THERMODYNAMIC ANALYSIS OF THE EQUATIONS OF A LINEAR VISCOELASTIC ANISOTROPIC SOLID

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Constraints on the coefficients of the equations of state of anisotropic linear viscoelasticity are obtained using the positive definiteness of two quadratic forms—the potential energy and the energy dissipation rate.

We will consider a mechanical system completely characterized by generalized coordinates q_i, $i = 1 \ldots n$. The equilibrium position is taken at the coordinate origin where $q_i = 0$.

The system is acted upon by the generalized external forces Q_i, $i = 1 \ldots n$, associated with the coordinates q_i. Of course, some of the forces Q_i may be equal to zero.

Let the system be displaced from the equilibrium position into some nonequilibrium state q_i. If we assume that the entropy of the system S is zero in the equilibrium position, then for the displaced system $S < 0$:

$$S=0 \quad \text{at} \quad q_i=0; \qquad S<0 \quad \text{at} \quad q_i \neq 0. \tag{1}$$

The entropy of a system under the action of some medium (or forces) is equal to the minimum work $R_{\min}$ (with a minus sign and divided by the temperature T) that must be done by the forces (or the medium) in order to transfer the body from the equilibrium position into the nonequilibrium state of (1):

$$S=-\frac{R_{\min}}{T}. \tag{2}$$

The minimum work is equal to the difference of the energies of these two states. If it is assumed that the temperature of the body is constant and the process is almost static, i.e., if we neglect the kinetic energy, then $R_{\min}= U-U_0-\sum\limits_{i=1}^{n} Q_i q_i$, where U is the potential energy in the nonequilibrium state, U_0 is the potential energy in the position $q_i=0$; $-\sum\limits_{i=1}^{n} Q_i q_i$ is the potential energy of the external forces.

It is always possible to assume that $U = 0$ at $q_i = 0$. Within the framework of linear theory

$$U=\frac{1}{2} a_{ij} q_i q_j, \tag{3}$$

where a_{ij} is a positive definite quadratic form. (The potential energy of the system is not negative.) Consequently, the entropy of the system is expressed as follows:

$$TS=-\frac{1}{2} a_{ij} q_i q_j+\sum_{i=1}^{n} Q_i q_i. \tag{4}$$

Onsager's theorem states that if we set $X_i=-\frac{\partial S}{\partial x_i}$, where x_i are coordinates determining the state of the system, and if a linear relation exists between X_i and x_i, then $X_i = \gamma_{ik}\dot{x}_k$, and $\gamma_{ik} = \gamma_{ki}$.

It is easy to see that $\gamma_{ik}\dot{x}_i\dot{x}_k$ is a nonnegative quadratic form, since

$$\dot{S}=\frac{\partial S}{\partial x_i}\dot{x}_i=X_i\dot{x}_i=\gamma_{ik}\dot{x}_i\dot{x}_k \geqslant 0.$$

Consequently,

$$T\frac{\partial S}{\partial q_i}=-a_{ij}q_j+Q_i=b_{ij}\dot{q}_j.$$

(The coefficients b_{ij} differ from γ_{ij} with respect to the constant factor T.)

$$Q_i=a_{ij}q_j+b_{ij}\dot{q}_j. \tag{5}$$

In what follows, a linear viscoelastic medium is a medium for which the relation between the generalized forces and coordinates is given by Eq. (5).

It is possible to show (see [1]) that $b_{ij}\dot{q}_i\dot{q}_j$ determines the dissipation of mechanical energy in the system and half that quantity is called the dissipation function:

$$D=\frac{1}{2} b_{ij}\dot{q}_i\dot{q}_j. \tag{6}$$

In this case the law of conservation of total energy

$$Q_i\dot{q}_i=\dot{U}+2D \tag{7}$$

is satisfied.

Equation (5) was obtained in [2] by somewhat different reasoning.

In viscoelastic systems the position of the system is not completely characterized by assigning a single total deformation ε_{ij}. In the general case, apart from the six components of the tensor ε_{ij}, it is necessary to introduce other coordinates q_i, which, in addition to ε_{ij}, also characterize the medium.

