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AN INCREMENTAL FORMULATION FOR THE LINEAR ANALYSIS OF THIN VISCOELASTIC STRUCTURES USING GENERALIZED VARIABLES

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SUMMARY

This paper is concerned with the development of an incremental formulation in the time domain for the displacement and stress analysis of quasistatic, linear, thin viscoelastic structures undergoing mechanical deformation. By representing the viscoelastic behaviour of a material by a discrete creep spectrum and by an incremental constitutive equations, the difficulty of retaining the stress history in computer solutions is avoided. A complete general formulation of linear viscoelastic stress analysis is developed in terms of increments of midsurface strains and curvatures and corresponding stress resultants.

KEY WORDS: linear viscoelasticity; discrete creep spectrum; incremental constitutive law; strain history; damping analysis; thin structures

INTRODUCTION

Stress and strain analysis of viscoelastic phenomena which can be observed in the behaviour of many real materials,[1,2] presents many difficulties for real problems of complex geometry. Analytic solutions to the equations of viscoelasticity are often obtained by an application of the correspondence principle.[3–6] This approach makes use of the observation that the equations of viscoelasticity can be converted to the equations of elasticity by means of Laplace transformation. Thus, if an explicit solution to the associated equations of elasticity can be found, then the solution of those of viscoelasticity can be obtained by inversion of the Laplace transform. This method is restricted to the narrow class of problems for which it is possible to find an explicit solution to the associated equations of elasticity. In order to obtain solutions to more complicated problems it is necessary to develop numerical rather than analytic techniques.[7–11]

Several formulations have already been proposed in the literature, using either differential models,[12,13] or integral constitutive equations, [14–16] or intermediate models.[11] Most numerical solutions cannot deal with complex problems, because these methods require the retaining of the complete past history of stress and strain in the memory of a digital computer. To overcome these difficulties, it will be shown in this paper how the finite element method of elastic analysis can be extended to deal with complex viscoelastic problems.

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An incremental formulation for the linear analysis of thin viscoelastic structures is developed. In contrast with several formulations proposed in the past,[13,17] the present approach is not restricted to isotropic response. In order to reduce computer storage requirement and reduce the computational effort, we formulate the constitutive equations in terms of middle surface strains and curvatures and their conjugated stress resultants. We begin with a discussion of a mechanical constitutive equations for thin structures undergoing small deformations and subjected to mechanical stresses. It is assumed that the viscoelastic behaviour of a material can be represented by a discrete creep tensor (a finite series of Kelvin elements coupled with an elastic and viscous response). Using this assumption of wide validity, the incremental constitutive equations can be performed analytically. Thus the difficulty of computer storage requirements is avoided and the complete past history of stresses is represented by some auxiliary tensors.

DISCRETE CREEP SPECTRUM

For the sake of simplicity, we will discuss only the case of small strains and displacements. According to the results of Mandel,[18] the creep tensor $\mathbf{G}(t)$ is written in terms of a discrete creep spectrum

$$\mathbf{G}(t) = \left[\mathbf{G}^{(0)} + \sum_{i=1}^{N} (1 - \mathrm{e}^{-t\cdot\lambda_i})\mathbf{G}^{(i)} + t\mathbf{G}^{(\infty)}\right] Y(t) \tag{1}$$

where $\mathbf{G}^{(0)}$, $\mathbf{G}^{(\infty)}$, and $\mathbf{G}^{(i)}$, $i = 1, \ldots, N$, are tensors of the fourth rank defined on $\mathbb{R}^3$ to be determined in order to represent any particular creep function of interest, $Y(t)$ is the Heaviside unit step function, and λ_i are positive scalars.

Subsequently the components of the creep tensor $\mathbf{G}(t)$ can be represented in terms of an exponential series,[18,19] e.g.

$$G_{\alpha\beta\gamma\delta}(t) = \left[G^{(0)}_{\alpha\beta\gamma\delta} + \sum_{i=1}^{N} (1 - \mathrm{e}^{-t\lambda^{(i)}_{\alpha\beta\gamma\delta}})G^{(i)}_{\alpha\beta\gamma\delta} + tG^{(\infty)}_{\alpha\beta\gamma\delta}\right] Y(t) \tag{2}$$

where $\lambda^{(i)}_{\alpha\beta\gamma\delta}$, $i = 1, \ldots, N$, are strictly positives and repeated indices do not imply summation convention.

The constitutive equations between the components $\sigma_{\alpha\beta}(t)$ of the stress tensor and the components of the strain tensor $e_{\alpha\beta}(t)$ for a linear viscoelastic material can be written in terms of creep functions:

$$e_{\alpha\beta}(t) = \sum_{\gamma}\sum_{\delta}\int_{-\infty}^{t} G_{\alpha\beta\gamma\delta}(t-\tau)\frac{\partial\sigma_{\gamma\delta}(\tau)}{\partial\tau}\mathrm{d}\tau \tag{3}$$

According to Love's first-order shell theory, the strain at any point of the shell is given by

$$e_{\alpha\beta}(\xi_3, t) = \varepsilon_{\alpha\beta}(t) + \xi_3\chi_{\alpha\beta}(t) \quad (\alpha, \beta = 1, 2) \tag{4}$$

where $\varepsilon_{\alpha\beta}(t)$ and $\chi_{\alpha\beta}(t)$ are the middle surface extensional strain and curvature, respectively.

For a state of plane stress the non-vanishing stress resultants are defined by

$$N_{\alpha\beta}(t) = \int_{-h/2}^{+h/2} \sigma_{\alpha\beta}(\xi_3, t)\,\mathrm{d}\xi_3, \quad M_{\alpha\beta}(t) = \int_{-h/2}^{+h/2} \sigma_{\alpha\beta}(\xi_3, t)\xi_3\,\mathrm{d}\xi_3 \tag{5}$$

$N_{\alpha\beta}$ and $M_{\alpha\beta}$ are here the generalized stresses (stress resultants) and h is the structure thickness assumed to be constant. Due to the thin shell assumption, the radii of curvature for the middle surface do not enter into (5).

Introducing the middle surface extensional strain and curvature into the constitutive equations, we get

$$\begin{Bmatrix} \varepsilon_{\alpha\beta}(t) \\ \chi_{\alpha\beta}(t) \end{Bmatrix} = \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} \int_{-\infty}^{t} G_{\alpha\beta\gamma\delta}(t-\tau) \frac{\partial}{\partial \tau} \begin{Bmatrix} a_1 N_{\gamma\delta}(\tau) \\ a_2 M_{\gamma\delta}(\tau) \end{Bmatrix} \mathrm{d}\tau \tag{6}$$

where $a_1 = 1/h$ and $a_2 = 12/h^3$.

Note that in accordance with the thin shell assumption, the constitutive equations are assumed to be independent of the transverse shear forces Q_α.

Let us consider the generalized Kelvin model shown in Figure 1 in which $K^{(0)}_{\alpha\beta\gamma\delta}$, $K^{(i)}_{\alpha\beta\gamma\delta}$ designate modulus of elasticity where $\eta^{(i)}_{\alpha\beta\gamma\delta}$, $\eta^{(\infty)}_{\alpha\beta\gamma\delta}$ are coefficients of viscosity and N is the number of Kelvin–Voigt elements used to approximate the material's behaviour. When the stress $\sigma_{\gamma\delta}(t)$ is applied on this mechanical model, the total pseudostrain obtained is noted $\bar{e}_{\alpha\beta\gamma\delta}(t)$ and can be written in the form

$$\bar{e}_{\alpha\beta\gamma\delta}(t) = \bar{\varepsilon}_{\alpha\beta\gamma\delta}(t) + \xi_3 \bar{\chi}_{\alpha\beta\gamma\delta}(t) \tag{7}$$

where $\bar{\varepsilon}_{\alpha\beta\gamma\delta}(t)$ and $\bar{\chi}_{\alpha\beta\gamma\delta}(t)$ are fourth-order tensors defined on $\mathbb{R}^3$ and are related to generalized stresses through the relation

$$\begin{Bmatrix} \bar{\varepsilon}_{\alpha\beta\gamma\delta}(t) \\ \bar{\chi}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} = \int_{-\infty}^{t} G_{\alpha\beta\gamma\delta}(t-\tau) \frac{\partial}{\partial \tau} \begin{Bmatrix} a_1 N_{\gamma\delta}(\tau) \\ a_2 M_{\gamma\delta}(\tau) \end{Bmatrix} \mathrm{d}\tau \tag{8}$$

$\bar{\varepsilon}_{\alpha\beta\gamma\delta}$, $\bar{\chi}_{\alpha\beta\gamma\delta}$ can be interpreted as the contribution of the complete history of the component of generalized stress tensor $\{N_{\gamma\delta}(\tau), M_{\gamma\delta}(\tau)$ with $\tau \leqslant t\}$ on the component of the generalized strain tensor $\{\varepsilon_{\alpha\beta}(t), \chi_{\alpha\beta}(t)\}$.

