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On thickness and orientational design with orthotropic materials

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Abstract. Recent results from sensitivity analysis for strain energy with anisotropic elasticity are applied to thickness and orientational design of laminated membranes. The first order gradients of the total elastic energy are primarily used in an optimality criteria based method. This traditional method is shown to give slow convergence with respect to design parameters, although the convergence of strain energy is very good. To gain a deeper insight into this rather general characteristic, second order derivatives are included and it is shown how they can be obtained by first order sensitivity analysis. Examples of thickness design only, orientational design only and combined thickness-orientational design are presented.

1 Introduction

Design with advanced materials, such as anisotropic laminates, is a challenging area for optimization. We shall here restrict ourselves to plane problems, as in the early work of Banichuk (1983) (which includes further early references). Recent work by Pedersen (1989, 1990) was conducted independently and the formulations are rather parallel. Similar research has been carried out by Landriani and Rovati (1991) and by Thomsen (1991). In the present paper we combine these orientational optimizations with thickness optimization. The further goal is to gain deeper insight into the redesign procedures based on optimality criteria.

The sensitivity analysis that proves local gradient determination relative to a fixed strain field is presented. The physical understanding of these results has many aspects outside the scope of the present paper. An early paper by Masur (1970) includes valuable information on this sensitivity analysis. With thickness as design parameter it implies that first order gradients of the total strain energy are given directly by the local specific strain energy and that second order gradients of the total strain energy are found from first order local gradients.

For orthotropic materials, a single optimization parameter controls the orientational design. This parameter includes information about material, as well as about the state of strain. It is used as an optimization criterion, and in principle, the optimization procedure is a non-gradient technique. In this way local extrema are avoided.

When the principal axes of an orthotropic material are equal to, say, the principal strain axes, it follows directly that the principal stress axes also equal those of material and strain. However, optimal orientations exist for which the principal axes of material differ from those of the principal strains. Even for this case it is proved by Pedersen (1990) that the principal axes of stress equal those of the principal strains.

Sensitivity analysis for thickness change is extended to include mutual sensitivities, i.e. change in energy density with respect to thickness changes not at the same point. A symmetry relation is proven.

A number of examples are given and discussed, including a uniformly loaded short cantilever, a bending loaded knee and a biaxially loaded hole.

2 Sensitivity analysis for energy in non-linear elasticity

Let us start with the *work equation*

$$W + W^C = U + U^C \,, \tag{1}$$

where W, W^C are physical and complementary work of the external forces and U, U^C are physical and complementary elastic energy, also named strain and stress energy, respectively. The work equation (1) holds for any design h and, therefore, for the total differential quotient with respect to h

$$\frac{\mathrm{d}W}{\mathrm{d}h} + \frac{\mathrm{d}W^C}{\mathrm{d}h} = \frac{\mathrm{d}U}{\mathrm{d}h} + \frac{\mathrm{d}U^C}{\mathrm{d}h} \,. \tag{2}$$

Now in the same way as h represents the design field, in general, ϵ represents the strain field and σ represents the stress field. Bearing in mind that as a function of h,ϵ we have W, U, while the complementary quantities W^C, U^C are functions of h, σ, we then obtain (2) in greater detail by means of

$$\frac{\partial W}{\partial h}+\frac{\partial W}{\partial \epsilon}\frac{\partial \epsilon}{\partial h}+\frac{\partial W^C}{\partial h}+\frac{\partial W^C}{\partial \sigma}\frac{\partial \sigma}{\partial h}= =\frac{\partial U}{\partial h}+\frac{\partial U}{\partial \epsilon}\frac{\partial \epsilon}{\partial h}+\frac{\partial U^C}{\partial h}+\frac{\partial U^C}{\partial \sigma}\frac{\partial \sigma}{\partial h}. \tag{3}$$

The *principles of virtual work* which hold for solids/structures in equilibrium are

$$\frac{\partial W}{\partial \epsilon}=\frac{\partial U}{\partial \epsilon} \tag{4}$$

for the physical quantities with strain variation, and for the complementary quantities with stress variation we have

$$\frac{\partial W^C}{\partial \sigma}=\frac{\partial U^C}{\partial \sigma}. \tag{5}$$

Inserting (4) and (5) in (3), we obtain

$$\frac{\partial U^C}{\partial h}-\frac{\partial W^C}{\partial h}=-\left(\frac{\partial U}{\partial h}-\frac{\partial W}{\partial h}\right), \tag{6}$$

and for design-independent loads

$$\left(\frac{\partial U^C}{\partial h}\right)_{\text{fixed stresses}}=-\left(\frac{\partial U}{\partial h}\right)_{\text{fixed strains}}, \tag{7}$$

as stated by Masur (1970). Note that the only assumption behind this is the design-independent loads $\partial W/\partial h=0$, $\partial W^C/\partial h=0$.

To further obtain a *physical interpretation* of $(\partial U/\partial h)_{\text{fixed strains}}$ [and by (7) of $(\partial U^C/\partial h)_{\text{fixed stresses}}$] we need the relation between the external work W and the strain energy U. Let us assume that this relation is given by the constant c

$$W=cU. \tag{8}$$

For linear elasticity and dead loads we have $c=2$, and in general we expect $c>1$.

Parallel to the analysis from (1) to (3) and on the basis (8), we obtain

$$\frac{\partial W}{\partial h}+\frac{\partial W}{\partial \epsilon}\frac{\partial \epsilon}{\partial h}=c\frac{\partial U}{\partial h}+c\frac{\partial U}{\partial \epsilon}\frac{\partial \epsilon}{\partial h}, \tag{9}$$

which for design-independent loads $\partial W/\partial h=0$ with virtual work (4), gives

$$\frac{\partial W}{\partial \epsilon}\frac{\partial \epsilon}{\partial h}=\frac{\partial U}{\partial \epsilon}\frac{\partial \epsilon}{\partial h}=\frac{c}{1-c}\frac{\partial U}{\partial h}, \tag{10}$$

and thereby

$$\frac{\mathrm{d}U}{\mathrm{d}h}=\frac{\partial U}{\partial h}+\frac{\partial U}{\partial \epsilon}\frac{\partial \epsilon}{\partial h}=\frac{1}{1-c}\left(\frac{\partial U}{\partial h}\right)_{\text{fixed strains}}. \tag{11}$$

Note, in this important result, with $c>1$, we have *different signs* for $\mathrm{d}U/\mathrm{d}h$ and $(\partial U/\partial h)_{\text{fixed strains}}$.

