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Some general optimal design results using anisotropic, power law nonlinear elasticity*

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Abstract Recent results on optimal design with anisotropic materials and optimal design of the materials themselves are in most cases based on the assumption of linear elasticity. We shall extend these results to the nonlinear model classified as powerlaw elasticity. These models return proportionality between elastic strain energy density and elastic stress energy density. This is shown to imply localized sensitivity analysis for the total elastic energy, and for a number of optimal design problems this immediately gives practical, general results.

For two- and three-dimensional problems the effective strain and the effective stress are defined from an energy consistent point of view, and it is shown that a definition generalizing the von Mises stress must be used. The optimization criterion of uniform energy density also holds for nonlinear materials, and several general conclusions can be based on this fact. Applications to size design illustrate this.

For stiffness optimization the ultimate optimal material design problem is addressed. The validity of recent results are extended to nonlinear materials, and a simple proof based on constraint on the Frobenius norm is given. We note that the optimal material is orthotropic, that principal directions of material, strain and stress are aligned, and that there is no shear stiffness. In reality, the constitutive matrix only has one nonzero eigenvalue and the material therefore has stiffness only in relation to the specified strain condition. Results related to orientational design with orthotropic materials are also focused on.

With respect to strength optimization, i.e. the more difficult problem with local constraints, we shall comment on the influence of the different strength criteria.

1 Introduction

The increasing use of anisotropic materials and the ability to tailor material to specific needs have presented a challenge to research on optimal design. Results have been available at least since the early eighties and some of these results are available in textbooks, see Haftka *et al.* (1990).

In addition to the traditional parameter classification of optimal design as size, shape and topology design, the advanced materials make it necessary to deal with orientational design and with micromechanics design of the material itself.

In most cases, the optimizations with the advanced materials are based on linear elasticity or on perfect plasticity. For early research on the sensitivity analysis related to nonlinear and transient problems, see Cardoso and Arora (1988) or Choi and Santos (1987). For a more recent overview, see Michaleris *et al.* (1994).

The goal of the present paper is to put forward some general results that can be used without numerical sensitivity analysis. That is, we want to deal with nonlinear problems but with the simplest possible extension from linear elasticity, which is the power-law nonlinear elasticity. Because this class of problems can also be used to describe plasticity theory in the deformation plasticity formulation and stationary creep, this simple extension covers a broad range of practical important problems.

A number of new results are included in the paper. From the localized sensitivity analysis the coalignment of principal directions of strain, stress and material, also follows for nonlinear elasticity. So do the extension of the ultimate optimal material design. For a class of problems without bounds on the design variables the optimal designs are independent of the power of the nonlinearity. We focus on the possible changes in optimal design when the von Mises stress constraint is applied. A more general discussion of the alternative effective stress/strain measures for anisotropic and/or compressive materials is included and the paper ends with comments on shape optimization for minimum energy concentration.

2 Analysis, effective stress/strain and energy densitities

The analysis of anisotropic, nonlinear elastic structures/continua will be presented in the secant formulation, as described by Pedersen and Taylor (1993). We shall here concentrate on the compliance matrix and the definition of effective stress. The preferred effective stress is defined in an energy-consistent way and the relation to the von Mises stress is pointed out. Then, the very important relation between strain and stress energy densities is established. This relation is not well-known, but was already mentioned by Hill (1956). We shall use the following notation: for the stress and strain vectors: $\{\sigma\}$ and $\{\varepsilon\}$, which for 3D-problems, each have 6 elements; for the scalar effective stress, strain the notation σ_e, ε_e; and for the constant reference modulus of elasticity E. The nonlinearity is described by the powers n or p, where $n = 1/p$.

2.1 The nondimensional compliance matrix

In the compliance description (see end of this section)

$\{\varepsilon\} = \sigma_e^{n-1} E^{-n} [\beta] \{\sigma\}\,,$

* Dedicated to Professor Franz Ziegler on the occasion of his 60-th birthday in December 1997

$$\sigma_e^2 := \{\sigma\}^T[\beta]\{\sigma\}\,, \tag{1}$$

the nondimensional matrix $[\beta]$ describes the anisotropy, and the only restriction on the matrix is that it must be symmetric and positive semidefinite, i.e

$$[\beta]^T = [\beta] \text{ and } [\beta] \geq 0\,, \tag{2}$$

Now, separating the stress vector into normal terms with index N and shear terms with index S, we have