It is seen from equations (6) and (8) that the generalized strains $\varepsilon_{\alpha\beta}(t)$ and $\chi_{\alpha\beta}(t)$ are given by

$$\forall t \in \mathbb{R}, \quad \begin{Bmatrix} \varepsilon_{\alpha\beta}(t) \\ \chi_{\alpha\beta}(t) \end{Bmatrix} = \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} \begin{Bmatrix} \bar{\varepsilon}_{\alpha\beta\gamma\delta}(t) \\ \bar{\chi}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} \tag{9}$$

In order to determine the components of the creep tensors $\mathbf{G}^{(0)}$, $\mathbf{G}^{(\infty)}$ and $\mathbf{G}^{(i)}$, $i = 1, \ldots, N$, we consider the Laplace transform creep function $f^*(p)$ of $f(t)$

$$f^*(p) = \left[\frac{1}{K^{(0)}_{\alpha\beta\gamma\delta}} + \frac{1}{p\eta^{(\infty)}_{\alpha\beta\gamma\delta}} + \sum_{i=1}^{N} \frac{1}{K^{(i)}_{\alpha\beta\gamma\delta} + p\eta^{(i)}_{\alpha\beta\gamma\delta}} \right] Y(p) \tag{10}$$

where p is the transform parameter. The inverse transform of $f^*(p)$ is given by

$$f(t) = \left[\frac{1}{K^{(0)}_{\alpha\beta\gamma\delta}} + \frac{t}{\eta^{(\infty)}_{\alpha\beta\gamma\delta}} + \sum_{i=1}^{N} \frac{1}{K^{(i)}_{\alpha\beta\gamma\delta}} (1 - \mathrm{e}^{-tK^{(i)}_{\alpha\beta\gamma\delta}/\eta^{(i)}_{\alpha\beta\gamma\delta}}) \right] Y(t) \tag{11}$$

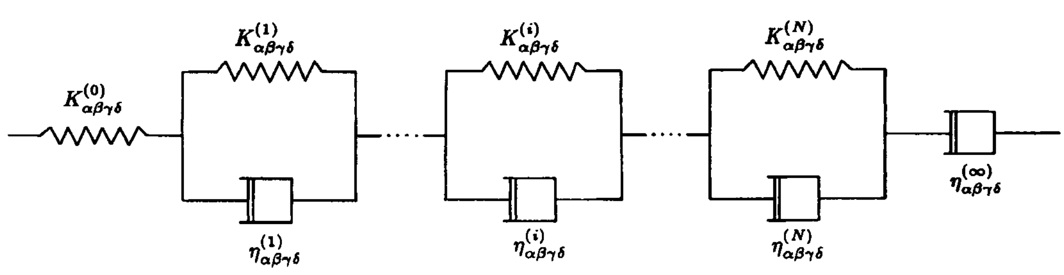

Figure 1. Generalized Kelvin model

The comparison between (2) and (11) gives

$$K^{(0)}_{\alpha\beta\gamma\delta} = 1/G^{(0)}_{\alpha\beta\gamma\delta}, \quad K^{(i)}_{\alpha\beta\gamma\delta} = 1/G^{(i)}_{\alpha\beta\gamma\delta}$$
$$\eta^{(\infty)}_{\alpha\beta\gamma\delta} = 1/G^{(\infty)}_{\alpha\beta\gamma\delta}, \quad \eta^{(i)}_{\alpha\beta\gamma\delta} = 1/(\lambda^{(i)}_{\alpha\beta\gamma\delta} G^{(i)}_{\alpha\beta\gamma\delta}) \tag{12}$$

It is apparent from (8) and (12), that the constitutive equations (8) can be represented by the mechanical model of Figure 1.

FORMULATION OF GOVERNING EQUATIONS

When the stress $\sigma_{\gamma\delta}(t)$ is applied on the mechanical model shown in Figure 1, the rate of the total pseudo-strain $\partial \bar{e}_{\alpha\beta\gamma\delta}(t)/\partial t$ is determined by

$$\frac{\partial \bar{e}_{\alpha\beta\gamma\delta}(t)}{\partial t} = \frac{\partial \bar{e}^{(0)}_{\alpha\beta\gamma\delta}(t)}{\partial t} + \sum_{i=1}^{N} \frac{\partial \bar{e}^{(i)}_{\alpha\beta\gamma\delta}(t)}{\partial t} + \frac{\partial \bar{e}^{(\infty)}_{\alpha\beta\gamma\delta}(t)}{\partial t} \tag{13}$$

where $\bar{e}^{(0)}_{\alpha\beta\gamma\delta}(t)$ is the total pseudo-strain in the spring, $\bar{e}^{(\infty)}_{\alpha\beta\gamma\delta}(t)$ the total pseudo-strain in the dashpot and $\bar{e}^{(i)}_{\alpha\beta\gamma\delta}(t)$ the total pseudo-strain in the ith Kelvin element.

These pseudo-strains are written in terms of generalized pseudo-strains as

$$\frac{\partial \bar{e}^{(0)}_{\alpha\beta\gamma\delta}}{\partial t} = \frac{\partial \bar{\varepsilon}^{(0)}_{\alpha\beta\gamma\delta}(t)}{\partial t} + \xi_3 \frac{\partial \bar{\chi}^{(0)}_{\alpha\beta\gamma\delta}(t)}{\partial t}$$
$$\frac{\partial \bar{e}^{(\infty)}_{\alpha\beta\gamma\delta}}{\partial t} = \frac{\partial \bar{\varepsilon}^{(\infty)}_{\alpha\beta\gamma\delta}(t)}{\partial t} + \xi_3 \frac{\partial \bar{\chi}^{(\infty)}_{\alpha\beta\gamma\delta}(t)}{\partial t} \tag{14}$$
$$\frac{\partial \bar{e}^{(i)}_{\alpha\beta\gamma\delta}}{\partial t} = \frac{\partial \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t)}{\partial t} + \xi_3 \frac{\partial \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t)}{\partial t}$$

Introducing equation (14) into (13), we obtain the first differential equation which govern the mechanical model:

$$\frac{\partial}{\partial t}\begin{Bmatrix} \bar{\varepsilon}_{\alpha\beta\gamma\delta}(t) \\ \bar{\chi}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} = \frac{1}{K^{(0)}_{\alpha\beta\gamma\delta}} \frac{\partial}{\partial t}\begin{Bmatrix} a_1 N_{\gamma\delta}(t) \\ a_2 M_{\gamma\delta}(t) \end{Bmatrix} + \sum_{i=1}^{N} \frac{\partial}{\partial t}\begin{Bmatrix} \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t) \\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} + \frac{1}{\eta^{(\infty)}_{\alpha\beta\gamma\delta}}\begin{Bmatrix} a_1 N_{\gamma\delta}(t) \\ a_2 M_{\gamma\delta}(t) \end{Bmatrix} \tag{15}$$

The second differential equation is determined by

$$\sigma_{\gamma\delta}(t) = \sigma^{(k)}_{\gamma\delta}(t) + \sigma^{(\eta)}_{\gamma\delta}(t) \tag{16}$$

where $\sigma^{(k)}_{\gamma\delta}(t)$ is the stress in the ith spring and $\sigma^{(\eta)}_{\gamma\delta}(t)$ the stress in the ith dashpot, which can be written in terms of generalized stresses as

$$\begin{Bmatrix} a_1 N_{\gamma\delta}(t) \\ a_2 M_{\gamma\delta}(t) \end{Bmatrix} = K^{(i)}_{\alpha\beta\gamma\delta}\begin{Bmatrix} \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t) \\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} + \eta^{(i)}_{\alpha\beta\gamma\delta} \frac{\partial}{\partial t}\begin{Bmatrix} \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t) \\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} \tag{17}$$

where $\bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t)$ and $\bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t)$ are fourth-order tensors to be determined in order to represent the middle surface strain and curvature. These two tensors correspond to the ith Kelvin element in the mechanical model, and are related to generalized pseudo-strains through the relation:

$$\begin{Bmatrix} \varepsilon^{(i)}_{\alpha\beta}(t) \\ \chi^{(i)}_{\alpha\beta}(t) \end{Bmatrix} = \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} \begin{Bmatrix} \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t) \\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t) \end{Bmatrix} \tag{18}$$

Using equations (9) and (18), we find that equation (15) can be rewritten as

$$\frac{\partial}{\partial t}\begin{Bmatrix}\varepsilon_{\alpha\beta}(t)\\ \chi_{\alpha\beta}(t)\end{Bmatrix}=\sum_{\gamma=1}^{2}\sum_{\delta=1}^{2}\left(\frac{1}{K^{(0)}_{\alpha\beta\gamma\delta}}\frac{\partial}{\partial t}\begin{Bmatrix}a_1N_{\gamma\delta}(t)\\ a_2M_{\gamma\delta}(t)\end{Bmatrix}+\sum_{i=1}^{N}\frac{\partial}{\partial t}\begin{vmatrix}\bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t)\\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t)\end{vmatrix}+\frac{1}{\eta^{(\infty)}_{\alpha\beta\gamma\delta}}\begin{Bmatrix}a_1N_{\gamma\delta}(t)\\ a_2M_{\gamma\delta}(t)\end{Bmatrix}\right) \tag{19}$$