For the case *of linear elasticity and dead loads* we have, with $c=2$ and adding (7),

$$\frac{\mathrm{d}U}{\mathrm{d}h}=-\left(\frac{\partial U}{\partial h}\right)_{\text{fixed strains}}=\left(\frac{\partial U}{\partial h}\right)_{\text{fixed stresses}}. \tag{12}$$

For the case *of non-linear elasticity* modelled in one dimension by

$$\sigma=E\epsilon^n \tag{13}$$

and still dead loads ($W^C=0$), we obtain $c=1+n$ and thereby

$$\frac{\mathrm{d}U}{\mathrm{d}h}=-\frac{1}{n}\left(\frac{\partial U}{\partial h}\right)_{\text{fixed strains}}=\frac{1}{n}\left(\frac{\partial U^C}{\partial h}\right)_{\text{fixed stresses}}. \tag{14}$$

3 Optimality criteria

We want to *minimize the elastic strain energy U*

$$\text{minimize}\quad \left(U=\sum_{e=1}^{N}U_e\right), \tag{15}$$

which is obtained as the sum of the element energies U_e for $e=1,2,\dots,N$. Two groups of design parameters are considered. The material orientations θ_e for $e=1,2,\dots,N$ assumed constant in each element and the element thicknesses t_e for $e=1,2,\dots,N$, also constant in each element. The constraint of our optimization problem is a given volume $\overline{V}$, i.e. by summation over element volumes V_e for $e=1,2,\dots,N$,

$$V-\overline{V}=\sum_{e=1}^{N}V_e-\overline{V}=0. \tag{16}$$

The *gradients of volume* are easily obtained for thicknesses

$$\frac{\partial V}{\partial t_e}=\frac{\partial V_e}{\partial t_e}=\frac{V_e}{t_e}, \tag{17}$$

and volume does not depend on material orientation

$$\frac{\partial V}{\partial \theta_e}=0. \tag{18}$$

The *gradients of elastic strain energy* are simplified by the results of Section 2 and are thereby localized

$$\frac{\partial U}{\partial h_e} = -\left(\frac{\partial U}{\partial h_e}\right)_{\text{fixed strains}} = -\left[\frac{\partial (u_e V_e)}{\partial h_e}\right]_{\text{fixed strains}}, \tag{19}$$

valid for $h_e = \theta_e$, as well as for $h_e = t_e$. The strain energy density u_e is introduced by $U_e = u_e V_e = u_e a_e t_e$ with a_e for the element area.

With fixed strains, the thickness has no influence on the strain energy density u_e. Thus with (17) and (19), we directly obtain

$$\frac{\partial U}{\partial t_e} = -\frac{U_e}{t_e} = -\frac{u_e V_e}{t_e}. \tag{20}$$

With respect to material orientation, the gradient is more complicated, because even with fixed strains the energy density u_e will depend on θ_e. A rather simple formula is derived by Pedersen (1989), in terms of principal strains ϵ_I, ϵ_{II} ($|\epsilon_I| > |\epsilon_{II}|$) - angle ψ from direction of ϵ_I to principal material direction - and material parameters C_2 and C_3

$$\frac{\partial U}{\partial \theta_e} = \left[V(\epsilon_I - \epsilon_{II})^2 \sin 2\psi \left(C_2 \frac{\epsilon_I + \epsilon_{II}}{\epsilon_I - \epsilon_{II}} + 4C_3 \cos 2\psi\right)\right]_e . \tag{21}$$

With the gradients determined by (17), (18), (20) and (21), we can now formulate optimality criteria, directly based on the *general optimality criterion* of proportional gradients, i.e. for our problems

$$\frac{\partial V}{\partial h_e} = A \frac{\partial U}{\partial h_e} \quad \text{for all } e, \tag{22}$$

where A is the factor of proportionality.

For *thickness optimization* this general criterion gives, with (17) and (20),

$$V_e/t_e = -A u_e V_e/t_e, \tag{23}$$

which means constant energy density, equal to the mean strain energy density $\overline{u}$

$$u_e = \overline{u} \quad \text{for all } e. \tag{24}$$

See also the early paper by Masur (1970) for this optimality criterion.

For the *material orientation optimization* we have an unconstrained problem and thus, from (21), the optimality criterion

$$\sin 2\psi \left(C_2 \frac{\epsilon_I + \epsilon_{II}}{\epsilon_I - \epsilon_{II}} + 4C_3 \cos 2\psi\right)_e = 0 \quad \text{for all } e. \tag{25}$$

Solutions to (24) and (25) may correspond to maximum, minimum or just stationarity. Furthermore, the extremum may be local or global. Finally, the existence of a design satisfying (24) and/or (25) is not proven, and a procedure for obtaining such a possible solution is still to be described. How is a thickness distribution that fulfils (24) obtained? We shall discuss a practical procedure, cf. Rozvany (1989), which is based on a number of approximations. Firstly, we neglect the mutual influences from element to element, i.e. each element is redesigned independently (but simultaneously)

$$(t_e)_{\text{next}} = t_e + (\Delta t_e). \tag{26}$$

Secondly, the optimal mean energy density $\overline{u}$ is taken as the present mean energy density $\tilde{u}$. Thirdly, the element energy U_e is assumed constant through the change Δt_e and then, from (24), we obtain

$$\frac{U_e}{V_e(1 + \Delta t_e/t_e)} = \tilde{u}, \quad \text{i.e.}$$

$$\Delta t_e = t_e(u_e - \tilde{u})/\tilde{u} \quad \text{or} \quad (t_e)_{\text{next}} = t_e u_e/\tilde{u}. \tag{27}$$

A relaxation power, say 0.8, is normally introduced in the formulation. It is natural to ask why the gradient of element energy is not taken into account,

$$(U_e)_{\text{next}} = U_e + \frac{\partial U_e}{\partial t_e} \Delta t_e, \tag{28}$$

but this is explained by the fact that although $\partial U/\partial t_e$ is known from (20), the gradient of the local energy (the element strain energy)

$$\frac{\partial U_e}{\partial t_e} = \left(\frac{\partial U_e}{\partial t_e}\right)_{\text{fixed strain}} + \left(\frac{\partial U_e}{\partial \epsilon}\right) \frac{\partial \epsilon}{\partial t_e} \tag{29}$$

is more difficult to determine. The two terms in (29) have different signs, and the other neglected terms $\partial U_e/\partial t_i$ for $e \neq i$ may also be of the same order. Although the procedure (27) mostly works rather satisfactorily, we shall extend our analysis to the coupled problem.