In an actual material the additional coordinates q_i may be given physical significance. Thus, for example, in [3] the equation of a standard solid was obtained by introducing, apart from the strain ε, the characteristic ξ characterizing the departure from equilibrium (ξ characterizes the difference of molecules with left and right orientation). In the same paper it was shown that it is also possible to introduce several coefficients ξ_i characterizing the number of deformed molecules of the different components, which leads to the formation of the equations of a viscoelastic body of more complicated structure. The coefficients ξ_i are generalized coordinates which, at the same time as the total strain, characterize the nonequilibrium viscoelastic system.

We will consider what constraints must be imposed on the equations of anisotropic viscoelasticity to satisfy basic relation (5), if among the external forces only the stresses act. We write the equations of anisotropic viscoelasticity in the form:

$$\sigma_i = a_{ij}\varepsilon_j + \int\limits_{-\infty}^{t} \sum_s b_{ij}{}^s e^{-\frac{t-\tau}{n_s}} \dot{\varepsilon}_j(\tau)\,d\tau + c_{ij}\dot{\varepsilon}_j. \tag{8}$$

In [4], on the basis of an assumption concerning the summation of strains over infinitely small springs and dashpots, the conditions for the coefficients of Eq. (8) were obtained in the one-dimensional case:

$$\sigma = E\varepsilon + \eta\dot{\varepsilon} + \int\limits_{-\infty}^{t} \sum_s b_s e^{-\frac{t-\tau}{n_s}} \dot{\varepsilon}\,d\tau. \tag{9}$$

These conditions are very simple: all E, η, b_s, and n_s must be real and nonnegative.

For an anisotropic material the analogous condition is not sufficient.

As an example we will consider the system of equations of the plane problem of an orthotropic viscoelastic body:

$$\sigma_{11} = a_{11}\varepsilon_1 + a_{12}\varepsilon_2 +$$

$$+ \int\limits_{-\infty}^{t} e^{-\frac{t-\tau}{n_{11}}} b_{11}\dot{\varepsilon}_1 d\tau + \int\limits_{-\infty}^{t} e^{-\frac{t-\tau}{n_{12}}} b_{12}\dot{\varepsilon}_2 d\tau;$$

$$\sigma_2 = a_{12}\varepsilon_1 + a_{22}\varepsilon_2 +$$

$$+ \int\limits_{-\infty}^{t} e^{-\frac{t-\tau}{n_{12}}} b_{12}\dot{\varepsilon}_1 d\tau + \int\limits_{-\infty}^{t} e^{-\frac{t-\tau}{n_{22}}} b_{22}\dot{\varepsilon}_2 d\tau. \tag{10}$$

Applying the Laplace transformation to (10)

$$\sigma_1 = a_{11}\varepsilon_1 + a_{12}\varepsilon_2 + b_{11}\frac{p\varepsilon_1}{\frac{1}{n_{11}}+p} + b_{12}\frac{p\varepsilon_2}{\frac{1}{n_{12}}+p};$$

$$\sigma_2 = a_{12}\varepsilon_1 + a_{22}\varepsilon_2 + b_{12}\frac{p\varepsilon_1}{\frac{1}{n_{12}}+p} + b_{22}\frac{p\varepsilon_2}{\frac{1}{n_{22}}+p}, \tag{11}$$

we solve them for ε_1 and ε_2. After reducing the equations to the common denominator, we obtain

$$\varepsilon_1 = A_{11}\sigma_1 + A_{12}\sigma_2 + \sigma_1\frac{Q_{11}{}^3(p)}{Q^4(p)} + \sigma_2\frac{Q_{12}{}^3(p)}{Q^4(p)};$$

$$\varepsilon_2 = A_{12}\sigma_1 + A_{22}\sigma_2 + \sigma_1\frac{Q_{12}{}^3(p)}{Q^4(p)} + \sigma_2\frac{Q_{22}{}^3(p)}{Q^4(p)}, \tag{12}$$

where $Q^3(p)$ and $Q^4(p)$ are third- and fourth-degree polynomials.