If we substitute equations (12) and (18) into equation (19), we obtain the second differential equation

$$\frac{\partial}{\partial t}\begin{Bmatrix}\varepsilon_{\alpha\beta}(t)\\ \chi_{\alpha\beta}(t)\end{Bmatrix}=\sum_{\gamma=1}^{2}\sum_{\delta=1}^{2}G^{(0)}_{\alpha\beta\gamma\delta}\frac{\partial}{\partial t}\begin{Bmatrix}a_1N_{\gamma\delta}(t)\\ a_2M_{\gamma\delta}(t)\end{Bmatrix}+\sum_{i=1}^{N}\frac{\partial}{\partial t}\begin{vmatrix}\varepsilon^{(i)}_{\alpha\beta}(t)\\ \chi^{(i)}_{\alpha\beta}(t)\end{vmatrix}+\sum_{\gamma=1}^{2}\sum_{\delta=1}^{2}G^{(\infty)}_{\alpha\beta\gamma\delta}\begin{Bmatrix}a_1N_{\gamma\delta}(t)\\ a_2M_{\gamma\delta}(t)\end{Bmatrix}\Bigg) \tag{20}$$

FINITE DIFFERENCE INTEGRATION

The solution process of a step-by-step nature can now be described for a general case in which loads are applied stepwise at various time intervals. Consider the time step $\Delta t_n = t_n - t_{n-1}$ (the subscript $n-1$ and n refer to the values at the beginning and end of the time step, respectively). Equation (20) can now be written in the form

$$\begin{Bmatrix}\Delta(\varepsilon_{\alpha\beta})_n\\ \Delta(\chi_{\alpha\beta})_n\end{Bmatrix}=\sum_{\gamma=1}^{2}\sum_{\delta=1}^{2}G^{(0)}_{\alpha\beta\gamma\delta}\begin{Bmatrix}a_1\Delta(N_{\gamma\delta})_n\\ a_2\Delta(M_{\gamma\delta})_n\end{Bmatrix}+\sum_{i=1}^{N}\begin{vmatrix}\Delta(\varepsilon^{(i)}_{\alpha\beta})_n\\ \Delta(\chi^{(i)}_{\alpha\beta})_n\end{vmatrix}+\sum_{\gamma=1}^{2}\sum_{\delta=1}^{2}G^{(\infty)}_{\alpha\beta\gamma\delta}\int_{t_{n-1}}^{t_n}\begin{Bmatrix}a_1N_{\gamma\delta}(\tau)\\ a_2M_{\gamma\delta}(\tau)\end{Bmatrix}\mathrm{d}\tau \tag{21}$$

We have assumed that the time derivative during each time increment is constant and is expressed by

$$\frac{\partial u_{\alpha\beta}}{\partial t}=\frac{u_{\alpha\beta}(t_n)-u_{\alpha\beta}(t_{n-1})}{\Delta t_n}=\frac{\Delta(u_{\alpha\beta})_n}{\Delta t_n}$$

$u_{\alpha\beta}$ represent generalized strains or stresses. A linear approximation is used for generalized stresses[19] and is expressed by

$$\begin{Bmatrix}N_{\gamma\delta}(\tau)\\ M_{\gamma\delta}(\tau)\end{Bmatrix}=\begin{Bmatrix}N_{\gamma\delta}(t_{n-1})\\ M_{\gamma\delta}(t_{n-1})\end{Bmatrix}+\frac{\tau-t_{n-1}}{\Delta t_n}\begin{Bmatrix}\Delta(N_{\gamma\delta})_n\\ \Delta(M_{\gamma\delta})_n\end{Bmatrix} \tag{22}$$

Substituting into equation (21) the linear approximation of generalized stress increments given by (22), we obtain

$$\begin{aligned}\begin{Bmatrix}\Delta(\varepsilon_{\alpha\beta})_n\\ \Delta(\chi_{\alpha\beta})_n\end{Bmatrix}&=\sum_{\gamma=1}^{2}\sum_{\delta=1}^{2}G^{(0)}_{\alpha\beta\gamma\delta}\begin{Bmatrix}a_1\Delta(N_{\gamma\delta})_n\\ a_2\Delta(M_{\gamma\delta})_n\end{Bmatrix}+\sum_{i=1}^{N}\begin{vmatrix}\Delta(\varepsilon^{(i)}_{\alpha\beta})_n\\ \Delta(\chi^{(i)}_{\alpha\beta})_n\end{vmatrix}\\ &\quad+\sum_{\gamma=1}^{2}\sum_{\delta=1}^{2}G^{(\infty)}_{\alpha\beta\gamma\delta}\left[\Delta t_n\begin{Bmatrix}a_1N_{\gamma\delta}(t_{n-1})\\ a_2M_{\gamma\delta}(t_{n-1})\end{Bmatrix}+\frac{1}{2}\Delta t_n\begin{Bmatrix}a_1\Delta(N_{\gamma\delta})_n\\ a_2\Delta(M_{\gamma\delta})_n\end{Bmatrix}\right]\end{aligned} \tag{23}$$

In order to determine the generalized strain increments $\Delta(\varepsilon_{\alpha\beta})_n$ and $\Delta(\chi_{\alpha\beta})_n$ from equation (23), we have to determine the generalized strain increments $\Delta(\varepsilon^{(i)}_{\alpha\beta})_n$ and $\Delta(\chi^{(i)}_{\alpha\beta})_n$ which correspond to the ith element in the mechanical model. For this reason, consider equation (17) which can be written as

$$\frac{1}{\lambda^{(i)}_{\alpha\beta\gamma\delta}}\frac{\partial}{\partial t}\begin{vmatrix}\bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t)\\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t)\end{vmatrix}+\begin{vmatrix}\bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t)\\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t)\end{vmatrix}=G^{(i)}_{\alpha\beta\gamma\delta}\begin{Bmatrix}a_1N_{\gamma\delta}(t)\\ a_2M_{\gamma\delta}(t)\end{Bmatrix} \tag{24}$$

The solution of this equation can be written as

$$\begin{Bmatrix} \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t_n) \\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t_n) \end{Bmatrix} - \begin{Bmatrix} \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t_{n-1}) \\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t_{n-1}) \end{Bmatrix} = [\mathrm{e}^{-\lambda^{(i)}_{\alpha\beta\gamma\delta}\Delta t_n} - 1] \begin{Bmatrix} \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t_{n-1}) \\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t_{n-1}) \end{Bmatrix}$$

$$+ \lambda^{(i)}_{\alpha\beta\gamma\delta} G^{(i)}_{\alpha\beta\gamma\delta} \mathrm{e}^{-\lambda^{(i)}_{\alpha\beta\gamma\delta} t_n} \int_{t_{n-1}}^{t_n} \mathrm{e}^{\lambda^{(i)}_{\alpha\beta\gamma\delta}\tau} \begin{Bmatrix} a_1 N_{\gamma\delta}(\tau) \\ a_2 M_{\gamma\delta}(\tau) \end{Bmatrix} \mathrm{d}\tau \tag{25}$$

The integral in equation (25) is called the spectrum integral which can be interpreted as the influence of the generalized stress on the creep spectrum; it is defined by

$$I_s = \int_{t_{n-1}}^{t_n} \mathrm{e}^{\lambda^{(i)}_{\alpha\beta\gamma\delta}\tau} \begin{Bmatrix} a_1 N_{\gamma\delta}(\tau) \\ a_2 M_{\gamma\delta}(\tau) \end{Bmatrix} \mathrm{d}\tau \tag{26}$$

If we integrate by parts using the linear approximation of generalized stresses given in equation (22), the spectrum integral I_s is then given by

$$I_s = \frac{1}{\lambda^{(i)}_{\alpha\beta\gamma\delta}} \mathrm{e}^{\lambda^{(i)}_{\alpha\beta\gamma\delta} t_n} \left[(1 - \mathrm{e}^{-\lambda^{(i)}_{\alpha\beta\gamma\delta}\Delta t_n}) \begin{Bmatrix} a_1 N_{\gamma\delta}(t_{n-1}) \\ a_2 M_{\gamma\delta}(t_{n-1}) \end{Bmatrix} + \left[1 - \frac{1}{\Delta t_n \lambda^{(i)}_{\alpha\beta\gamma\delta}} (1 - \mathrm{e}^{-\lambda^{(i)}_{\alpha\beta\gamma\delta}\Delta t_n}) \right] \begin{Bmatrix} a_1 \Delta(N_{\gamma\delta})_n \\ a_2 \Delta(M_{\gamma\delta})_n \end{Bmatrix} \right] \tag{27}$$

Consequently, when we substitute equation (27) into equation (25), we obtain the generalized strain increments corresponding to the ith Kelvin element in the rheological model:

$$\begin{Bmatrix} \Delta(\varepsilon^{(i)}_{\alpha\beta})_n \\ \Delta(\chi^{(i)}_{\alpha\beta})_n \end{Bmatrix} = \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} [\mathrm{e}^{-\lambda^{(i)}_{\alpha\beta\gamma\delta}\Delta t_n} - 1] \begin{Bmatrix} \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t_{n-1}) \\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t_{n-1}) \end{Bmatrix}$$

$$+ \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} G^{(i)}_{\alpha\beta\gamma\delta} (1 - \mathrm{e}^{-\lambda^{(i)}_{\alpha\beta\gamma\delta}\Delta t_n}) \begin{Bmatrix} a_1 N_{\gamma\delta}(t_{n-1}) \\ a_2 M_{\gamma\delta}(t_{n-1}) \end{Bmatrix}$$

$$+ \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} G^{(i)}_{\alpha\beta\gamma\delta} \left[1 - \frac{1}{\Delta t_n \lambda^{(i)}_{\alpha\beta\gamma\delta}} (1 - \mathrm{e}^{-\lambda^{(i)}_{\alpha\beta\gamma\delta}\Delta t_n}) \right] \begin{Bmatrix} a_1 \Delta(N_{\gamma\delta})_n \\ a_2 \Delta(M_{\gamma\delta})_n \end{Bmatrix} \tag{28}$$