4 Mutual sensitivities

The redesign procedure by (27) neglects the mutual sensitivities, i.e. the change in element energy due to change in the thickness of the other elements. This is caused by the overall change of strain field. These sensitivities can be calculated by classical sensitivity analysis. Assume that the analysis is related to a finite element model

$$[S]\{D\} = \{A\}, \tag{30}$$

where $\{A\}$ are the given nodal actions, $\{D\}$ the resulting nodal displacements and $[S] = \sum_e [S_e]$ the system stiffness matrix accumulated over the element stiffness matrices $[S_e]$ for $e = 1, 2, \ldots, N$.

Let h_e be an element design parameter without influence on $\{A\}$. We then obtain

$$[S]\frac{\partial\{D\}}{\partial h_e} = -\frac{\partial[S]}{\partial h_e}\{D\} = \{P_e\}, \tag{31}$$

where the right-hand side $\{P_e\}$ is a pseudo load, equivalent to design change. Knowing $\partial\{D\}/\partial h_e$ it is straightforward to calculate $\partial U_i/\partial h_e$. Generally, the computational effort involved corresponds to one additional load for each design parameter.

Then, with all the gradients $\partial U_e/\partial t_i$ available, we can formulate a procedure for simultaneous redesign of all element thicknesses,

$$\{t\}_{\text{next}} = \{t\} + \{\Delta t\}, \tag{32}$$

that takes the mutual sensitivities into account. In agreement with the optimality criteria (24) we change towards equal energy density $\tilde{u}$ in all elements. Formulated in terms of strain energy per area, we want

$$u_e t_e + \sum_{i=1}^{N} \frac{\partial(u_e t_e)}{\partial t_i}\Delta t_i = \tilde{u}(t_e + \Delta t_e) \text{ for } e = 1, 2, \ldots, N, \tag{33}$$

or in matrix notation

$$\{ut\} + [\nabla(ut)]\{\Delta t\} = \tilde{u}(\{t\} + \{\Delta t\}), \tag{34}$$

with the solution

$$\{\Delta t\} = \left[[\nabla(ut)] - \tilde{u}[I]\right]^{-1}\{(\tilde{u} - u)t\}. \tag{35}$$

The gradient matrix $[\nabla(ut)]$ consists of the quantities $\partial(u_e t_e)/\partial t_i$. Note that with the assumption of fixed element energy, the strain energy per area is unchanged, i.e. $[\nabla(ut)] = [0]$ and we obtain the simple redesign formula (27).

An alternative formulation would be Newton-Raphson iterations directly on energy densities

$$(u_e - \tilde{u}) + \sum_{i=1}^{N} \frac{\partial(u_e - \tilde{u})}{\partial t_i}\Delta t_i = 0 \quad \text{for } e = 1, 2, \ldots, N, \tag{36}$$

or, in matrix notation,

$$[\nabla u]\{\Delta t\} = \tilde{u}\{1\} - \{u\}. \tag{37}$$

Here, the gradient matrix $[\nabla u]$ constitutes $\partial u_e/\partial t_i$. An interesting formulation is obtained when we multiply every row e with area a_e and obtain

$$[\nabla(ua)]\{\Delta t\} = \{(\tilde{u} - u)a\}. \tag{38}$$

The present matrix is now symmetric, which, to the author's knowledge, is not well-known. Remembering that $u_e a_e = U_e/t_e$, we prove this directly from (20)

$$\frac{\partial^2 U}{\partial t_e \partial t_i} = -\frac{\partial(U_e/t_e)}{\partial t_i}, \quad \frac{\partial^2 U}{\partial t_i \partial t_e} = -\frac{\partial(U_i/t_i)}{\partial t_e}. \tag{39}$$

Therefore, as $\partial^2 U/(\partial t_e \partial t_i) = \partial^2 U/(\partial t_i \partial t_e)$, we have

$$[\nabla(ua)]^T = [\nabla(ua)]. \tag{40}$$

5 Optimal solutions

Three different problems will be used as illustrative examples: a uniformly loaded cantilever, a bending loaded "knee", and a biaxially loaded hole. For all these problems the optimizations are performed in at least three ways: optimization of thickness distribution only, optimization of material orientation only, and simultaneous optimization of thickness and orientation. All the results show that the two kinds of optimization parameters supplement each other, thus using the combined optimization really improves the stiffness. Thickness optimization redistributes the stresses to almost uniform energy density. Orientational optimization changes the stress distribution only slightly, but improves the stiffness by strain minimization. Thus, if stress concentration is also of concern, orientational optimization without thickness redistribution will not give the desired result.

In all the examples we have used the same way of visualization of the results. The *design is characterized* by thickness and orientation, which is shown by hatching the triangular finite elements in the direction of the "stiffest" principal material direction and with the hatch density proportional to the thickness. Dark areas are, therefore, areas with large thicknesses. The *response is characterized* by the distribution of the strain energy density and by the direction of the "larger" principal stress. The technique of hatching is again used with hatch direction equal to principal stress direction and with hatch density proportional to strain energy density. Dark areas are, therefore, areas with energy concentration.

5.1 Cantilever example

This first example is a relatively rough model of a uniformly loaded cantilever, but it gives the main aspects also obtained by the more detailed models shown later.

