ and $t = 0$

$$\int_\Omega \text{div } \{[\nabla(y-g) \bullet \nabla\psi]W\} + \Delta\psi Y'_W + \psi\Delta Y'_W + \nabla\psi \bullet \nabla Y'_W \, dx = 0$$

$$\Rightarrow \int_\Omega \nabla(y-g) \bullet \nabla P'_V W \bullet n d\Gamma + \int_\Omega \Delta P'_V Y'_W + P'_V \Delta Y'_W + \nabla P'_V \bullet \nabla Y'_W \, dx = 0 \tag{22}$$

and $\phi = \hat{\Phi}_W|_{\Omega_t} = Y'_W$ in equation (11) with $t = 0$

$$\int_\Omega \text{div } \{[\nabla\Phi \bullet \nabla p]V\} + \Delta\Phi P'_V + \Phi\Delta P'_V + \nabla\Phi \bullet \nabla P'_V + \Phi Y'_V dx = 0$$

$$\Rightarrow \int_\Gamma \nabla Y'_W \bullet \nabla p \; V \bullet n \; d\Gamma + \int_\Gamma \Delta Y'_W P'_V + Y'_W \Delta P'_V + \nabla Y'_W \bullet \nabla P'_V + Y'_W Y'_V \; dx = 0 \tag{23}$$

This yields the following identity

$$\int_\Gamma \nabla(y-g) \bullet \nabla P'_V \; W \bullet n \; d\Gamma = \int_\Gamma \nabla Y'_W \bullet \nabla p \; V \bullet n \; d\Gamma + \int_\Gamma Y'_W Y'_V \; dx \tag{24}$$

or

$$\int_\Gamma \frac{\partial}{\partial n}(y-g)\frac{\partial P'_V}{\partial n} W \bullet n \; d\Gamma = \int_\Gamma \frac{\partial Y'_W}{\partial n}\frac{\partial p}{\partial n} V \bullet n \; d\Gamma + \int_\Gamma Y'_W Y'_V \; dx. \tag{25}$$

As a result

$$\begin{aligned} d^2 J(\Omega; V; W) = \int_\Gamma & \{\text{div } [\frac{1}{2}(g-y_d)^2 + \frac{\partial}{\partial n}(y-g)\frac{\partial p}{\partial n}]V\} W \bullet n \; d\Gamma \\ & + \int_\Gamma \frac{\partial p}{\partial n}\,[\frac{\partial Y'_W}{\partial n} V \bullet n + \frac{\partial Y'_V}{\partial n} W \bullet n] \; d\Gamma + \int_\Gamma Y'_W Y'_V \; dx \end{aligned} \tag{26}$$

where Y'_V is the unique solution of

$$\Delta Y'_V = 0 \;\text{ in }\; \Omega, \quad Y'_V = -\frac{\partial}{\partial n}(y-g)V \bullet n \;\text{ on }\; \Gamma. \tag{27}$$

5.2. Verification of the hypothese. In section 5.1, we have boldly applied the conclusion of the theorem of Correa and Seeger to a Lagrangian which contains a cost functional which is not necessarily convex. This means that the corresponding Lagrangian functional does not necessarily have saddle points. Yet, the conclusions of the theorem extend to semiconvex cost functionals (section 5.2.1). The verification of the hypotheses will essentially be the same as for the gradient in section 4.2 (section 5.5.2).

5.2.1. Semiconvex cost functionals. Consider a Lagrangian functional of the form

$$G(t,x,y) = F(t,x) + b(t,x,y) \tag{28}$$

for a family of continuous bilinear forms $b(t,x,y)$ on $X \times Y$ and continuous cost functionals $F(t,x)$ on X. Formally, the saddle points equations are given by

$$x^t \in X,\ b(t,x^t,y) = 0,\ \forall y \in Y \tag{29}$$

$$y^t \in Y,\ dF(t,x^t;x) + b(t,x,y^t) = 0,\ \forall x \in X. \tag{30}$$

When $G(t,x,y)$ is convex in x and concave in y, (29)-(30) characterize the saddle points $X(t) \times Y(t) \subset X \times Y$ of $G(t,\cdot,\cdot)$. So, when $F(t,x)$ is not convex in x, equations (29)-(30) need not characterize saddle points of $G(t,\cdot,\cdot)$.

We say that the functional $F(t,x)$ is semiconvex in x if there exists a family of continuous convex functionals $C(t,x)$ on X such that $F(t,x) + C(t,x)$ is convex in x. This means that $F(t,\cdot) + C(t,\cdot)$ and $C(t,\cdot)$ both have directional derivatives and hence $F(t,\cdot)$ also has a directional derivative: the following limit exists

$$dF(t,x;x') = \lim_{\theta \searrow 0} \frac{F(t,x+\theta x') - F(t,x)}{\theta} \tag{31}$$

Denote by $X(t)$ the set of all solutions

$$x^t \in X, \quad b(t,x^t,y) = 0, \quad \forall y \in Y \tag{32}$$

and assume that

$$\forall x^t \in X(t), \quad F(t,x^t) = J(t), \tag{33}$$

that is $F(t,x^t)$ is only a function of t. We use $J(t)$ as the definition of our cost function.