$$\{\sigma\}^T = \left\{\{\sigma\}_N^T\ \{\sigma\}_S^T\right\} = \left\{\{\sigma_{11}\ \sigma_{22}\ \sigma_{33}\}\left\{\sqrt{2}\sigma_{12}\ \sqrt{2}\sigma_{13}\ \sqrt{2}\sigma_{23}\right\}\right\}\,, \tag{3}$$

and, accordingly, we have the following $[\beta]$ matrix:

$$[\beta] = \begin{bmatrix} [\beta]_{NN} & [\beta]_{NS} \\ [\beta]_{NS}^T & [\beta]_{SS} \end{bmatrix}\,,$$

with the submatrices

$$[\beta]_{NN} = \begin{bmatrix} \beta_{1111} & \beta_{1122} & \beta_{1133} \\ \beta_{1122} & \beta_{2222} & \beta_{2233} \\ \beta_{1133} & \beta_{2233} & \beta_{3333} \end{bmatrix}\,,$$

$$[\beta]_{NS} = \sqrt{2}\begin{bmatrix} \beta_{1112} & \beta_{1113} & \beta_{1123} \\ \beta_{2212} & \beta_{2213} & \beta_{2223} \\ \beta_{3312} & \beta_{3313} & \beta_{3323} \end{bmatrix}\,,$$

$$[\beta]_{SS} = 2\begin{bmatrix} \beta_{1212} & \beta_{1213} & \beta_{1223} \\ \beta_{1213} & \beta_{1313} & \beta_{1323} \\ \beta_{1223} & \beta_{1323} & \beta_{2323} \end{bmatrix}\,. \tag{4}$$

In the orthotropic directions the *orthotropic case* is characterized by the simpler form

$$[\beta]_{SS} = 2\begin{bmatrix} \beta_{1212} & 0 & 0 \\ 0 & \beta_{1313} & 0 \\ 0 & 0 & \beta_{2323} \end{bmatrix}\,, \quad [\beta]_{NS} = [0]\,, \tag{5}$$

and is thus described by only 9 parameters. The *isotropic case* is described by only 2 parameters with

$$[\beta]_{NN} = \begin{bmatrix} \beta_{1111} & \beta_{1122} & \beta_{1122} \\ \beta_{1122} & \beta_{1111} & \beta_{1122} \\ \beta_{1122} & \beta_{1122} & \beta_{1111} \end{bmatrix}\,,$$

$$[\beta]_{SS} = \begin{bmatrix} \beta_{1111}-\beta_{1122} & 0 & 0 \\ 0 & \beta_{1111}-\beta_{1122} & 0 \\ 0 & 0 & \beta_{1111}-\beta_{1122} \end{bmatrix}\,,$$

$$[\beta]_{NS} = [0]\,. \tag{6}$$

For the following analysis we shall list the *conditions of incompressibility* of an orthotropic description for any stress state, which is

$$\begin{aligned} &\beta_{1111} + \beta_{1122} + \beta_{1133} = 0\,, \\ &\beta_{1122} + \beta_{2222} + \beta_{2233} = 0\,, \\ &\beta_{1133} + \beta_{2233} + \beta_{3333} = 0\,, \end{aligned} \tag{7}$$

while incompressibility in relation to hydrostatic pressure is obtained by the single condition

$$\beta_{1111} + \beta_{2222} + \beta_{3333} + 2(\beta_{1122} + \beta_{1133} + \beta_{2233}) = 0\,. \tag{8}$$

2.2 The von Mises effective stress

In traditional plasticity theory the effective stress is not defined as shown in (1), but instead by means of the von Mises stress σ_M defined by

$$\sigma_M^2 := \frac{3}{2}\{s\}^T\{s\}\,, \tag{9}$$

where $\{s\}$ is the vector of deviatoric stresses, i.e. the hydrostatic pressure is eliminated. Pedersen (1987) shows that this deviatoric stress vector can be obtained by a projection with the projection matrix $[\mathbf{P}]$ ($[\mathbf{P}]^T = [\mathbf{P}]$ and $[\mathbf{P}]\,[\mathbf{P}] = [\mathbf{P}]$)

$$\{s\} = [\mathbf{P}]\{\sigma\}\,,$$

$$[\mathbf{P}] := \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}\,. \tag{10}$$