In Fig. 1 we show to the left (mainly) the design (mate-

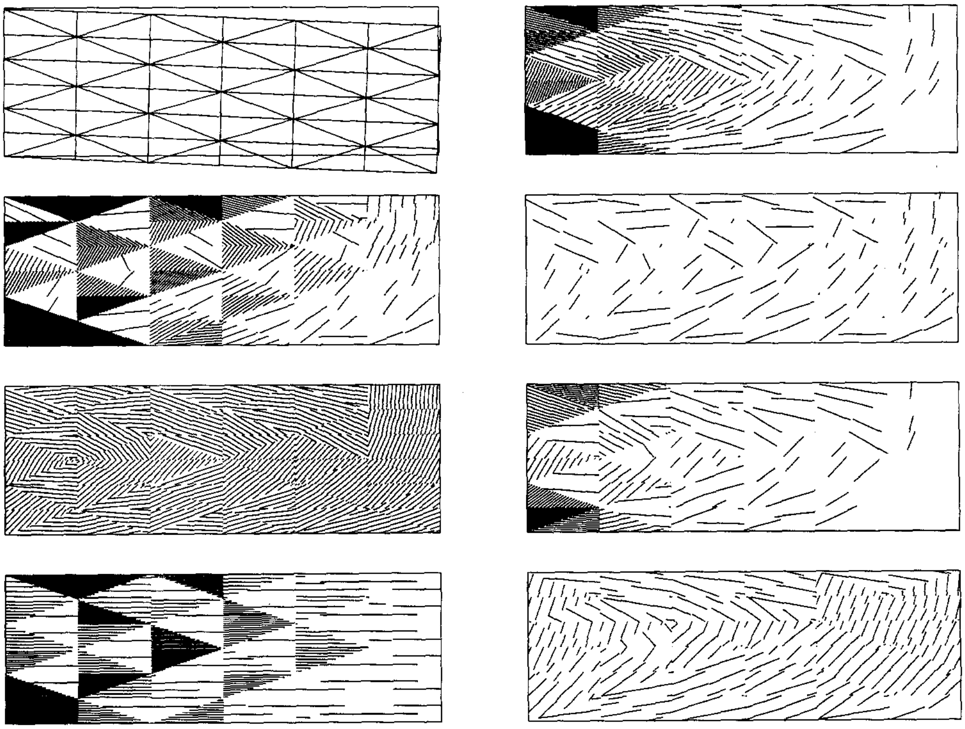

Fig. 1. Uniformly loaded cantilever -72 elements. Orthotropic material (80, 10, 0, 40, 0, 5).
1. row: Model and displacements with uniform thickness = 1; principal stress directions: mean $u = 661$; max $u = 3997$
2. row: Thickness and orientational optimization min $t = 0.01$; max $t = 7.7$; mean $u = 140$; max $u = 141$
3. row: Orientational optimization only ($t = 1$); mean $u = 327$; max $u = 2576$
4. row: Thickness optimization only; min $t = 0.01$; max $t = 6.3$; mean $u = 374$; max $u = 375$

rial orientation and thickness distribution) and to the right the corresponding, resulting stress directions and energy level (level by line intensity). In relation to each design, we give the mean strain energy density (measure of the overall stiffness/the compliance) and the maximum strain energy density (measure of the stress/strain concentration).

The first design (the upper one) has uniform thickness and the stiff material direction uniformly in the length direction. Therefore, the finite element model and displacement are shown instead of the design. The stress/energy visualized illustrates the high concentration $u_{\text{max}}/u_{\text{mean}} = 3997/661$. The second row shows how much can be gained by simultaneous thickness and orientational design, i.e. almost uniform energy density, and compliance improved by a factor of $4.7 = 661/140$. The third and fourth rows show only orientational and thickness optimization, respectively. We see that the improvement in compliance, $2.0 = 661/327$ and $1.8 = 661/374$, is almost the same, but only the thickness optimization levels out the energies. Thus, the general conclusion from the result in Fig. 1 is that both thickness and orientational optimization need to be performed.

It is natural to ask how sensitive the results just shown are to the analysis model, i.e. the finite element model. Therefore, the same physical problem is solved with a 720 element model (1440 design variables) to study the change from the 72 element model. The new results are shown in Fig. 2 with the same order of presentation as in Fig. 1. We see that the energy level for all the solutions has increased for the refined model and that especially the maximum energy level 32919/3997 is much better modelled. However, the general picture of the optimized designs is unchanged and the compliance improvements, 4.3=787/181, 1.8=787/432, 1.9=787/414, are also rather unchanged. Thus, at least for initial design improvements, the simple 72 element model is good enough.

Absolute size limits on the thicknesses naturally affect the solutions. The designs in Figs. 1 and 2 are based on a minimum thickness of 0.01 (mean thickness = 1), which was always active. However, no maximum thickness was active, and we see that the refined model offers the possibility of "introducing rips" with max $t = 25.6$ (practically non-realistic), as compared with max $t = 7.7$ in the model in Fig. 1. In Fig. 3 we show the results with enforced maximum thickness equal to 8, 5 and 3, respectively and compare this with the repeated result from Fig. 2. Note that especially the maximum energy is influenced by active maximum thickness.