INCREMENTAL CONSTITUTIVE EQUATIONS

The incremental constitutive equations in generalized variables can now be obtained from equation (21). Substituting equation (28) into (21) and integrating the last term of equation (21) using the linear approximation of generalized stress given in (22), we find

$$\begin{Bmatrix} \Delta(\varepsilon_{\alpha\beta})_n \\ \Delta(\chi_{\alpha\beta})_n \end{Bmatrix} = \begin{Bmatrix} (\tilde{\varepsilon}_{\alpha\beta})_{n-1} \\ (\tilde{\chi}_{\alpha\beta})_{n-1} \end{Bmatrix} + \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} \mathscr{M}_{\alpha\beta\gamma\delta} \begin{Bmatrix} a_1 \Delta(N_{\gamma\delta})_n \\ a_2 \Delta(M_{\gamma\delta})_n \end{Bmatrix} \tag{29}$$

where $\mathscr{M}_{\alpha\beta\gamma\delta}$ is a fourth-order tensor which can be interpreted as the compliance tensor, it is given by

$$\mathscr{M}_{\alpha\beta\gamma\delta} = G^{(0)}_{\alpha\beta\gamma\delta} + \sum_{i=1}^{N} G^{(i)}_{\alpha\beta\gamma\delta} \left[1 - \frac{1}{\Delta t_n \lambda^{(i)}_{\alpha\beta\gamma\delta}} (1 - \mathrm{e}^{-\lambda^{(i)}_{\alpha\beta\gamma\delta}\Delta t_n}) \right] + \frac{1}{2} G^{(\infty)}_{\alpha\beta\gamma\delta} \cdot \Delta t_n$$

$(\tilde{\varepsilon}_{\alpha\beta})_{n-1}$ and $(\tilde{\chi}_{\alpha\beta})_{n-1}$ represent the influence of the complete past history of generalized stresses. They are given by

$$\begin{Bmatrix} (\tilde{\varepsilon}_{\alpha\beta})_{n-1} \\ (\tilde{\chi}_{\alpha\beta})_{n-1} \end{Bmatrix} = \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} \left[\Delta t_n G^{(\infty)}_{\alpha\beta\gamma\delta} + \sum_{i=1}^{N} G^{(i)}_{\alpha\beta\gamma\delta}\left(1 - e^{-\lambda^{(i)}_{\alpha\beta\gamma\delta}\Delta t_n}\right) \right] \begin{Bmatrix} a_1 N_{\gamma\delta}(t_{n-1}) \\ a_2 M_{\gamma\delta}(t_{n-1}) \end{Bmatrix}$$
$$+ \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} \sum_{i=1}^{N} \left[e^{-\lambda^{(i)}_{\alpha\beta\gamma\delta}\Delta t_n} - 1 \right] \begin{Bmatrix} \bar{\varepsilon}^{(i)}_{\alpha\beta\gamma\delta}(t_{n-1}) \\ \bar{\chi}^{(i)}_{\alpha\beta\gamma\delta}(t_{n-1}) \end{Bmatrix}$$

The incremental constitutive law given in equation (29) can now be inverted to obtain

$$\begin{Bmatrix} a_1 \Delta(N_{\alpha\beta})_n \\ a_2 \Delta(M_{\alpha\beta})_n \end{Bmatrix} = - \begin{Bmatrix} (\tilde{N}_{\alpha\beta})_{n-1} \\ (\tilde{M}_{\alpha\beta})_{n-1} \end{Bmatrix} + \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} \mathscr{D}_{\alpha\beta\gamma\delta} \begin{Bmatrix} \Delta(\varepsilon_{\gamma\delta})_n \\ \Delta(\chi_{\gamma\delta})_n \end{Bmatrix} \tag{30}$$

where $\mathscr{D}_{\alpha\beta\gamma\delta} = (\mathscr{M}_{\alpha\beta\gamma\delta})^{-1}$ is the inverse of the compliance tensor.

$$\begin{Bmatrix} (\tilde{N}_{\alpha\beta})_{n-1} \\ (\tilde{M}_{\alpha\beta})_{n-1} \end{Bmatrix} = \sum_{\gamma=1}^{2} \sum_{\delta=1}^{2} \mathscr{D}_{\alpha\beta\gamma\delta} \begin{Bmatrix} (\tilde{\varepsilon}_{\gamma\delta})_{n-1} \\ (\tilde{\chi}_{\gamma\delta})_{n-1} \end{Bmatrix} \tag{31}$$

$(\tilde{N}_{\alpha\beta})_{n-1}$ and $(\tilde{M}_{\alpha\beta})_{n-1}$ represent the influence of the complete past history of strains on the components of the generalized stress increments.

Equation (30), which represent the incremental constitutive law in generalized variables, will now be introduced in a finite element discretization in order to obtain the response of thin structures.

VARIATIONAL FORMULATION IN VISCOELASTICITY

In the context of a finite element model the governing equations of the discretized system are derived from the principle of virtual displacements.[20-23] Assuming a linear quasistatic viscoelastic structure, the total internal virtual work is equal to the total external virtual work; i.e. we have in generalized variables:

$$\int_{{}^eA} \langle \delta_t \Delta(\varepsilon_{\alpha\beta})_n, \delta_t \Delta(\chi_{\alpha\beta})_n \rangle \begin{Bmatrix} \Delta(N_{\alpha\beta})_n \\ \Delta(M_{\alpha\beta})_n \end{Bmatrix} \mathrm{d}^e A = \int_{{}^eV} \Delta(f_i^{\mathrm{v}})_n \delta_t \Delta(u_i)_n \,\mathrm{d}V \tag{32}$$

where $\Delta(u_i)_n$ is the increment displacement field between t_{n-1} and t_n, $\Delta(f_i^{\mathrm{v}})_n$ is the incremental body forces per unit volume, δ_t is the variation symbol, ${}^eA$ and ${}^eV$ are the area and the volume of the element. In (32) the term corresponding to surface traction is omitted for the sake of simplicity.

The generalized strains are derived from shape functions in a standard manner[22,23] assuming small displacements within the elements.

$$\begin{Bmatrix} \Delta(\varepsilon_{\alpha\beta})_n \\ \Delta(\chi_{\alpha\beta})_n \end{Bmatrix} = [B]\{\Delta(q^e)_n\} \tag{33}$$

where $\Delta(q^e)_n$ is the element displacement increment and $[B]$ is the strain–displacement transformation matrix.

Introducing the constitutive law (30) into the equilibrium equations (32) and using the finite element approximation (33), the equilibrium equations can be rewritten as

$$[K_{\mathrm{T}}]_n\{\Delta(q^e)_n\} = \{\Delta F\}_n + \{\Delta F^{\mathrm{vis}}\}_{n-1} \tag{34}$$

Table I. Global incremental procedure

1. Given a state at time t_n: $\{q\}_n$, $\{\varepsilon_{\alpha\beta}\}_n$, $\{\chi_{\alpha\beta}\}_n$, $\{N_{\alpha\beta}\}_n$, $\{M_{\alpha\beta}\}_n$, $\{\tilde{\varepsilon}_{\alpha\beta}\}_n$, $\{\tilde{\chi}_{\alpha\beta}\}_n$ compute the compliance moduli $\mathscr{M}_{\alpha\beta\gamma\delta}$ and the tangent moduli $\mathscr{D}_{\alpha\beta\gamma\delta}$
2. Compute the pseudo-stresses $\{\tilde{N}_{\alpha\beta}\}_n$ and $\{\tilde{M}_{\alpha\beta}\}_n$ from equation (31)
3. Compute the viscous load vector increment $\{\Delta F^{\text{vis}}\}_n$ from equation (36)
4. Update the viscoelastic stiffness matrix $[K_{\text{T}}]_n$
5. Assemble and solve the equilibrium equations (34) to obtain the displacement increment $\{\Delta q\}_n$
6. Compute the strain increment $\{\Delta\varepsilon_{\alpha\beta}\}_n$ and $\{\Delta\chi_{\alpha\beta}\}_n$ from equation (33)
7. Use the result of Step 2 to compute the stress increments: $\Delta\{N_{\alpha\beta}\}_n$ and $\Delta\{M_{\alpha\beta}\}_n$ from equation (30)
8. Compute the pseudo-strains $\{\tilde{\varepsilon}_{\alpha\beta}\}_{n+1}$ and $\{\tilde{\chi}_{\alpha\beta}\}_{n+1}$
9. Update the state:

$$\{q\}_{n+1} = \{q\}_n + \{\Delta q\}_n$$
$$\{N_{\alpha\beta}\}_{n+1} = \{N_{\alpha\beta}\}_n + \Delta\{N_{\alpha\beta}\}_n$$
$$\{M_{\alpha\beta}\}_{n+1} = \{M_{\alpha\beta}\}_n + \Delta\{M_{\alpha\beta}\}_n$$
$$\{\varepsilon_{\alpha\beta}\}_{n+1} = \{\varepsilon_{\alpha\beta}\}_n + \Delta\{\varepsilon_{\alpha\beta}\}_n$$
$$\{\chi_{\alpha\beta}\}_{n+1} = \{\chi_{\alpha\beta}\}_n + \Delta\{\chi_{\alpha\beta}\}_n$$