Inserting (10) in (9), we have the von Mises stress in terms of the total stresses

$$\sigma_M^2 = \frac{3}{2}\{\sigma\}^T[\mathbf{P}]\{\sigma\}\,, \tag{11}$$

and comparing this to the definition of σ_e^2 by (1), we *only have* $\sigma_M^2 = \sigma_e^2$ *for* the specific compliance matrix that corresponds to an *isotropic and incompressible* material,

$$[\beta]_{\text{isotropic and incompressible}} = \begin{bmatrix} 1 & -0.5 & -0.5 & 0 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.5 \end{bmatrix}\,. \tag{12}$$

For other materials there will generally be a difference between the von Mises stress σ_M and our energy-based effective stress σ_e,

$$\sigma_e^2 - \sigma_M^2 = \{\sigma\}^T \begin{bmatrix} \beta_{1111}-1 & \beta_{1122}+0.5 & \beta_{1133}+0.5 & 0 & 0 & 0 \\ \beta_{1122}+0.5 & \beta_{2222}-1 & \beta_{2233}+0.5 & 0 & 0 & 0 \\ \beta_{1133}+0.5 & \beta_{2233}+0.5 & \beta_{3333}-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\beta_{1212}-1.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\beta_{1313}-1.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\beta_{2323}-1.5 \end{bmatrix} \{\sigma\}\,, \tag{13}$$

here related to an orthotropic description.

In optimal design the solution will naturally depend on the chosen objective and constraints. Therefore, with $\sigma_e \neq \sigma_M$, we obtain solutions related to the specific choice. We shall return to this and just here point out that the more general results are based on the energy related definition σ_e.

2.3 The Hill strength measure

The Hill (1950) strength reference F_{Hill} for anisotropic materials is

$$F_{\text{Hill}} = F(\sigma_{22}-\sigma_{33})^2 + G(\sigma_{33}-\sigma_{11})^2 + H(\sigma_{11}-\sigma_{22})^2 + 2L\sigma_{23}^2 + 2M\sigma_{13}^2 + 2N\sigma_{12}^2\,, \tag{14}$$

which, in matrix notation with the definition (3), is written

$F_{\text{Hill}} =$

$$\{\sigma\}^T \begin{bmatrix} G+H & -H & -G & 0 & 0 & 0 \\ -H & F+H & -F & 0 & 0 & 0 \\ -G & -F & F+G & 0 & 0 & 0 \\ 0 & 0 & 0 & N & 0 & 0 \\ 0 & 0 & 0 & 0 & M & 0 \\ 0 & 0 & 0 & 0 & 0 & L \end{bmatrix} \{\sigma\} . \quad (15)$$

We see that for hydrostatic stress $\{\sigma\}^T = \overline{\sigma}\{1\,1\,1\,0\,0\,0\}$ we obtain $F_{\text{Hill}} = 0$ such that this measure is insensitive to hydrostatic pressure. Note also that the Hill measure has only 6 parameters (plus one constraint for hydrostatic pressure), while the orthotropic $[\beta]$ matrix has 9 parameters.

2.4 *Strain energy density and stress energy density*

As shown by Pedersen and Taylor (1993), the secant description (1) can also be stated,

$$\{\sigma\} = \varepsilon_e^{p-1} E[\alpha]\{\varepsilon\} ,$$
$$\varepsilon_e^2 := \{\varepsilon\}^T[\alpha]\{\varepsilon\} , \quad (16)$$

where the nondimensional constitutive matrix $[\alpha]$ is just the inverse of the compliance matrix $[\beta]$ (here assumed positive definite)

$$[\alpha] = [\beta]^{-1} , \quad (17)$$

with submatrix definitions in complete analogy to (4)-(6).

The integrated strain energy density u will be

$$u = E\frac{1}{p+1}\varepsilon_e^{p+1} , \quad (18)$$

and the stress energy density u^C will be

$$u^C = \frac{1}{E^n}\frac{1}{n+1}\sigma_e^{n+1} , \quad (19)$$

where $n = 1/p$. It follows (most easily from $u + u^C = \sigma_e\varepsilon_e$) that we have the following important relation:

$$u^C = pu , \quad (20)$$

and this relation gives rise to simplified sensitivity analysis as well as rather general optimality criteria.