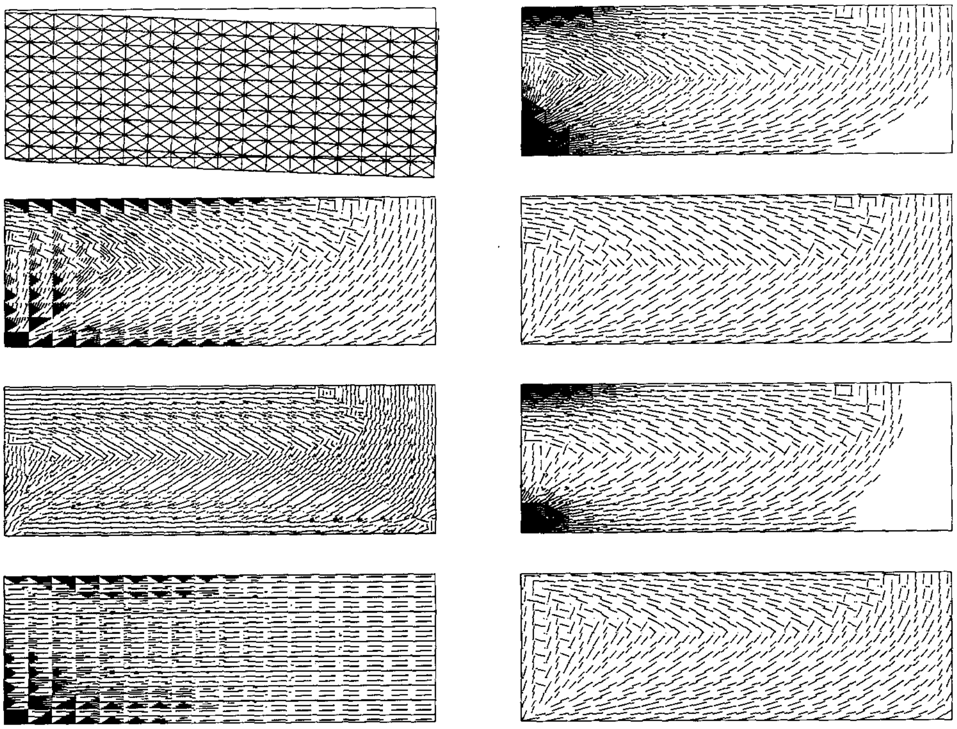

Fig. 2. Uniformly loaded cantilever -720 elements. Orthotropic material (80, 10, 0, 40, 0, 5).
1. row: Model and displacements with uniform thickness = 1; principal stress directions: mean u = 787; max u = 32919
2. row: Thickness and orientational optimization min t = 0.01; max t = 25.6; mean u = 181; max u = 199
3. row: Orientational optimization only (t = 1); mean u = 432; max u = 17661
4. row: Thickness optimization only; min t = 0.01; max t = 21.9 mean u = 414; max u = 450

5.2 Bending knee example

Now let us turn to quite a different problem, here termed a bending loaded knee. In Fig. 4, the 1408 element model (2816 design parameters) is shown, with the stress/energy solution for uniform design and optimized design, respectively. The optimal thickness distribution and the material orientation are again illustrated by hatch intensity and orientation. The improvement in compliance is by a factor of 4.8 = 726/151 and almost uniform energy density is obtained.

Again, the ratio of maximum thickness to mean thickness 39.1 is practically non-realistic, and in Fig. 5 we show solutions with maximum thickness set at 5 and 2.5, respectively.

Solutions with orientational design only and thickness design only are shown in Fig. 6. Being along the same lines as for the cantilever example, this figure is hopefully self-explanatory.

5.3 Hole example

The last example relates to a biaxially loaded hole. First, we shall optimize the thickness distribution based on isotropic material. The shape of the hole is the optimal one for min-max stress, i.e. an ellipse with the ratio of axes equal to that of the boundary stresses (Kristensen and Madsen 1976), here 3:2.

In Fig. 7 we see that the uniform thickness distribution gives a rather uniform energy distribution for this hole shape. As expected, the optimal thickness distribution is almost like a rip around the hole, and the compliance improvement is small: from 856 to 789. However, energy concentration is almost eliminated: from 3093 to 823.

With orthotropic material, the results shown in Fig. 8 are rather parallel. Here, the hole shape is different from the ellipse in Fig. 7, but still optimal with respect to min-max energy density. For a report on how this is obtained, see the recent M.Sc. thesis by Tobiesen and Jensen (1990).

Dealing with anisotropic material we now also optimize the material orientation for this problem. However, this violates the symmetry assumptions of the total hole model and the results in Fig. 9 should only be interpreted as quarter hole results. While the thickness optimization in Fig. 8 improved the stiffness only slightly: 1.1 = 3209/2900, the orientational optimization has a large effect: 2.2 = 3209/1429. As was seen in the other exam-

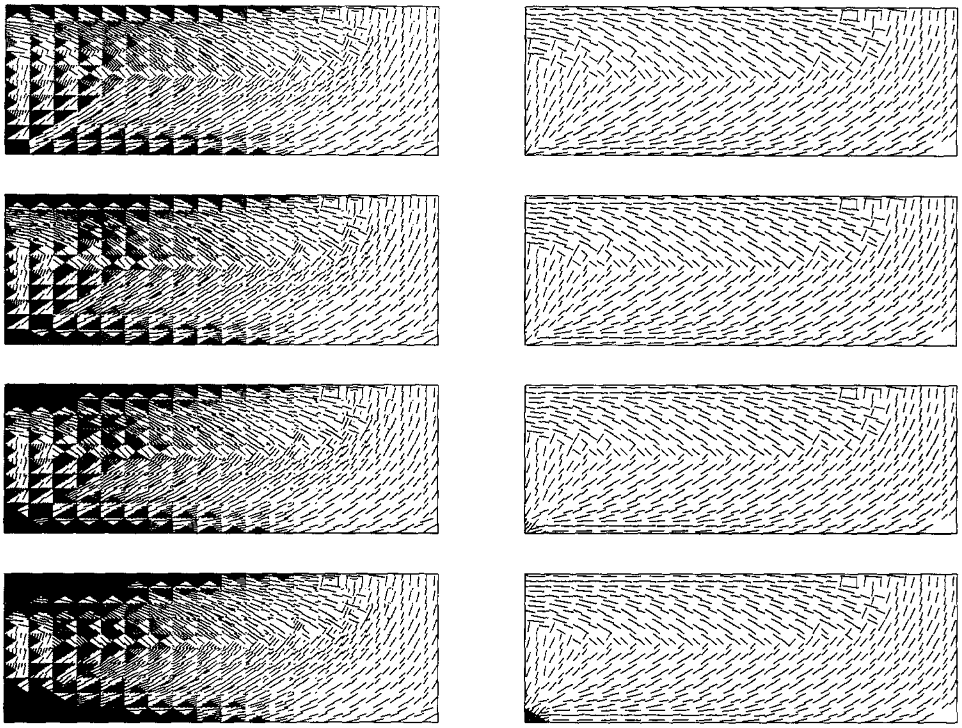

Fig. 3. Uniformly loaded cantilever -720 elements. Maximum thicknesses.
1. row: Thickness and orientational optimization min $t = 0.01$; max $t = 25.6$; mean $u = 181$; max $u = 199$
2. row: min $t = 0.01$; max $t = 8$; mean $u = 190$; max $u = 487$
3. row: min $t = 0.01$; max $t = 5$; mean $u = 201$; max $u = 932$
4. row: min $t = 0.01$; max $t = 3$; mean $u = 222$; max $u = 2170$

ples, the simultaneous optimization has the combined effect: $2.5 = 3209/1299$.