It is easy to find relations for the coefficients of the starting equations (10) for which $Q^4(p)$ has complex roots; consequently, the resolvents in the solved equations will contain the sine and cosine of the function, i.e., under the action of constant stresses the strains are sinusoidal, which is physically impossible. For example, if we set $\frac{a_{11}}{a_{12}} = \frac{a_{11}}{b_{12}} = \frac{10}{3}$ and the relaxation times in directions 1 and 2 are the same ($n_{11} = n_{22}$), then, with variation of the relaxation time at a Poisson's ratio $\frac{n_{12}}{n_{11}} > \frac{39}{32}$, complex roots appear in the resolvents. At the same time, on the basis of [2] we can formulate a lemma: if we have a system of equations

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_k \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_k \\ \cdot \\ q_n \end{bmatrix} \begin{bmatrix} & & \\ & \alpha_{ij} + p\beta_{ij} & \\ & & \end{bmatrix}, \tag{13}$$

where α_{ij} and β_{ij} are symmetric nonnegative quadratic forms, and p denotes the symbol of differentiation, its solution will be

$$q_i = \sigma_j\gamma_{ij} + \int\limits_{-\infty}^{t} \gamma_{ij}{}^s e^{-\frac{t-\tau}{\lambda_s}} \sigma_j(\tau)\,d\tau, \tag{14}$$

where λ_s are positive real numbers.

Consequently, the positiveness of U and D together with Onsager's theorem give a sufficient condition for the nonappearance of complex and positive roots in the resolvents of solutions (14). However, it is possible to show with an example that these conditions are not necessary.

We will now consider the conditions under which system of equations (8) is equivalent to system (5) or in developed form to (13). In operator form, Eqs. (8) are written:

$$\sigma_i = \left(a_{ij} + c_{ij}p + \sum_s b_{ij}{}^s \frac{p}{\frac{1}{n_s}+p} \right) \varepsilon_j. \tag{15}$$

Equations (13) can be somewhat simplified by a linear transformation of the coordinates q_i, reducing the bottom right-hand part of the matrix to diagonal form:

$$\left[\begin{array}{c} \sigma_i \\ \hline \\ 0 \end{array}\right] = \left[\begin{array}{c} \varepsilon_j \\ \hline \\ q_r \end{array}\right] \left[\begin{array}{c:cc} \alpha_{ij} + p\beta_{ij} & \multicolumn{2}{c}{\alpha_{ir} + p\beta_{ir}} \\ \hdashline & p + \frac{1}{\eta_{k+1}} & 0 \\ \alpha_{rj} + p\beta_{rj} & 0 & p + \frac{1}{\eta_r} \end{array}\right];$$

$$i, j = 1 \ldots k; \quad r = k+1 \ldots n \quad \text{or} \quad r = k+s. \tag{16}$$

We eliminate from Eqs. (16) the additional coordinates q_r:

$$\sigma_i = (\alpha_{ij} + p\beta_{ij})\varepsilon_j + (\alpha_{ir} + p\beta_{ir})q_r;$$

$$q_r = -\frac{\varepsilon_j(\alpha_{rj}+p\beta_{rj})}{p+\frac{1}{\eta_r}}$$

(there is no summation over the subscript r of the coefficient η_r either here or in what follows) and obtain

$$\sigma_i = \left[\alpha_{ij}+p\beta_{ij}-\frac{(\alpha_{rj}+p\beta_{rj})(\alpha_{ir}+p\beta_{ir})}{p+\frac{1}{\eta_r}}\right]\varepsilon_j;$$

$$\sigma_i = \Bigg[\alpha_{ij}+p\beta_{ij}-$$

$$-\frac{\alpha_{rj}\alpha_{ir}+p(\alpha_{ir}\beta_{rj}+\beta_{ir}\alpha_{rj})+p^2\beta_{ir}\beta_{rj}}{p+\frac{1}{\eta_r}}\Bigg]\varepsilon_j=$$