10. Go to Step 1.

where

$$[K_{\text{T}}]_n = \int_{^eA} [B]^{\text{T}}[C_n][B]\,\mathrm{d}^eA \tag{35}$$

$$\{\Delta F^{\text{vis}}\}_{n-1} = \int_{^eA} [B]^{\text{T}} \begin{Bmatrix} (1/a_1)(\tilde{N}_{\alpha\beta})_{n-1} \\ (1/a_2)(\tilde{M}_{\alpha\beta})_{n-1} \end{Bmatrix} \mathrm{d}^eA \tag{36}$$

$\{\Delta F\}_n = \{F\}_n - \{F\}_{n-1}$ is the external load vector increment, $\{\Delta F^{\text{vis}}\}_{n-1}$ is the viscous load vector increment corresponding to the complete past history of strains and $[C_n]$ is the generalized constitutive matrix which can be determined numerically by the inverse of the compliance tensor (see Table I).

An efficient code for the linear analysis of thin viscoelastic structures (VISCO) is developed. The software can be effectively employed for plane viscoelasticity problems, axially symmetric structures, thin plates and shells.

Plane viscoelasticity problems are idealized by a series of triangular elements CST (Constant Strain Triangle),[23] axially symmetric structures are discretized by conical frustum-shaped elements,[21] while plates and shells are idealized by the DKT elements (Discrete Kirchhoff Triangle).[24]

ILLUSTRATIVE EXAMPLES

The numerical method developed in the previous sections will be illustrated by several examples. Newton, millimeter and hour are the units used for the force, length and time, respectively. One hour time is divided into ten time steps for the analysis of the problems.

Plane stress plate

The first example is a plane stress plate. The geometrical and loading details are given in Figure 2. The structure is divided into 160 plane stress finite elements.

The applied stress $\sigma_x(t)$ is taken to be

$$\sigma_x(t) = \sigma_0 Y(t), \quad t \leqslant 2$$

$$\sigma_x(t) = 0, \quad t > 2$$

where $\sigma_0 = 250$ MPa. The creep function for the material is represented by equation (11) with $N = 2$ and the constants of the creep function are assumed as in Table II.

It is a constant stress problem. Strains vary with time (creep) but are constants with geometry. The exact solution for the problem is

$$u(x, t) = [1{\cdot}41417 + 0{\cdot}04 \times t - 0{\cdot}9249 \times e^{-9{\cdot}55t} - 0{\cdot}4693 \times e^{-38{\cdot}60t}] \cdot \sigma_0, \quad t \leqslant 2$$

The results obtained using the present method with the exact solution are given in Table III. It is seen from Table III, that the present method gives very accurate results. In Figure 3 numerical results for the axial displacement are plotted which shows very good agreement with the exact solution.

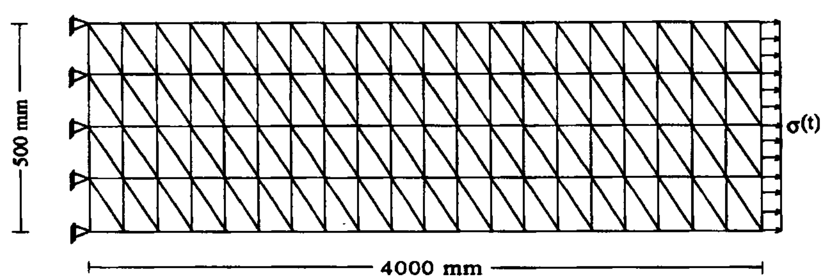

Figure 2. Plane stress finite element

Table II. Constants used in creep function

K_0	K_1	K_2	η_1	η_2	η^∞
2×10^5	4325	8523	452·67	220·82	10^5

Table III. Strains in plane stress problem

t (h)	ε_{xx} Exact	ε_{xx} FEM
0·0	0·0012500	0·0012500
0·4	0·088121	0·088105
0·8	0·090358	0·090358
1·2	0·091385	0·091385
1·6	0·092386	0·092386
2·0	0·093386	0·093386
2·5	—	0·0059399
3·0	—	0·0051319

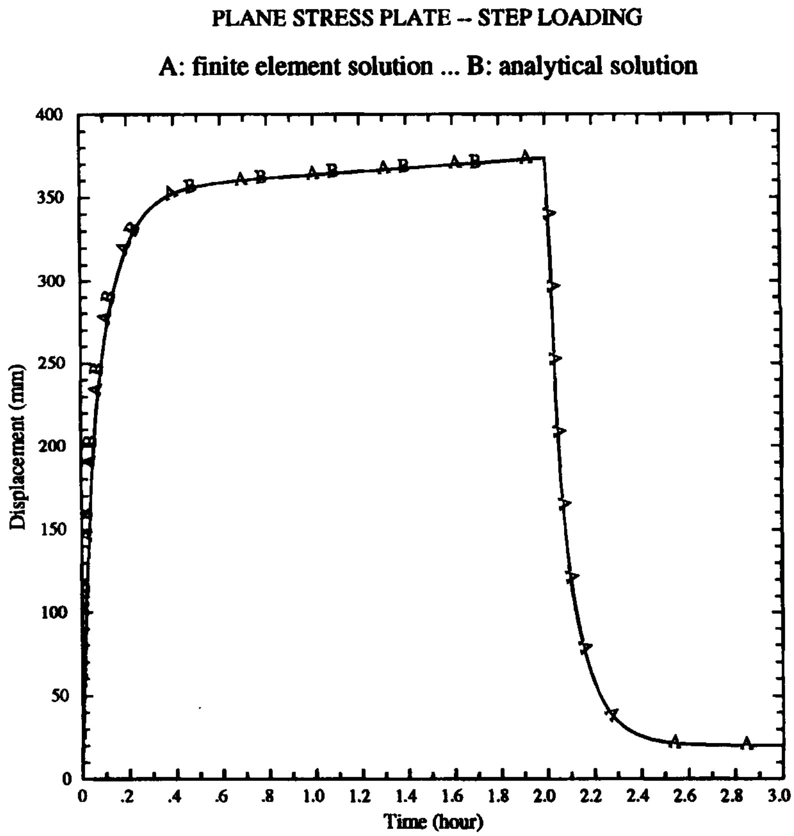

Figure 3. Horizontal displacement versus time in a plane stress plate

Steady state harmonic oscillation

The second example will be illustrated by the problem shown schematically in Figure 4. AA' is a perfectly flexible footing, subject to a harmonic function of time, acting on a layer of viscoelastic material which rests on a rough rigid base CC'. The plate contains a finite crack of length a, the surface of the crack is free of all tractions.

The applied stress $\sigma_y(t)$ is a harmonic function of time and is taken to be

$$\sigma_y(t) = \sigma_0(1 + \cos\omega t)\,Y(t), \quad t \leqslant 5$$

$$\sigma_y(t) = 0, \quad 10 > t > 5$$

$$\sigma_y(t) = \sigma_0(1 + \cos\omega t)\,Y(t), \quad t \geqslant 10$$

where $\sigma_0 = 0{\cdot}35$ MPa and the frequency of the oscillation $\omega = 2\pi$ rad/h.

The problem was analysed using the network of constant strain triangles shown in Figure 4.

The viscoelastic response of the material can be represented by the mechanical model shown in Figure 1 with $N = 1$, where the constants of the creep function shown in Figure 5 are assumed as in Table IV.

We are interested in the steady-state response. We assume that the forcing function $\sigma_y(t)$ has been acting on the structure for an indefinitely long time and that all initial transient disturbances