An alternative to starting with (1) and then derive (19) is to start with (19) and then derive (1). From

$$\{\varepsilon\} = \frac{\mathrm{d}u^C}{\mathrm{d}\{\sigma\}} = \frac{1}{E^n}\frac{1}{n+1}(n+1)\sigma_e^n\frac{\mathrm{d}\sigma_e}{\mathrm{d}\{\sigma\}} , \quad (21)$$

and then with the definition of σ_e^2

$$2\sigma_e\frac{\mathrm{d}\sigma_e}{\mathrm{d}\{\sigma\}} = 2[\beta]\{\sigma\} , \quad (22)$$

we have

$$\{\varepsilon\} = \sigma_e^{n-1}E^{-n}[\beta]\{\sigma\} . \quad (23)$$

3 Potentials and sensitivity analysis - localized and nonlocalized

The general equation of energy equilibrium is

$$U + U^C + U_{\text{ext}} = 0 , \quad (24)$$

with strain and stress energy from the corresponding densities

$$U = \int u\,\mathrm{d}V , \quad U^C = \int u^C\,\mathrm{d}V , \quad (25)$$

and the external potential by

$$U_{\text{ext}} = -\left(\int T_i v_i\,\mathrm{d}A + \int p_i v_i\,\mathrm{d}V\right) , \quad (26)$$

with surface tractions T_i, volume forces p_i and A surrounding the volume V.

With material resulting in the density relation (20) everywhere in the structure, equilibrium (24) will be

$$(1+p)U = -U_{\text{ext}} . \quad (27)$$

Defining the total potential Π by

$$\Pi := U + U_{\text{ext}} , \quad (28)$$

we have [again based on (20)]

$$U = \frac{1}{p}U^C = -\frac{1}{p}\Pi = \frac{-1}{1+p}U_{\text{ext}} , \quad (29)$$

and by $p > 0$ and $U > 0$ we have $U^C > 0$, $\Pi < 0$ and $U_{\text{ext}} < 0$. From this follows that design for the different extremum problems

$$\min U = \frac{1}{p}\min U^C = -\frac{1}{p}\max \Pi = \frac{-1}{1+p}\max U_{\text{ext}} \quad (30)$$

are equivalent and that their values at the solutions are as given.

The derivative of the strain energy U with respect to the arbitrary design parameter h follows from (29)

$$\frac{\mathrm{d}U}{\mathrm{d}h} = -\frac{1}{p}\frac{\mathrm{d}\Pi}{\mathrm{d}h} =$$
$$-\frac{1}{p}\left[\left(\frac{\partial \Pi}{\partial h}\right)_{\text{fixed strains}} + \left(\frac{\partial \Pi}{\partial \varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}h}\right)\right] , \quad (31)$$

and for *design-independent external loads* we have from (28) with $(\partial U_{\text{ext}}/\partial h)_{\text{fixed strains}} = 0$

$$\left(\frac{\partial \Pi}{\partial h}\right)_{\text{fixed strains}} = \left(\frac{\partial U}{\partial h}\right)_{\text{fixed strains}} , \quad (32)$$

which, with stationary total potential $\partial\Pi/\partial\varepsilon = 0$ (virtual work principle), gives the important result

$$\left(\frac{\mathrm{d}U}{\mathrm{d}h}\right) = -\frac{1}{p}\left(\frac{\partial U}{\partial h}\right)_{\text{fixed strains}} . \quad (33)$$

For a local design parameter h_i that only changes the design in the domain i, this gives the possibility of a *localized determination* of the sensitivity for the total elastic energy U

$$\frac{\mathrm{d}U}{\mathrm{d}h_i} = -\frac{1}{p}\left[\frac{\partial(\overline{u}_i V_i)}{\partial h_i}\right]_{\text{fixed strains}} , \quad (34)$$

where $\overline{u}_i$ is the mean energy density in the domain of h_i and where V_i is the corresponding volume. We note that the only difference between linear ($p = 1$) and nonlinear material is the factor $1/p$.

In optimal design for global stiffness the simple result (34) can be applied. However, in optimal design for local strength, the sensitivity analysis is more complicated. Let us therefore next relate sensitivity to u (at a specific point) and again with respect to the arbitrary design parameter h

$$\frac{\mathrm{d}u}{\mathrm{d}h} = \left(\frac{\partial u}{\partial h}\right)_{\text{fixed strains}} + \left(\frac{\partial u}{\partial \varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}h}\right) . \quad (35)$$