6 Conclusion

Optimization problems with a single active constraint (thickness design with given volume) or without constraints (orientational design) can be solved by simple iterative redesigns based on derived optimality criteria.

For the thickness design this redesign procedure is studied by deriving higher order sensitivities. Second order sensitivities of total strain energy are evaluated as first order sensitivities of local (element) specific strain energy.

For the orientational design a normal gradient technique will generally not work because many local optima exist. Therefore, design changes in each redesign must be based on a criterion that identifies the orientation giving a global minimum of the strain energy.

For the optimal material orientation we obtain coinciding principal stresses and strain directions. This is used as a "test optimality criterion" and can also be utilized during iteration.

Optimization of thickness distribution for anisotropic materials (and even a class of non-linearity as well) is no more complicated than for simple linear isotropic materials. The criterion of uniform energy density still holds.

References

Banichuk, N.V. 1983: *Problems and methods of optimal structural design.* New York: Plenum Press

Kristensen, E.S.; Madsen, N.F. (1976): On the optimum shape of fillets in plates subjected to multiple in-plane loading cases. *Int. J. Numer. Meth. Eng.* **10**, 1007-1019

Landriani, G.S.; Rovati, M. (1991): Optimal design for 2-D structures made of composite materials. *ASME J. Engrg. Mat. Techn.* (to appear)

Masur, E.F. (1970): Optimum stiffness and strength of elastic structures. *J. Eng. Mech. Div. ASCE* **EM5**, 621-649

Pedersen, P. (1989): On optimal orientation of orthotropic materials. *Struct. Optim.* **1**, 101-106

Pedersen, P. (1990): Bounds on elastic energy in solids of orthotropic materials. *Struct. Optim.* **2**, 55-63

Rozvany, G.I.N. (1989): *Structural design via optimality criteria.* Dordrecht: Kluwer

Thomsen, J. (1991): Optimization of composite discs. *Struct. Optim.* (accepted)

Tobiesen, L.; Jensen, S.H. (1990): *Optimal hulfacon i ortotrop skive.* M. Sc. Thesis, DTH (in Danish)

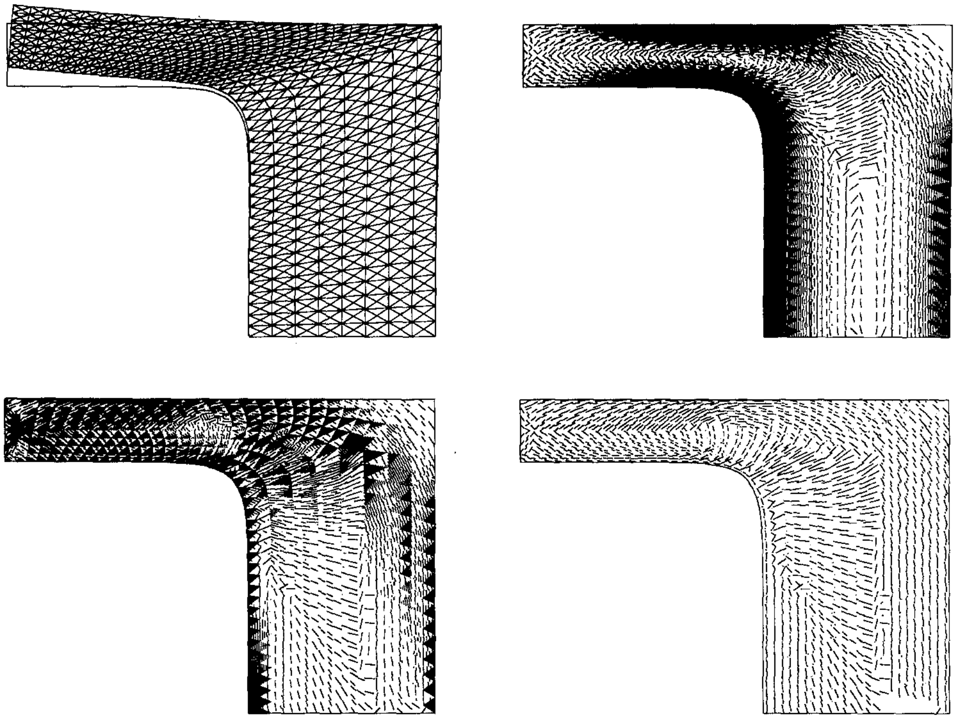

Fig. 4. Bending loaded knee -1408 elements. Orthotropic material (400, 30, 0, 100, 0, 75).
1. row: Model and displacements with uniform thickness = 1; principal stress directions: mean $u = 726$; max $u = 5002$
2. row: Thickness and orientational optimization min $t = 0.01$; max $t = 39.1$; mean $u = 151$; max $u = 161$