$$=\Bigg\{\alpha_{ij}+\beta_{ij}p-\Bigg[\alpha_{tr}\alpha_{rj}\eta_r+\beta_{ir}\beta_{rj}p-$$

$$-\frac{p}{p+\frac{1}{\eta_r}}\left(\alpha_{ir}\alpha_{rj}\eta_r+\frac{\beta_{ir}\beta_{rj}}{\eta_r}+\beta_{rj}\alpha_{ir}-\beta_{ir}\alpha_{rj}\right)\Bigg]\Bigg\}\varepsilon_j=$$

$$=\Bigg[\alpha_{ij}+\beta_{ij}p-\alpha_{ir}\alpha_{rj}\eta_r-\beta_{ir}\beta_{rj}p+$$

$$+\frac{p}{p+\frac{1}{\eta_r}}\left(\alpha_{ir}\sqrt{\eta_r}-\frac{\beta_{ir}}{\sqrt{\eta_r}}\right)\left(\alpha_{rj}\sqrt{\eta_r}-\frac{\beta_{rj}}{\sqrt{\eta_r}}\right)\Bigg]\varepsilon_j. \quad (17)$$

Comparing expression (17) with Eq. (15), we arrive at the conclusion that all η_r must be equal to the relaxation time n_s, while the parenthetical expressions

$$\left(\alpha_{ir}\sqrt{\eta_r}-\frac{\beta_{ir}}{\sqrt{\eta_r}}\right) \text{ and } \left(\alpha_{rj}\sqrt{\eta_r}-\frac{\beta_{rj}}{\sqrt{\eta_r}}\right) \quad (18)$$

are determined from a comparison with the coefficients $b_{ij}{}^s$ in Eq. (15).

Thus, it appears that α_{ir} and β_{ir} cannot be uniquely determined starting from the coefficients of Eq. (15); only expressions (18) are determined. Therefore, Eqs. (16) may be written in infinitely many ways; it is also possible arbitrarily to assign either β_{ir} or α_{ir}, with a view to maximum simplification of system (16). Hence, in what follows we assume that $\beta_{ir} = 0$. Then Eqs. (17) take the form:

$$\sigma_i = \left[(\alpha_{ij}-\alpha_{ir}\alpha_{rj}\eta_r)+\beta_{ij}p+\frac{p}{p+\frac{1}{\eta_r}}\alpha_{ir}\alpha_{rj}\eta_r\right]\varepsilon_j. \quad (19)$$

If all the η_r are different, then, comparing (19) with (15) and substituting k + s for r, we have

$$\eta_{k+s}=n_s; \quad \alpha_{i\,k+s}=\sqrt{\frac{b_{ii}{}^s}{n_s}} \text{ (no summation)};$$

$$\beta_{ij}=c_{ij}; \quad \alpha_{ij}=a_{ij}+\sum_s b_{ij}{}^s$$

$$\text{(no summation over i, j).} \quad (20)$$

In this case, since all the η_r are assumed to be different, the constraint

$$b_{ij}{}^s=\sqrt{b_{ii}{}^s b_{jj}{}^s} \quad (21)$$

is imposed on the coefficients $b_{ij}{}^s$. Below we will show that this constraint is not mandatory.

From Eqs. (20) it is convenient to find the quadratic forms α_{ij} and β_{ij} and, consequently, the potential and dissipated energy for any one-dimensional linear viscoelastic solid, where the transverse strains b_{ij} are not considered, or in cases in which the coefficients $b_{ij}{}^s$ satisfy Eq. (21).