The first term will only be nonzero when h has direct influence on the material at the specific point for u. The second term will always be nonzero, i.e. change in strain field will change the field of energy density. Traditional sensitivity analysis with pseudo loads can still be performed. In the secant formulation (16), we will after iteration (redefining the ε_e field) have the global stiffness matrix [S] which, in a finite element formulation, gives the result by

$$[\mathbf{S}]\{\mathbf{D}\} = \{\mathbf{A}\} , \quad (36)$$

where $\{\mathbf{A}\}$ contains the given nodal loads and $\{\mathbf{D}\}$ the corresponding nodal displacements. With design-independent loads, $\mathrm{d}\{\mathbf{A}\}/\mathrm{d}h = \{\mathbf{0}\}$, we obtain $\mathrm{d}\{\mathbf{D}\}/\mathrm{d}h$ from

$$[\mathbf{S}]\frac{\mathrm{d}\{\mathbf{D}\}}{\mathrm{d}h} = -\frac{\mathrm{d}[\mathbf{S}]}{\mathrm{d}h}\{\mathbf{D}\}, \tag{37}$$

and to find $\mathrm{d}[\mathbf{S}] = \sum \mathrm{d}[\mathbf{S}]_i$, we need for each element the sensitivity of the constitutive secant matrix. With the constitutive secant matrix $\varepsilon_e^{p-1}E[\boldsymbol{\alpha}]$, this can be found from

$$\frac{\mathrm{d}(\varepsilon_e^{p-1}[\boldsymbol{\alpha}])}{\mathrm{d}h} = (p-1)\varepsilon_e^{p-3}[\boldsymbol{\alpha}]\left(\{\boldsymbol{\varepsilon}\}^T[\boldsymbol{\alpha}]\frac{\mathrm{d}\{\boldsymbol{\varepsilon}\}}{\mathrm{d}h} + \frac{1}{2}\{\boldsymbol{\varepsilon}\}^T\frac{\mathrm{d}[\boldsymbol{\alpha}]}{\mathrm{d}h}\{\boldsymbol{\varepsilon}\}\right) + \varepsilon_e^{p-1}\frac{\mathrm{d}[\boldsymbol{\alpha}]}{\mathrm{d}h}, \tag{38}$$

and we note that the term $\mathrm{d}\{\boldsymbol{\varepsilon}\}/\mathrm{d}h$ must be iteratively determined from $\mathrm{d}\{\mathbf{D}\}/\mathrm{d}h$.

The general conclusion is that the strength sensitivities are much more involved than the stiffness sensitivities, but can be found from analytical or semianalytical sensitivity analysis.

4 Optimal design for stiffness

Among the different formulations for optimal design of global stiffness, we choose to minimize the elastic energy U with a given amount of mass $\overline{M}$, i.e.

$$\min U \text{ for } M = \overline{M}. \tag{39}$$

4.1 Homogeneous relations

The total mass M we write as the sum of masses M_i for the design subdomains (say domains of one finite element)

$$M = \sum_i M_i, \tag{40}$$

and we assume that the design parameter h_i only influences the design in subdomain i. Furthermore, we shall assume that M_i is dependent on h_i in a homogeneous way, i.e.

$$M_i = h_i^m \mathcal{M}_i, \tag{41}$$

where $\mathcal{M}_i$ is independent of h_i. We shall later show that this assumption is not very restrictive. Inserting (41) in (40), we have

$$M = \sum_i h_i^m \mathcal{M}_i, \tag{42}$$

and thus we have the simple sensitivity result

$$\frac{\mathrm{d}M}{\mathrm{d}h_i} = m h_i^{m-1}\mathcal{M}_i = \frac{mM_i}{h_i}. \tag{43}$$

The strain energy U we also write as the sum over energies U_i in the design subdomains

$$U = \sum_i U_i. \tag{44}$$

Here, too, we assume a homogeneous relation, but only relative to a fixed strain field

$$U_i = h_i^n \mathcal{U}_i, \tag{45}$$

i.e. $\mathcal{U}_i$ is explicitly independent of h_i. We have

$$U = \sum_i h_i^n \mathcal{U}_i, \tag{46}$$

and then, when using (33), we have the simple sensitivity result

$$\frac{\mathrm{d}U}{\mathrm{d}h_i} = -\frac{1}{p}\left(\frac{\partial U}{\partial h_i}\right)_{\text{fixed strains}} = -\frac{1}{p}\left(\frac{\partial U_i}{\partial h_i}\right)_{\text{fixed strains}} = -\frac{n}{p}h_i^{n-1}\mathcal{U}_i = -\frac{n}{p}U_i/h_i. \tag{47}$$