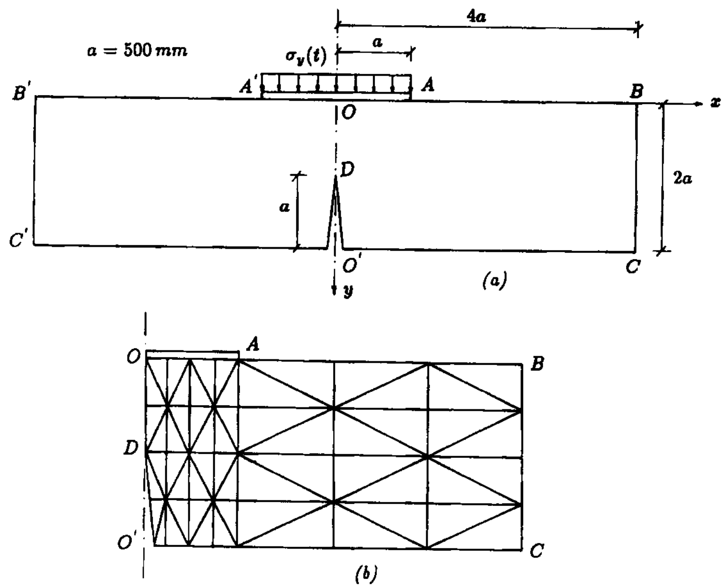

Figure 4. Steady-state harmonic oscillation problem

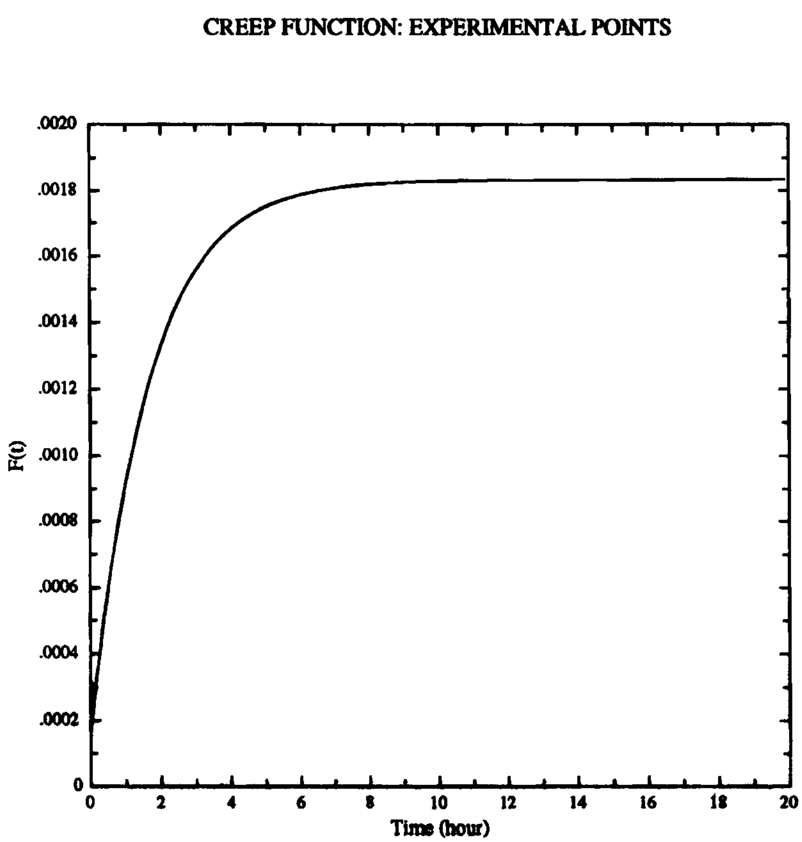

Figure 5. Creep function of the material

have died out. The purpose of the dynamic analysis in this case is, therefore, to investigate the viscoelastic response as a sequence of static perturbations. In such a case we assume that the vibration amplitude is sufficiently small.

The results of the numerical process are shown in Figures 6–8. In Figure 6, numerical results for the vertical stress σ_y near the crack tip and at the central point of loading $x = y = 0$ are plotted.

In Figure 7, the horizontal displacement at the centre of the crack is plotted for various times, where Figure 8 shows the vertical displacement at the centre of the footing.

It may be seen that, after an initial settlement, creep takes place but ceases as times becomes large. This behaviour may be explained in terms of the rheological model shown in Figure 1 where an applied loading would cause an initial compression of the single spring followed by viscous flow in the dashpot until load is taken by the two springs.

Table IV. Constants used in creep function

K_0	K_1	η_1
6000	600	1000

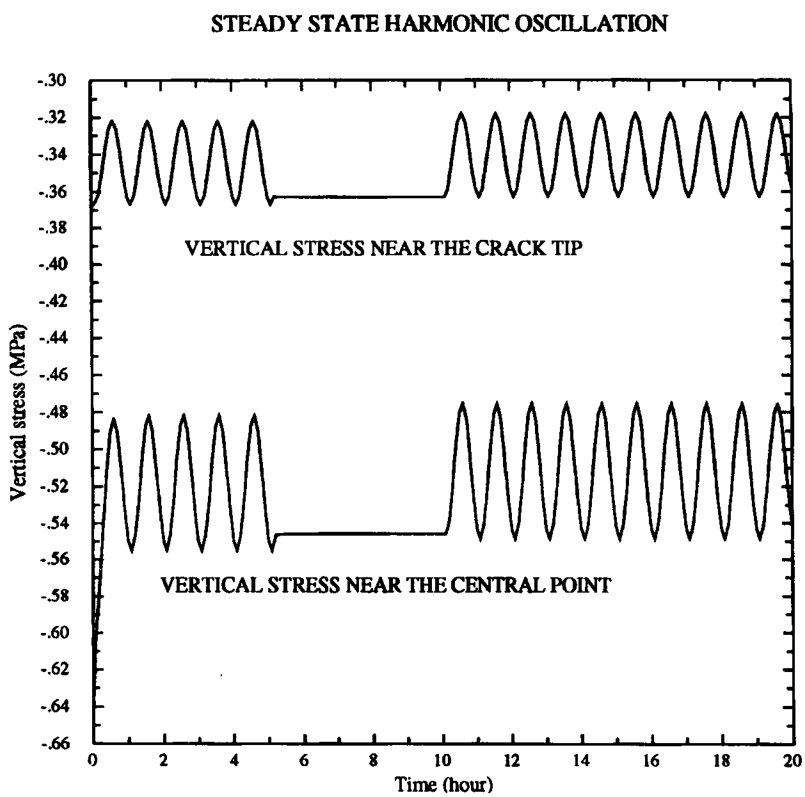

Figure 6. Vertical stress near the crack tip and at the central point

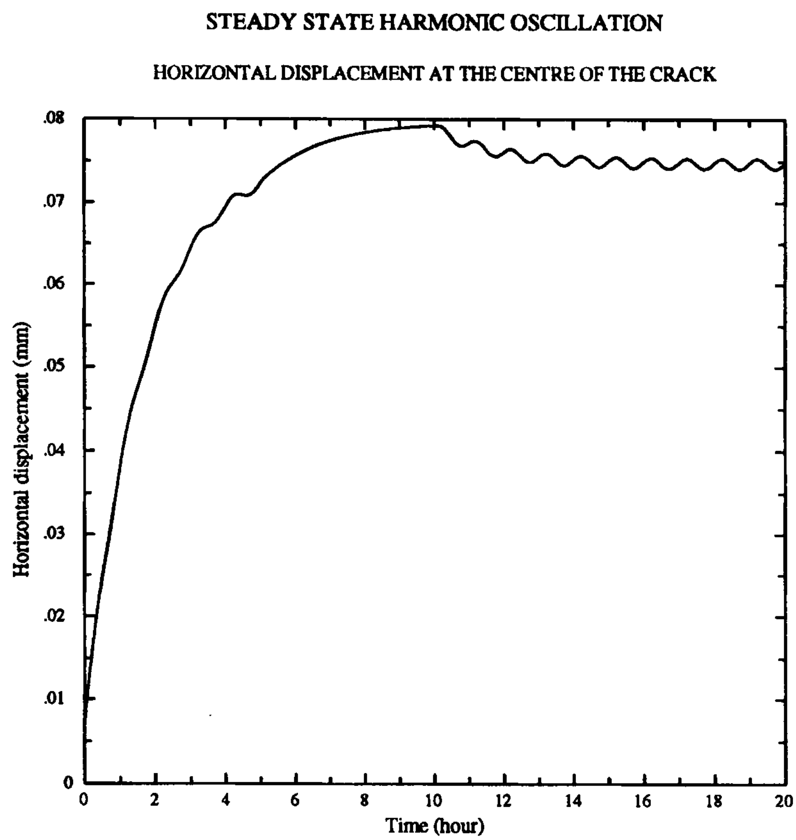

Figure 7. Horizontal displacement at the centre of the crack

Circular cylindrical shell

The cylindrical shell is fixed at one end and loaded at the free end. The loading case considered is that of unit radial end loading at the free end while the other end is built on.

Figure 9 shows the cylinder and the manner in which it has been idealized. It should be noted that the mesh is graded so that there are more elements near the loaded point since it is in this region that the stresses and deflections change most rapidly.

The material properties used are assumed as in Table V.

Figure 10 shows how the radial deflection varies with axial position for various times. The stresses and strains near free end are given in Table VI for various values of t.

It is observed from the results that the stresses do not change with time, whereas the strains keep on building up leading to the strain failure.

The variation in the meridional moment along the tube is plotted in Figure 11 for the 24-element idealization. It is found that there is no stress variation with time. The meridional moment is compared with the theoretical solution given in Reference 25 which shows very good agreement with the theoretical solution.