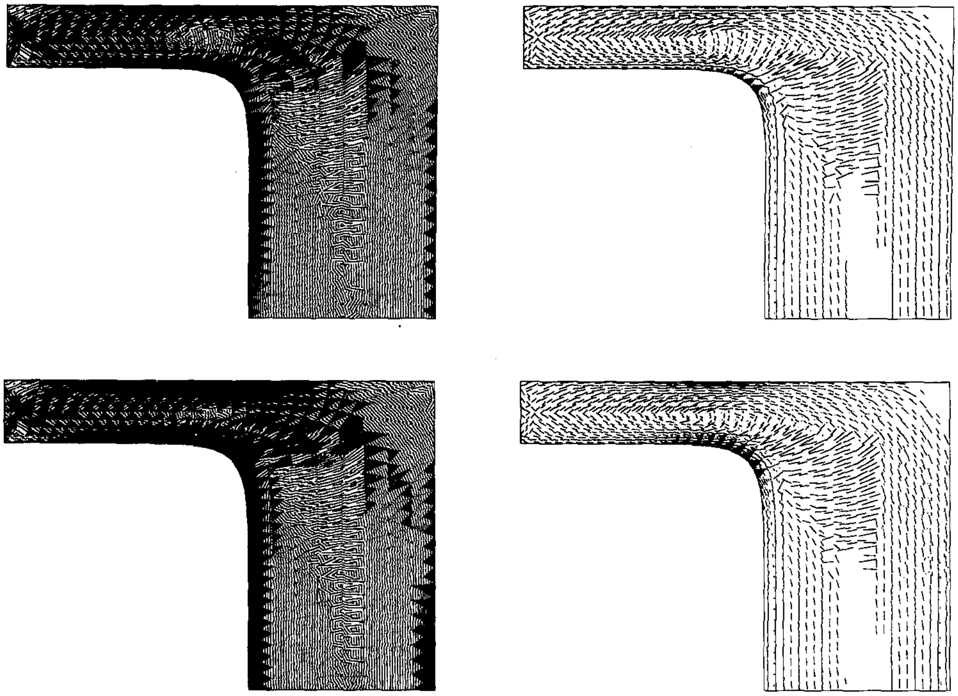

Fig. 5. Bending loaded knee -1408 elements. Maximum thicknesses.
1. row: Thickness and orientational optimization min $t = 0.5$; max $t = 5$; mean $u = 203$; max $u = 864$
2. row: Thickness and orientational optimization min $t = 0.5$; max $t = 2.5$; mean $u = 245$; max $u = 2217$

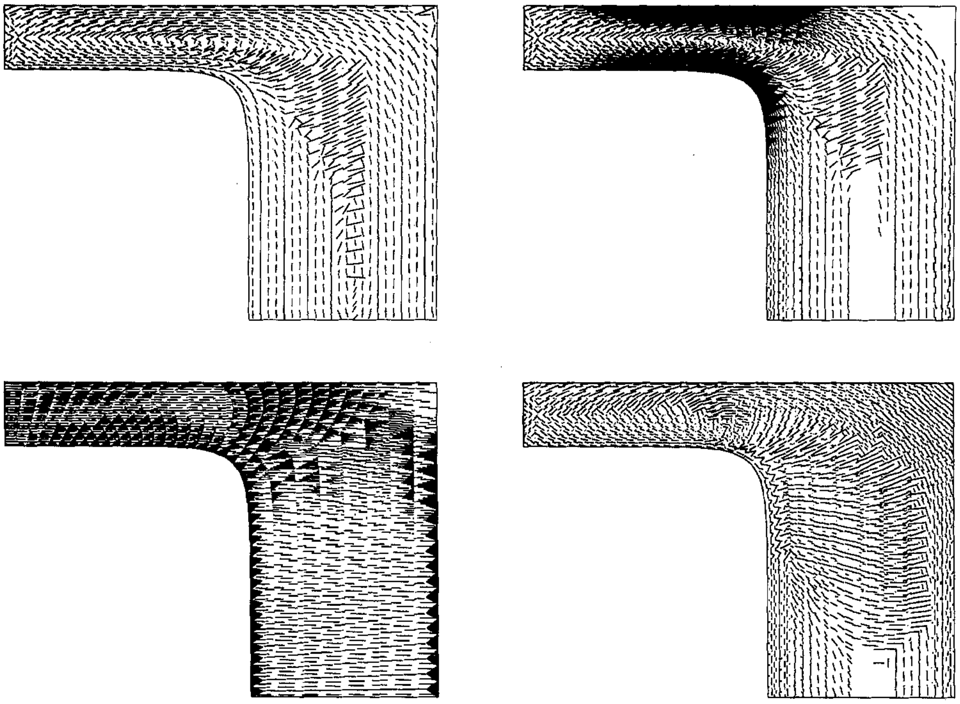

Fig. 6. Bending loaded knee -1408 elements. Individual optimizations.
1. row: Orientational optimization only ($t = 1$); mean $u = 452$; max $u = 12053$
2. row: Thickness optimization only; min $t = 0.01$; max $t = 28.0$; mean $u = 287$; max $u = 328$