We will consider the example of a standard solid:

$$\sigma=\varepsilon\left[E+(H-E)\frac{p}{p+\frac{1}{n}}\right].$$

Equations (20) take the form:

$$\eta=n; \quad \alpha_{12}=\sqrt{\frac{H-E}{n}}; \quad \beta_{11}=0; \quad \alpha_{11}=H. \quad (22)$$

Equations (16) are written thus:

$$\begin{bmatrix}\sigma\\0\end{bmatrix}=\begin{bmatrix}\varepsilon\\q\end{bmatrix}\begin{bmatrix}H & \sqrt{\frac{H-E}{n}}\\ \sqrt{\frac{H-E}{n}} & \frac{1}{n}+p\end{bmatrix};$$

$$\sigma=\varepsilon H+q\sqrt{\frac{H-E}{n}};$$

$$0=\varepsilon\sqrt{\frac{H-E}{n}}+q\left(\frac{1}{n}+p\right). \quad (23)$$

The tables of α_{ij} and β_{ij} take the form:

$$\alpha_{ij}=\begin{bmatrix}H & \sqrt{\frac{H-E}{n}}\\ \sqrt{\frac{H-E}{n}} & \frac{1}{n}\end{bmatrix}; \quad \beta_{ij}=\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}.$$

Owing to the positive definiteness of the form α_{ij}, the following natural constraints are imposed on the coefficients of the starting equation: $H > 0$; $n > 0$; $H - E > 0$; $E > 0$. Now

$$U=\frac{1}{2}(\alpha_{11}\varepsilon^2+2\alpha_{12}\varepsilon q+\alpha_{22}q^2)=$$

$$=\frac{1}{2}\left(H\varepsilon^2+\sqrt{\frac{H-E}{n}}\varepsilon q+q^2\frac{1}{n}\right);$$

$$2D=q^2 \quad (24)$$

Finding q from the first of Eqs. (23) and substituting it in (24), we obtain

$$U=\frac{(\sigma-E\varepsilon)^2+\varepsilon^2E(H-E)}{H-E};$$

$$2D=\frac{n}{H-E}(\dot{\sigma}-\dot{\varepsilon}H)^2.$$

These expressions coincide with those obtained in the case of summation of the accumulated energy over the springs and the dissipated energy over the dashpots in [4].

We will consider the case of equal η_r. We denote by η_ρ^s those η_r that are equal to n_s. The corresponding α_{ir} are denoted by $\alpha_{i\rho}^s$. Then, collecting like terms in (19), we have

$$\sigma_i=\Bigg(\alpha_{ij}-\alpha_{ir}\alpha_{jr}\eta_r+$$

$$+\beta_{ij}p+\sum_s\frac{pn_s}{p+\frac{1}{n_s}}\sum_s\alpha_{i\rho}^s\alpha_{j\rho}^s\Bigg)\varepsilon_j.$$

Comparing with (15), we obtain

$$b_{ij}^s=\sum_\rho\alpha_{i\rho}^s\alpha_{j\rho}^s;$$

$$b_{ii}^s=\sum_\rho\alpha_{i\rho}^s\alpha_{i\rho}^s;\qquad b_{jj}^s=\sum_\rho\alpha_{j\rho}^s\alpha_{j\rho}^s, \tag{25}$$

i.e., b_{ij}^s is the scalar product of two vectors $\alpha_{i\rho}^s$ and $\alpha_{j\rho}^s$ (summation over ρ). Hence it follows that

$$(b_{ij}^s)^2\leqslant b_{ii}^s b_{jj}^s. \tag{26}$$

Inequality (26) gives the constraints on the viscoelastic Poisson's ratios. In particular, if in the direction of the deformations with subscript i there is no term with relaxation time n_s ($b_{ii}^s = 0$), then there will also be no such relaxation time in the transverse directions ($b_{ij}^s = 0$) for any j. If the material is isotropic, then the transverse deformation relaxation time is the same as for longitudinal deformation.

$$\sigma_1=a_{11}\varepsilon_1+a_{12}\varepsilon_2+b_{11}\frac{p}{p+\frac{1}{n}}\varepsilon_1+b_{12}\frac{p}{p+\frac{1}{n}}\varepsilon_2;$$

$$\sigma_2=a_{12}\varepsilon_1+a_{22}\varepsilon_2+b_{12}\frac{p}{p+\frac{1}{n}}\varepsilon_1+b_{22}\frac{p}{p+\frac{1}{n}}\varepsilon_2, \tag{27}$$

where $b_{12}^2 < b_{11}b_{22}$.