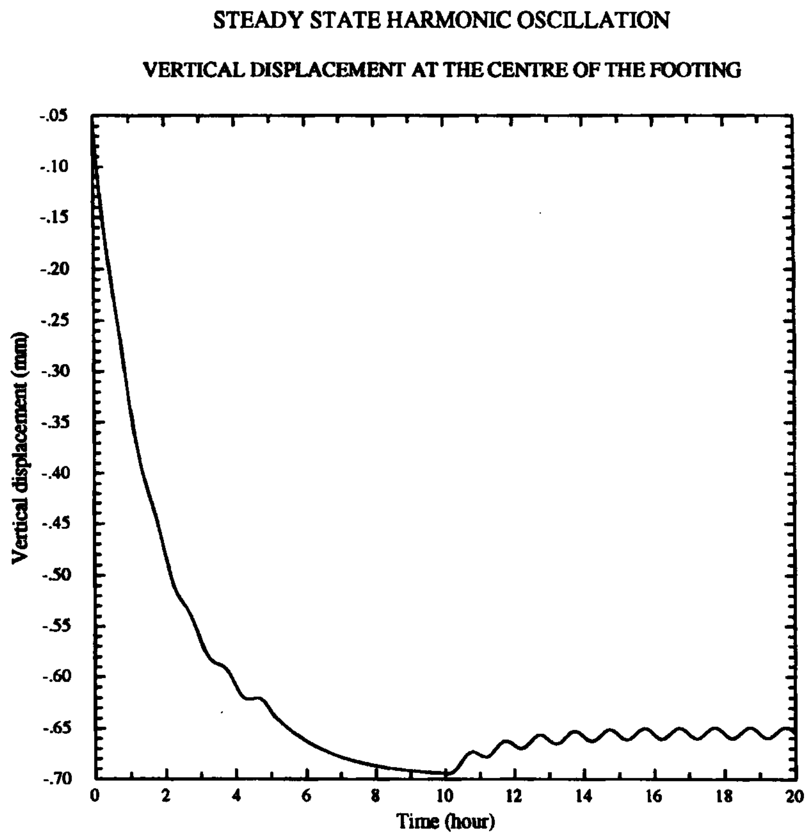

Figure 8. Vertical displacement at the central point of loading

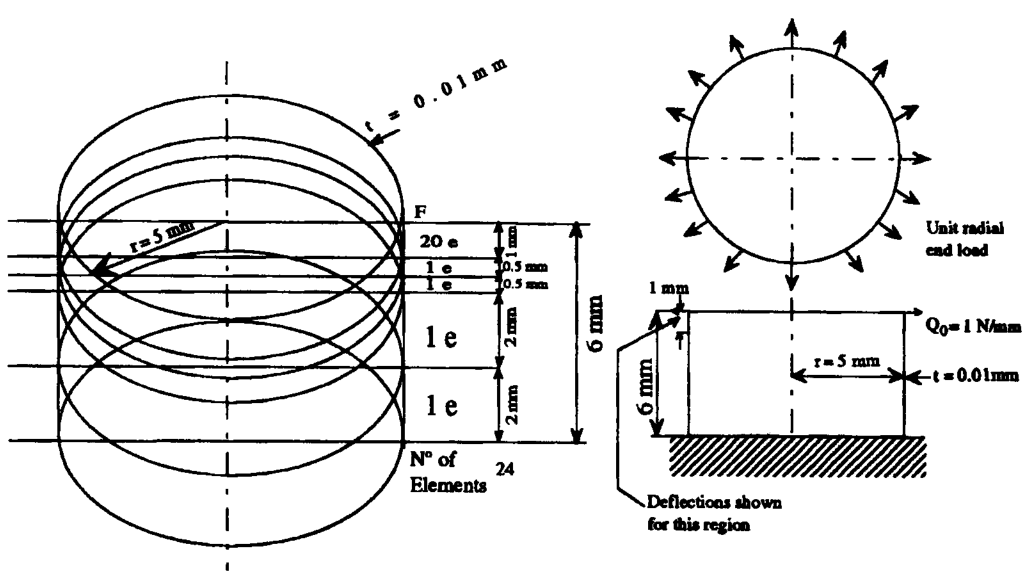

Figure 9. Idealization of cylinder using axisymmetric shell elements

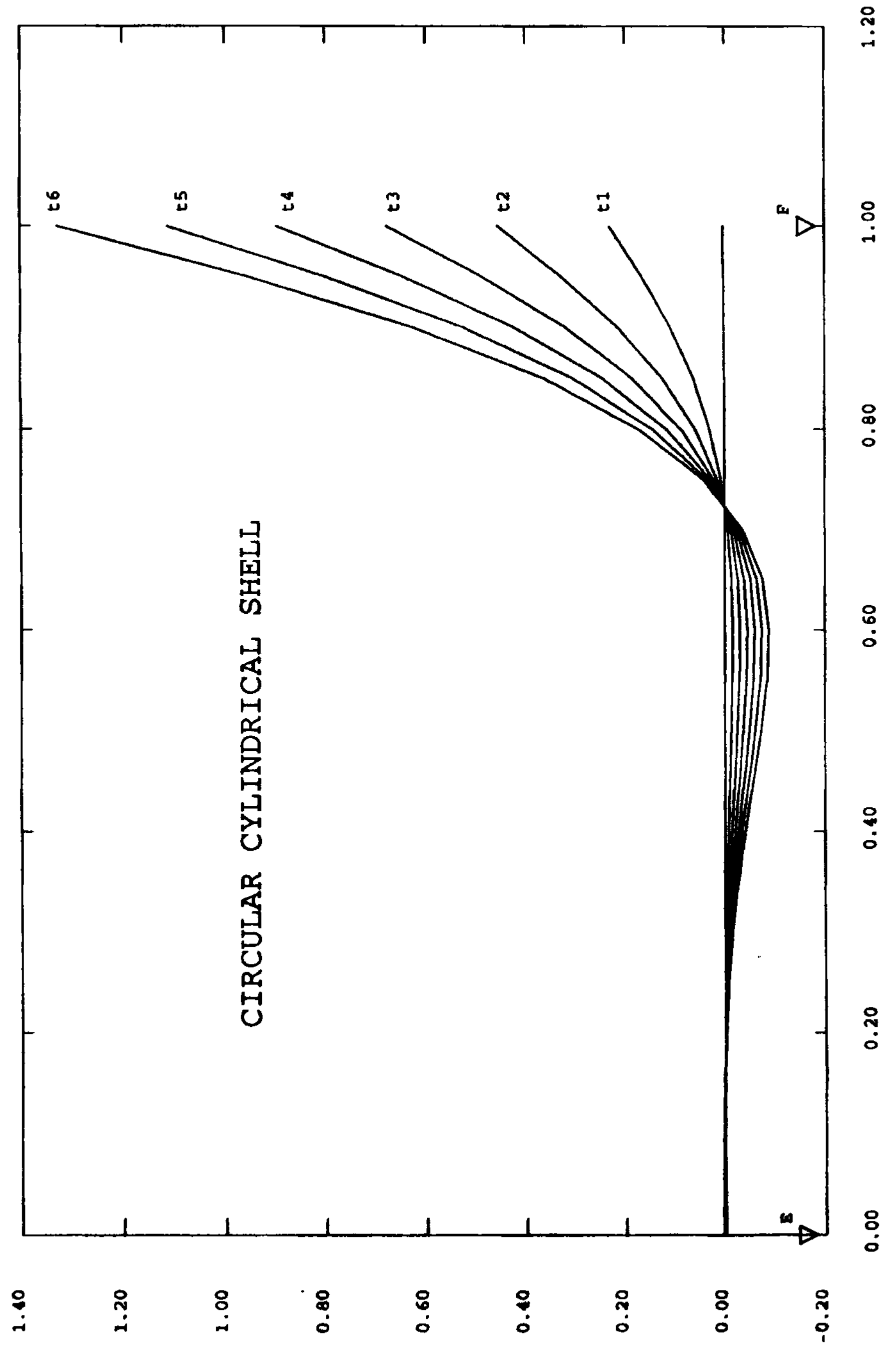

Figure 10. Variation in radial deflection in tube having radial end load

Table V. Constants used in creep function

K_0	K_1	K_2	η_1	η_2	η^{∞}
1×10^7	1157	1589	2×10^6	$2{\cdot}54 \times 10^4$	0

Table VI. Stresses and displacement near free end

t (h)	U_r (mm)	N_{11} (N/mm^2)	N_{22} (N/mm^2)	M_{11} (N/mm^2)	ε_{11} (%)	ε_{22} (%)
0·2	0·2304	2·66	5·81	$2{\cdot}12 \times 10^3$	− 1·186	4·602
0·4	0·4522	2·66	5·81	$2{\cdot}12 \times 10^3$	− 2·344	9·090
0·6	0·6772	2·66	5·81	$2{\cdot}12 \times 10^3$	− 3·487	13·525
0·8	0·8964	2·66	5·81	$2{\cdot}12 \times 10^3$	− 4·615	17·905
1·0	1·1130	2·66	5·81	$2{\cdot}12 \times 10^3$	− 5·730	22·230
1·2	1·3269	2·66	5·81	$2{\cdot}12 \times 10^3$	− 6·830	26·503