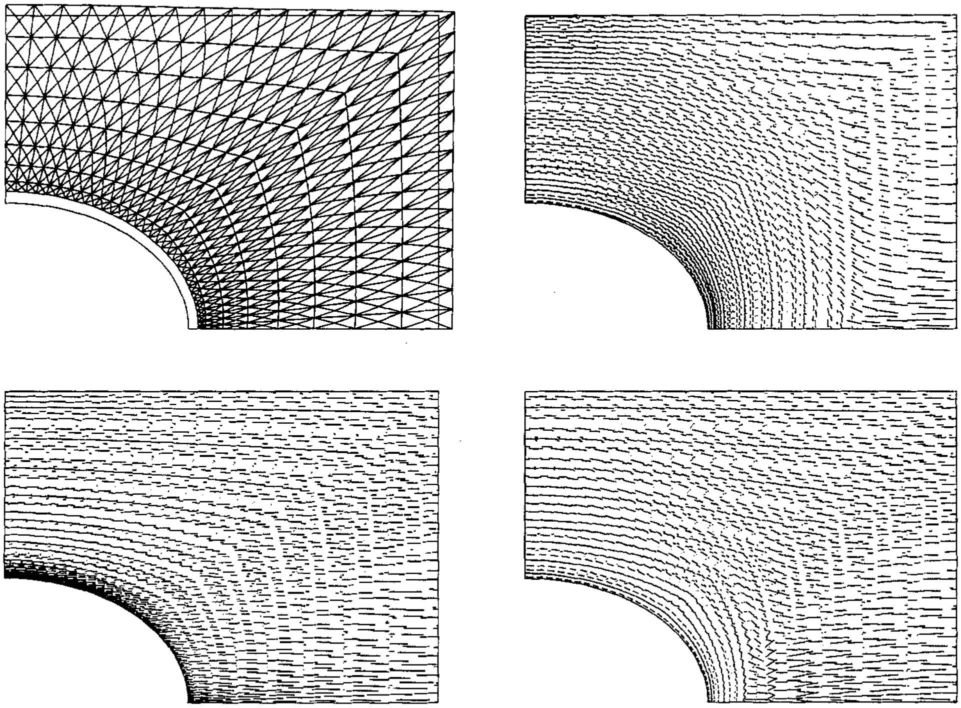

Fig. 7. Biaxially loaded hole -1024 elements isotropic material.
1. row: Model and displacements with uniform thickness = 1; principal stress directions: mean $u = 856$; max $u = 3093$
2. row: Thickness optimization min $t = 0.15$; max $t = 27.3$; mean $u = 789$; max $u = 823$

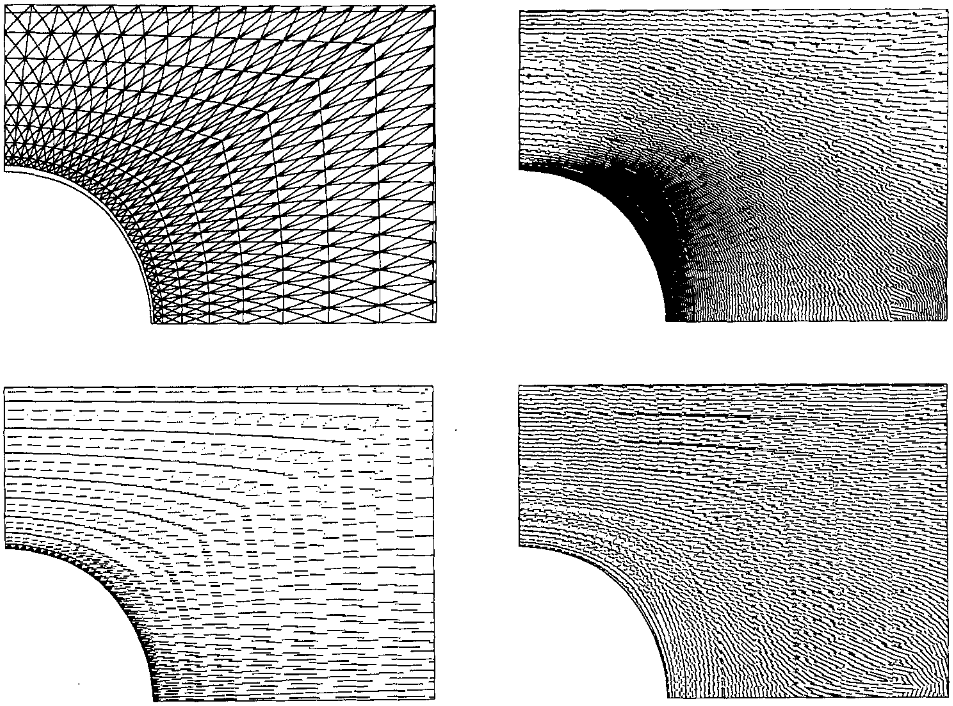

Fig. 8. Biaxially loaded hole -1024 elements. Orthotropic material (108, 3.7, 0, 12.4, 0, 4.7).
1. row: Model and displacements with uniform thickness = 1; principal stress directions: mean u = 3209; max u = 12370
2. row: Thickness optimization min t = 0.04; max t = 21.8; mean u = 2900; max u = 3320

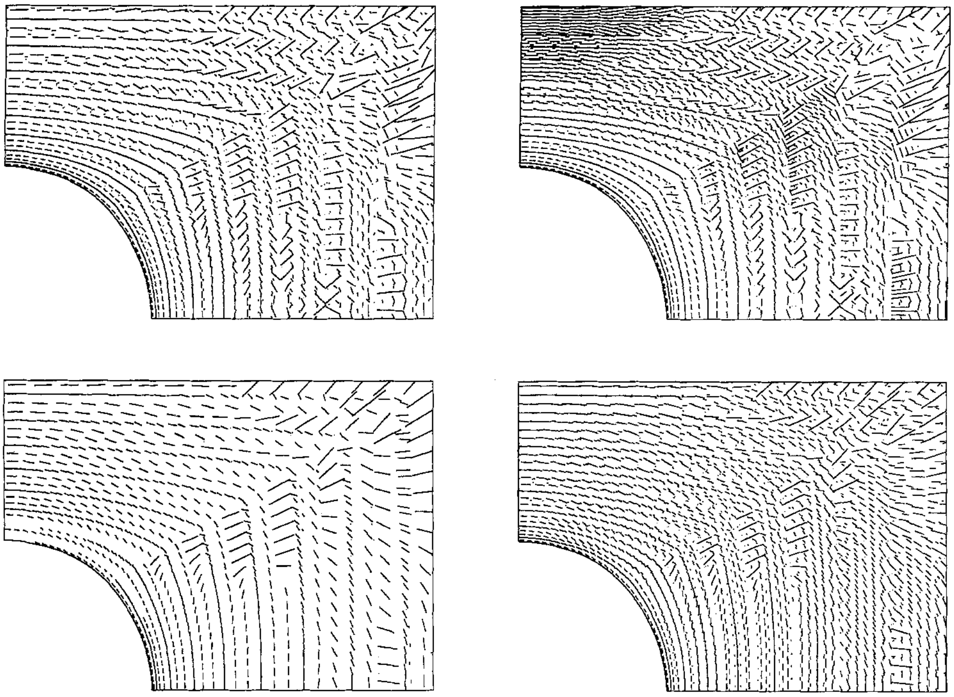

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Fig. 9. Biaxially loaded hole -1024 elements. Orthotropic material -quarter hole.
1. row: Thickness and orientational optimization min t = 0.01; max t = 7.7; mean u = 1299; max u = 1436
2. row: Orientational optimization only (t = 1); mean u = 1425; max u = 3887