The Lagrange function $\mathcal{L}$ corresponding to the optimization problem (39) is $\mathcal{L} = U - \lambda(M - \overline{M})$, where λ is a Lagrangian multiplier with dimension energy per mass. Stationarity of $\mathcal{L}$ with respect to all design variables h_i gives

$$\lambda\frac{\mathrm{d}M}{\mathrm{d}h_i} = \frac{\mathrm{d}U}{\mathrm{d}h_i}, \tag{48}$$

with the same constant λ for all design subdomains i. Inserting (43) and (47) in (48), we have

$$\lambda m\frac{M_i}{h_i} = -\frac{n}{p}\frac{U_i}{h_i} = -\frac{n}{p}\frac{\overline{u}_i}{\overline{\rho}_i}\frac{M_i}{h_i}, \tag{49}$$

where we have introduced the subdomain mean energy density $\overline{u}_i$ and the subdomain mean mass density $\overline{\rho}_i$ from

$$U_i = \overline{u}_i V_i = \overline{u}_i M_i/\overline{\rho}_i, \tag{50}$$

with V_i being the subdomain volume. From (49) we read the general result

$$\overline{u}_i/\overline{\rho}_i = U_i/M_i = \tilde{\lambda}, \tag{51}$$

where $\tilde{\lambda} = -\lambda mp/n$ is just a new constant. In words, the result (51) reads:

> For optimal stiffness design with homogeneous assumptions the ratio between subdomain energy and subdomain mass should be the same in all the design subdomains. (52)

Let us relate the homogeneous assumptions to some specific models. For size optimization of *trusses* with bar areas a_i as design variables, we have given mass densities $\overline{\rho}_i = \rho_i = \rho$ and uniform energy density in the bars $\overline{u}_i = u_i$. We thus have

$$u_i = \overline{u}, \tag{53}$$

where $\overline{u}$ is the total mean energy density. For thickness design of *membrane* problems of given mass density $\overline{\rho}_i = \rho_i = \rho$, we again obtain the result (53) if we are modelling with constant stress/strain/energy density elements, and if the design subdomains have nonuniform energy densities, we obtain

$$\overline{u}_i = \overline{u}, \tag{54}$$

again with $\overline{u}$ being the total mean energy density. For *3-D continua* problems the design variables can be the subdomain mean mass density $\overline{\rho}_i$ and then the optimality criterion is as stated in (51)-(52).

For *beams* based on Bernoulli-Euler theory we have the subdomain energy U_i in a prismatic part (constant cross-sectional moment of inertia I_i)

$$U_i = \frac{1}{2}EI_i\int(\mathrm{d}^2y/\mathrm{d}x^2)\,\mathrm{d}x, \tag{55}$$

and thus with the beam width as design variable we have in (45) $n = 1$, with height $n = 3$ and with area almost $n = 2$. For all these cases we have $m = 1$ in (41). For plates in pure bending we obtain similar results, but for coupled membrane/bending cases the homogeneous relations no longer hold.

The general result holds for 1D, 2D and 3D problems, for anisotropy as well as for isotropy, and is also independent of the parameter of the nonlinearity $n = 1/p$. With uniform energy density we obtain uniform effective strain, as follows from (18), and uniform effective stress (19). Then by (1) or (16) we obtain a uniform constitutive secant matrix. For problems without lower and upper bounds on the design variables; problems where the necessary optimality condition (53) is satisfied everywhere, we can therefore conclude that:

The optimal mass density distribution in independent of the power in the constitutive relation. (56)

Pedersen and Taylor (1993) illustrated this for thickness distributions in two-dimensional problems. Note that it only holds for problems with *one load case.*

4.2 Orientational design

For linear elastic, orthotropic materials the orientational design to obtain maximum stiffness was treated by Pedersen (1991). It is shown that significant stiffness increase can be obtained, but that only with thickness/density design can we obtain the uniform energy density that is the basis for the statement (56). However, the general results that follow from the sensitivity analysis are valid. This implies that for power-law, nonlinear elastic materials as well, the optimal orientation of orthotropic materials will result in:

Alignment of principal strains and stresses, and in most cases also alignment with the material principal axes, independent of the power in the constitutive equation. (57)

The resulting optimal orientation will not necessarily be independent of the power because the principal strain/stress direction may change with the power of the nonlinearity.

4.3 Ultimate optimal material design

In ultimate optimal material design we represent the material properties in the most general form possible for an elastic continuum, namely the unrestricted set of elements of positive semidefinite constitutive matrices. Cost is then measured on the basis of invariants of the matrices.