Now, assume that $F(t,\cdot)$ is semiconvex and that

$$\forall x^t \in X(t),\ C(t,x^t) = J_0(t), \tag{34}$$

that is $C(t,x^t)$ is only a function of t. We, again, use $J_0(t)$ as the definition of the cost function associated with C. Finally, let

$$J_C(t) = F(t,x^t) + C(t,x^t) \tag{35}$$

which is also only a function of t. Then, it is obvious that

$$J(t) = J_C(t) - J_0(t) \tag{36}$$

and that (if $dJ_C(0)$ and $dJ_0(0)$ exist)

$$dJ(0) = dJ_C(0) - dJ_0(0). \tag{37}$$

So, we are back to the use of CORREA-SEEGER [1] for both J_C and J_0. Construct the Lagrangians

$$G_C(t,x,y) = F(t,x) + C(t,x) + b(t,x,y) \tag{38}$$

$$G_0(t,x,y) = C(t,x) + b(t,x,y) \tag{39}$$

and assume that if all the hypotheses are verified both for G_0 and G_C

$$J_0(t) = \min_{x\in X} \max_{y\in Y} G_0(t,x,y) \quad dJ_0(0) = \min_{x\in X(0)} \max_{y\in Y_0(0)} \partial_t G_0(0,x,y) \tag{40}$$

$$J_C(t) = \min_{x\in X} \max_{y\in Y} G_C(t,x,y) \quad dJ_C(0) = \min_{x\in X(0)} \max_{y\in Y_C(0)} \partial_t G_C(0,x,y), \tag{41}$$

where the saddle point equations for J_0 are given by

$$x^t \in X,\ B(t,x^t,y) = 0,\ \forall y, \tag{42}$$

$$y_0^t \in Y,\ dC(t,x^t;x) + b(0,x,y_0^t) = 0,\ \forall x, \tag{43}$$

and the saddle point equations for $J_C(t)$

$$x^t \in X,\ B(t,x^t,y) = 0,\ \forall y, \tag{44}$$

$$y_C^t \in Y,\ dC(t,x^t;x) + dF(t,x^t;x) + b(0,x,y_C^t) = 0,\ \forall x. \tag{45}$$

Assume that

$$dJ_0(0) = \partial_t C(0,x^0) + \partial_t b(0,x^0,y_0^0),\ \forall x^0 \in X(0),\ \forall y_0^0 \in Y_0(0), \tag{46}$$

and that $dC(0,x^0;x)$ and $dF(0,x^0;x)$ are independent of the point x^0 chosen in $X(0)$. By substracting (43) from (45), construct the new variable $y^0 = y_C^0 - y_0^0$, which is a solution of

$$y^0 \in Y,\ dF(0,x^0;x) + b(0,x,y^0) = 0,\ \forall x, \tag{47}$$

and the set

$$Y(0) = \{y_C^0 - y_0^0 : y_C^0 \in Y_C(0),\ y_0^0 \in Y_0(0)\}.$$

Now,

$$\begin{aligned} dJ_C(0) &= \operatorname*{Min}_{x\in X(0)} \operatorname*{Max}_{y\in Y_C(0)} \partial_t F(0,x) + \partial_t C(0,x) + \partial_t b(0,x,y) \\ &= \operatorname*{Min}_{x\in X(0)} \operatorname*{Max}_{\substack{y_0^0\in B_0(0)\\ y_0\in Y(0)}} \{\partial_t F(0,x) + \partial_t b(0,x,y^0) + \partial_t C(0,x) + \partial_t b(0,x,y_0^0)\} \end{aligned}$$

But $\forall\ (x^0,y_0^0) \in X(0)\times Y_0(0)$

$$\partial_t C(0,x^0) + \partial_t b(0,x^0,y_0^0) = dJ_0(0),$$

and finally,

$$dJ(0) = dJ_C(0) - dJ_0(0) = \operatorname*{Min}_{x\in X(0)} \operatorname*{Max}_{y\in Y(0)} [\partial_t F(0,x) + \partial_t b(0,x,y)]. \tag{48}$$

where the saddle point $(x^0, y^0) \in X(0) \times Y(0)$ are the solution of the "formal saddle point equations" (29)-(30) for $t = 0$.

In section 4.2, the cost functional is semiconvex since there exists a constant $C > 0$ large enough such that

$$dJ(\Omega_t(W); V) + C[||y_t||^2_{H^4(\Omega_t)} + ||p_t||^2_{H^4(\Omega_t)}] \tag{49}$$

is convex and continuous on $H^4(\Omega_t) \times H^4(\Omega_t)$. The functional

$$C(t, \phi, \psi) = c||\phi||^2_{H^4(\Omega_t)} + ||\psi||^2_{H^4(\Omega_t)} \tag{50}$$

is clearly convex and continuous on $H^4(\Omega_t) \times H^4(\Omega_t)$. This provides a complete justification to the use of the conclusions of Correa and Seeger.

5.2.2. Verification of the hypotheses. We have chosen to work in $H^4(\mathbb{R}^N) \times H^4(\mathbb{R}^N) \times H^3(\mathbb{R}^N) \times H^3(\mathbb{R}^N)$ and introduced appropriate hypotheses on f, y_d and g in section 5.1. From this point on the technique is the same as the one in section 4.2 for the gradient. Therefore, we shall not repeat it here.

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