We divide b_{11} and b_{22} into the components:

$$b_{11}=X_1+X_2;\quad b_{22}=Y_1+Y_2;\quad b_{12}=\sqrt{X_1Y_1}+\sqrt{X_2Y_2}. \tag{28}$$

One unknown can be arbitrarily assigned. System (28) is satisfied, for example, by the values of the unknowns: $X_2=0$; $X_1=b_{11}$; $Y_1=\frac{b_{12}^2}{b_{11}}$; $Y_2=\frac{b_{22}b_{11}-b_{12}^2}{b_{11}}$. Consequently, (27) may be rewritten:

$$\sigma_1=a_{11}\varepsilon_1+a_{12}\varepsilon_2+\left(b_{11}\frac{p}{p+\frac{1}{n}}\varepsilon_1+b_{12}\frac{p}{p+\frac{1}{n}}\varepsilon_2\right)+$$

$$+\left(0\frac{p}{p+\frac{1}{n}}\varepsilon_1+0\frac{p}{p+\frac{1}{n}}\varepsilon_2\right);$$

$$\sigma_2=a_{12}\varepsilon_1+a_{22}\varepsilon_2+$$

$$+\left(b_{12}\frac{p}{p+\frac{1}{n}}\varepsilon_1+\frac{b_{12}^2}{b_{11}}\frac{p}{p+\frac{1}{n}}\varepsilon_2\right)+$$

$$+\left(0\frac{p}{p+\frac{1}{n}}\varepsilon_1+\frac{b_{22}b_{11}-b_{12}^2}{b_{11}}\frac{p}{p+\frac{1}{n}}\varepsilon_2\right).$$

The system of equivalent equations (16) has the form:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ q_3\\ q_4\end{bmatrix}\begin{bmatrix}a_{11}+b_{11} & a_{12}+b_{12} & \sqrt{\frac{b_{11}}{n}} & 0\\ a_{12}+b_{12} & a_{22}+b_{22} & \sqrt{\frac{b_{12}^2}{b_{11}n}} & \sqrt{\frac{b_{22}b_{11}-b_{12}^2}{b_{11}n}}\\ \sqrt{\frac{b_{11}}{n}} & \sqrt{\frac{b_{12}^2}{b_{11}n}} & p+\frac{1}{n} & 0\\ 0 & \sqrt{\frac{b_{22}b_{11}-b_{12}^2}{b_{11}n}} & 0 & p+\frac{1}{n}\end{bmatrix} \tag{29}$$

The coefficients α_{ij} can be found by dividing the b_{ij}^s in the starting equation (15) into parts: $b_{ij}^s = \sum_\rho X_{ij\rho}^s$, so that the individual components satisfy relations (21):

$$\sqrt{X_{ii\rho}^s X_{jj\rho}^s}=X_{ij\rho}^s,$$

after which it is possible to employ Eqs. (20).

As an example we will consider the equations of an orthotropic body in the plane case with a single relaxation time:

Having obtained the coefficients α_{ij} and β_{ij}, from Eqs. (29), after simple but rather lengthy transformations, we obtain expressions for the accumulated energy U and the rate of energy dissipation 2D:

$$2D=\frac{n}{b_{11}b_{22}-b_{12}^2}\{b_{22}\dot{\sigma}_1^2+b_{11}\dot{\sigma}_2^2+[(a_{11}+b_{11})^2b_{22}+$$

$$+b_{11}(a_{12}^2-b_{12}^2)-2b_{12}a_{11}(a_{12}+b_{12})]\dot{\varepsilon}_1^2+$$