With reference to the paper by Bendsøe *et al.* (1994), we shall now extend the results obtained in that paper to be valid also for power-law elasticity. If we choose as cost constraint the Frobenius norm (length of a matrix) of the constitutive matrix, then the proof even for 3D-problems is rather direct. With localized sensitivity analysis (33) in a fixed strain field, the minimum total strain energy U implies the maximum strain energy density u. The strain energy density depends homogeneously on the squared effective strain $u = [E/(p+1)](\varepsilon_e^2)^{(p+1)/2}$ as seen from (18). The problem formulation can therefore be stated

$$\max_{[\alpha]}\{\varepsilon\}^T[\boldsymbol{\alpha}]\{\varepsilon\}\,,$$
$$\text{for Frobenius }([\boldsymbol{\alpha}]) = 1\,. \tag{58}$$

In the *invariant formulation* of $[\boldsymbol{\alpha}](=[\boldsymbol{\beta}]^{-1})$ we can choose the coordinate system of principal strains

$$\{\varepsilon\}^T = \{\{\varepsilon_I\,\varepsilon_{II}\,\varepsilon_{III}\}\{0\,0\,0\}\}\,, \tag{59}$$

and obtain

$$\{\varepsilon\}^T[\alpha]\{\varepsilon\} = \{\varepsilon_I\,\varepsilon_{II}\,\varepsilon_{III}\}\begin{bmatrix}\alpha_{1111} & \alpha_{1122} & \alpha_{1133}\\ \alpha_{1122} & \alpha_{2222} & \alpha_{2233}\\ \alpha_{1133} & \alpha_{2233} & \alpha_{3333}\end{bmatrix}\begin{Bmatrix}\varepsilon_I\\ \varepsilon_{II}\\ \varepsilon_{III}\end{Bmatrix}. \tag{60}$$

Now, the Frobenius norm of a matrix is defined as the square root of the *sum of the squares* of all the elements of the matrix (equal to the length of the contracted matrix). It thus follows directly that the matrix elements not involved in (60) must optimally be zero.

This means directly that also for the nonlinear, power-law materials we have

- the optimal material is orthotropic;
- principal directions of material, strain and stress are aligned;
- there is no shear stiffness. (61)

This result for linear elastic material was proven by Bendsøe *et al.* (1994), based also on a constraint on the trace of the constitutive matrix. Here, the extension to nonlinear elastic material follows directly from the localized sensitivity result (34). For simplicity of proof we have chosen the Frobenius norm as the constraint.

The further analysis relates then only to the submatrix in (60). To fulfil the condition of being positive semidefinite, we have

$$\alpha_{1111} \geq 0\,,\quad \alpha_{2222} \geq 0\,,\quad \alpha_{3333} \geq 0\,,$$
$$\alpha_{1111}\alpha_{2222} \geq \alpha_{1122}^2\,,\quad \alpha_{1111}\alpha_{3333} \geq \alpha_{1133}^2\,,$$
$$\alpha_{2222}\alpha_{3333} \geq \alpha_{2233}^2\,. \tag{62}$$

The problem formulation (58) can now be written as

$$\max_{[\alpha]\geq 0} \varepsilon_e^2 = \alpha_{1111}\varepsilon_I^2 + \alpha_{2222}\varepsilon_{II}^2 + \alpha_{3333}\varepsilon_{III}^2 +$$
$$2\alpha_{1122}\varepsilon_I\varepsilon_{II} + 2\alpha_{1133}\varepsilon_I\varepsilon_{III} + 2\alpha_{2233}\varepsilon_{II}\varepsilon_{III}\,, \tag{63}$$

constrained by

$$F^2 - 1 = \alpha_{1111}^2 + \alpha_{2222}^2 + \alpha_{3333}^2 + 2\alpha_{1122}^2 +$$
$$2\alpha_{1133}^2 + 2\alpha_{2233}^2 - 1 = 0\,.$$