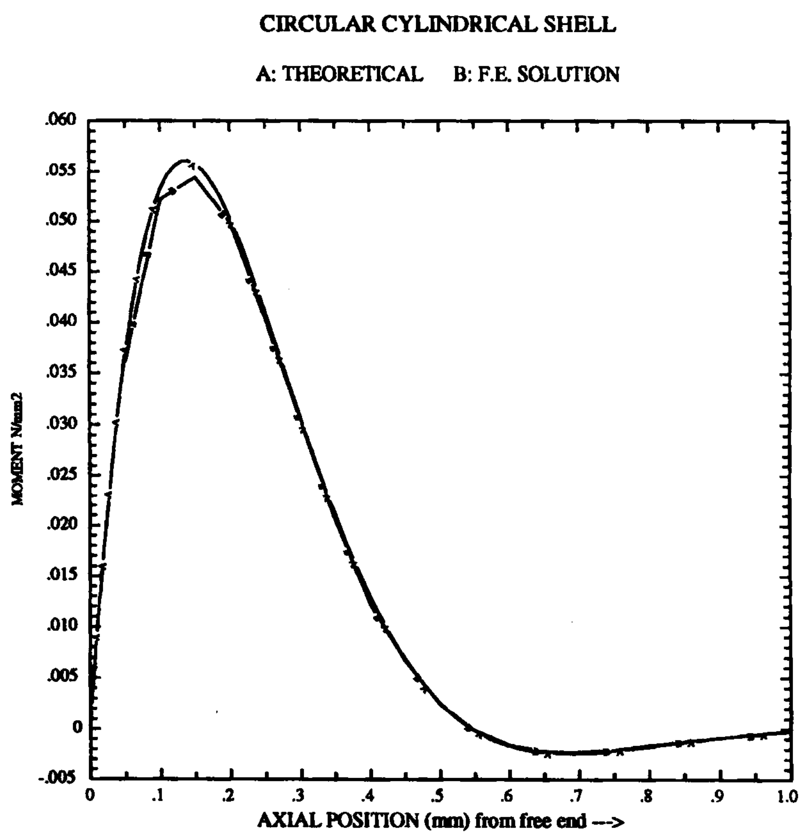

Figure 11. Variation in meridional moment in tube having radial end load

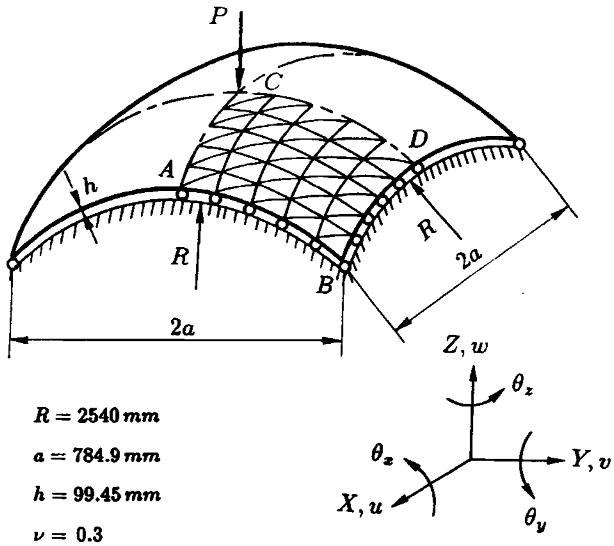

Figure 12. Spherical shell subjected to apex load $R = 2540$ mm, $a = 784,9$ mm, $h = 99,45$ mm, $\nu = 0,3$

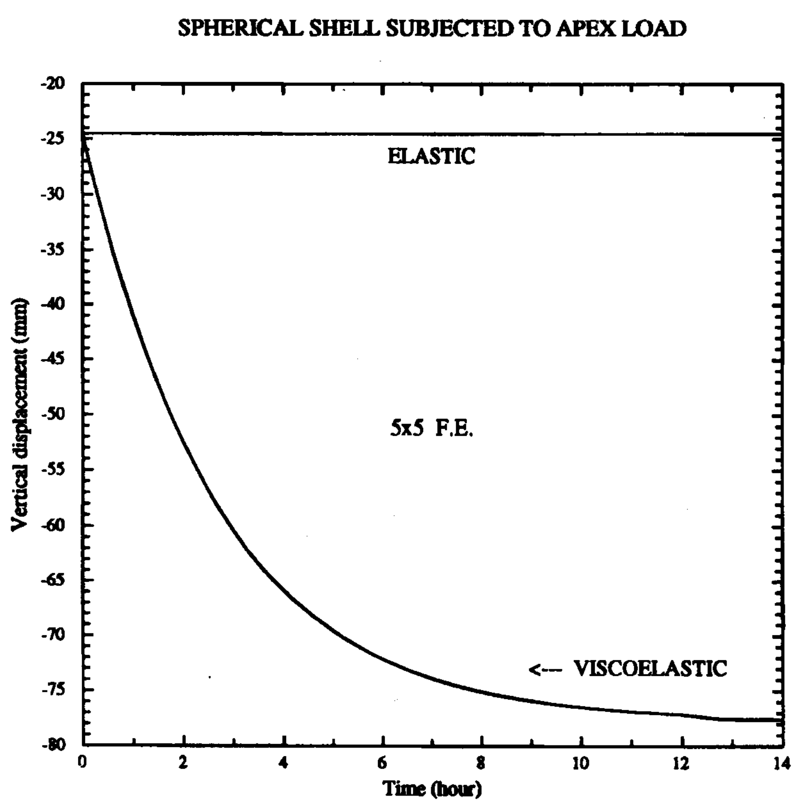

Figure 13. History of the displacement at the centre of the shell

Spherical shell

The shell described in Figure 12, which has been studied in the elastic range by a number of investigators,[26–28] was analysed for its viscoelastic displacement response using a 5×5 mesh. All edges are hinged and immovable. The shell is first submitted to an apex load: $P(t) = 44$ kN. For symmetry reasons, only one quarter of this shell is considered. The symmetry conditions then impose a zero displacement u and rotation θ_y along the line CD, a zero displacement v and rotation θ_x along the line AC.

The material properties used are assumed as in Table VII.

We present in Figure 13 a time history of the vertical displacement w observed at the centre of the shell. It may be seen that creep takes place but ceases as time becomes large.

Table VII. Constants used in creep function

K_0	K_1	η_1
68·95	31·72	84·57

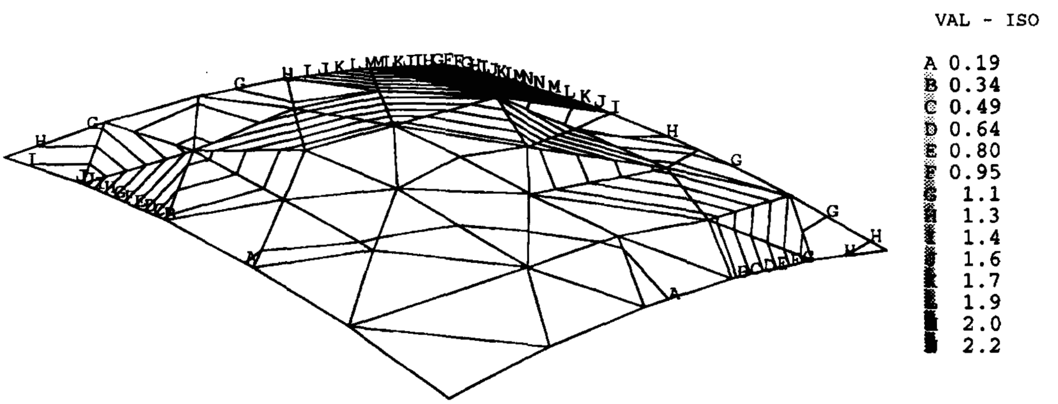

Figure 14. Isolines of Von Mises after the first increment

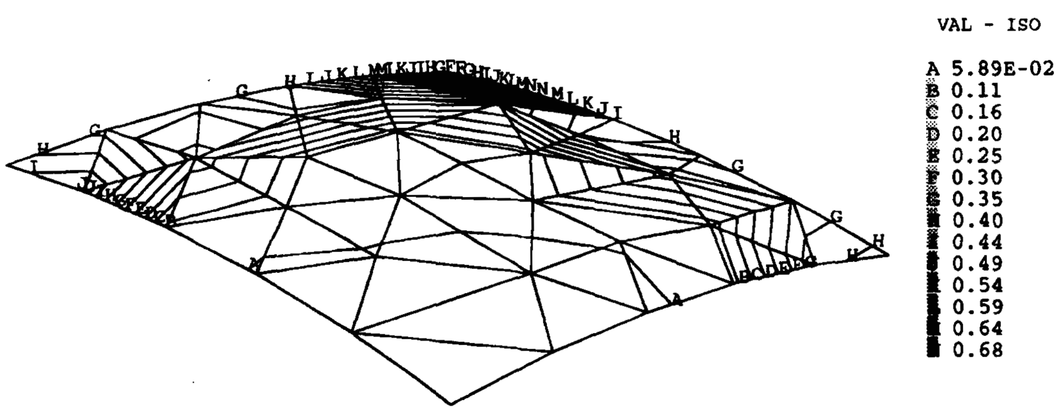

Figure 15. Isolines of Von Mises after the 140th increment

The shell is now subjected to a step displacement applied on the centre of the shell: $W(t) = -40$ mm $\forall t \geqslant 0$. Figures 14 and 15 show the isolines of Von Mises after the first and the 140th increment. As expected, we observe a stress relaxation in time which is consistent with the fading memory hypothesis.

CONCLUSION

A new formulation is presented in the time domain for the displacement and stress analysis of quasistatic, linear, viscoelastic thin structures undergoing mechanical deformation. The procedure is simple and maintains all the merit and accuracy of the finite element method. The response is obtained by using a discrete creep spectrum and an incremental constitutive equations. Numerical results of very good accuracy are achieved. The formulation can easily be extended to deal with aging materials, thermoviscoelastic and dynamic analysis.

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