$$+[(a_{22}+b_{22})^2b_{11}+b_{22}(a_{12}^2-b_{12}^2)-2b_{12}a_{22}\times$$

$$\times(a_{12}+b_{12})]\,\dot{\varepsilon}_2^2+2[(a_{12}+b_{12})b_{12}-$$
$$-(a_{11}+b_{11})b_{22}]\,\dot{\varepsilon}_1\dot{\sigma}_1+2[(a_{22}+b_{22})b_{12}-$$
$$-(a_{12}+b_{12})b_{22}]\,\dot{\varepsilon}_2\dot{\sigma}_1+2[(a_{11}+b_{11})b_{12}-$$
$$-(a_{12}+b_{12})b_{11}]\,\dot{\varepsilon}_1\dot{\sigma}_2+2[(a_{12}+b_{12})b_{22}-$$
$$-(a_{22}+b_{22})b_{12}]\,\dot{\varepsilon}_2\dot{\sigma}_2+2[(a_{11}+b_{11})(a_{12}+b_{12})b_{22}-$$
$$-(a_{12}+b_{12})^2b_{12}+(a_{22}+b_{22})(a_{12}b_{11}-a_{11}b_{12})]\,\dot{\varepsilon}_1\dot{\varepsilon}_2\}. \quad (30)$$

It is easy to show that if conditions (26) are satisfied, the system of integral equations (8) is equivalent to a system of differential equations of order n = s + 1.

As an example we will consider the case when all the n_s are different and s = 1. Then the system of differential equations has the following form:

$$\dot{\sigma}_i+\frac{1}{n_i}\sigma_i=A_{ij}\varepsilon_j+B_{ij}\dot{\varepsilon}_j+C_{ij}\ddot{\varepsilon}_j$$

$$\text{(no summation over i)}, \quad (31)$$

where the coefficients n_i, A_{ij}, B_{ij}, and C_{ij} are expressed in terms of the coefficients of Eqs. (8) as follows: $A_{ij}=\frac{a_{ij}}{n_i}$; $B_{ij}=b_{ij}+\frac{c_{ij}}{n_i}+a_{ij}$; $C_{ij}=c_{ij}$; $n_i=n_s$.

Condition (26) now takes the following form:

$$B_{ij}-n_iA_{ij}-\frac{C_{ij}}{n_i}=$$
$$=\sqrt{B_{ii}-n_iA_{ii}-\frac{C_{ii}}{n_i}}\sqrt{B_{jj}-n_iA_{jj}-\frac{C_{jj}}{n_i}} \quad (32)$$

(no summation).

The coefficients of the quadratic forms α_{ij} and β_{ij} are found from the following equations (see (22)):

$$\eta_r=n_i; \qquad \alpha_{ir}=\sqrt{\frac{1}{n_i}\left(B_{ii}-n_iA_{ii}-\frac{C_{ii}}{n_i}\right)};$$
$$\beta_{ij}=C_{ij}; \qquad \alpha_{ij}=B_{ij}-\frac{C_{ij}}{n_i}.$$

We note that, by using Eqs. (16), it always is possible to express q_r in terms of ε_i and σ_i, so that the operator p is not in the denominator. Therefore the functions U and D for an anisotropic material can always be expressed in terms of the stresses and strains and their derivatives. This was noted in [4] in relation to the one-dimensional case. In the isotropic case, of course, it is more convenient to find the energy U and the dissipation rate D separately for the volume strain and for the deviator component:

$$\sigma=a\varepsilon+c\dot{\varepsilon}+\sum_s b^s\frac{p}{p+\frac{1}{n_s}}\varepsilon;$$

$$S_{ij}=a_1e_{ij}+c_1\dot{e}_{ij}+\sum_s b_1{}^s\frac{p}{p+\frac{1}{n_s}}e_{ij}.$$

CONCLUSIONS

1. If the equations of state for a linear viscoelastic medium are given in the form of integral equations with kernel $\sum\limits_s b_{ij}{}^s\times e^{\frac{-t-\tau}{n_s}}$, then the satisfaction of constraint (26) is a sufficient condition of the existence of the two quadratic forms that play the part of potential energy and dissipation function.

2. Satisfaction of constraint (26) is a sufficient condition of the equivalence of the integral equations with exponential kernels and a system of linear differential equations of order not greater than n = s + 1.

3. If constraint (26) is satisfied, it is possible to calculate the potential energy and energy dissipation rate, whose positive definiteness imposes constraints on the coefficients of the starting equations. Satisfaction of the above-mentioned constraint is sufficient for sinusoidal strains not to develop under the action of constant stresses.

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