The Lagrange function for this problem is $\mathcal{L} = \varepsilon_e^2 - \overline{\lambda}(F^2-1)$ with a Lagrangian multiplier $\overline{\lambda}$. Stationarity of $\mathcal{L}$ gives $\mathrm{d}\varepsilon_e^2/\mathrm{d}\alpha_{iijj} = \overline{\lambda}\mathrm{d}F^2/\mathrm{d}\alpha_{iijj}$ for all $iijj$ (no summation), which gives the result

$$\frac{\varepsilon_I^2}{\alpha_{1111}} = \frac{\varepsilon_{II}^2}{\alpha_{2222}} = \frac{\varepsilon_{III}^2}{\alpha_{3333}} =$$
$$\frac{\varepsilon_I\varepsilon_{II}}{\alpha_{1122}} = \frac{\varepsilon_I\varepsilon_{III}}{\alpha_{1133}} = \frac{\varepsilon_{II}\varepsilon_{III}}{\alpha_{2233}}\,, \tag{64}$$

and we can finally write the resulting constitutive matrix in the directions of principal strain/stress

$$[\boldsymbol{\alpha}]_{\text{optimal}} = \frac{1}{(\varepsilon_I + \varepsilon_{II} + \varepsilon_{III})^2}\cdot$$
$$\begin{bmatrix}\varepsilon_I^2 & \varepsilon_I\varepsilon_{II} & \varepsilon_I\varepsilon_{III} & 0 & 0 & 0\\ \varepsilon_I\varepsilon_{II} & \varepsilon_{II}^2 & \varepsilon_{II}\varepsilon_{III} & 0 & 0 & 0\\ \varepsilon_I\varepsilon_{III} & \varepsilon_{II}\varepsilon_{III} & \varepsilon_{III}^2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}. \tag{65}$$

This result is now valid also for power-law, nonlinear elastic materials. We note that the matrix (65) has only one nonzero eigenvalue and that the material therefore only has stiffness in relation to the specified strain condition.

In papers by Sigmund (1994, 1995) for the linear elastic case, it is shown how such optimal material can be identified by an inverse homogenization solution.

For the ultimate optimal material, the effective strain ε_e the strain energy density u and the Frobenius norm F will be

$$\varepsilon_e^2 = \varepsilon_I^2 + \varepsilon_{II}^2 + \varepsilon_{III}^2\,,$$

$$u = E\frac{1}{p+1}\left(\varepsilon_I^2 + \varepsilon_{II}^2 + \varepsilon_{III}^2\right)^{(p+1)/2},$$
$$F = 1. \qquad (66)$$

We can obtain the same effective strain and energy density with an isotropic, zero Poisson's ratio material $[\boldsymbol{\alpha}] = [\mathbf{I}]$, but then the corresponding Frobenius norm will be $F = 6$, i.e. the material cost is six times greater. As shown by Bendsøe *et al.* (1994), the zero Poisson's ratio material may be valuable in numerical calculation, because of the degeneracy of the ultimate optimal material.

5 Optimal design for strength

Optimal design for strength is in most cases formulated as a minmax problem, i.e. minimize by design changes the strain/stress state which is worst (maximum) in the structure. The measure of the strain/stress state is normally a scalar quantity, and its specific definition is by no means unique. Some of the quantities used are

- strain or stress energy density: u or u^C
- von Mises stress: $\sigma_M^2 = \frac{3}{2}\{\sigma\}^T[\mathbf{P}]\{\sigma\}$
- $F_{\text{Hill}} = \{\sigma\}^T[\mathbf{H}]\{\sigma\}$
- $F_{\text{Tsai-Wu}} = \{\sigma\}^T[\mathbf{T}]\{\sigma\} + \{\mathbf{W}\}^T\{\sigma\}$

with the last criterion from Tsai and Wu (1971). More generally, some damage parameters may be used. The optimal design will depend on the chosen reference quantity.

Let us, as has also been done by Taylor (1969) and by Masur (1970), prove that if energy density is chosen, then *the stiffest design will also be the strongest.* (The stiffest design gives minimum U, i.e. minimum mean strain energy density $\overline{u}$; the strongest design gives minimum of maximum strain energy density). The uniform energy density gives minimum total energy, and with $*$ for any alternative design, we have

$$\sum u_i^* V_i^* \geq \sum \overline{u} V_i = \overline{u} V = \sum \overline{u} V_i^*, \qquad (67)$$

because the total volume $V = \sum V_i = \sum V_i^*$ is the same. Then we have

$$\sum (u_i^* - \overline{u}) V_i^* \geq 0, \qquad (68)$$

which, with nonnegative volumes $V_i^* \geq 0$, implies that not all u_i^* can be less than $\overline{u}$, corresponding to the solution for the optimal stiffness problem. So again restricting to problems where the necessary optimality condition (53) is satisfied everywhere, we can conclude that

min max u is obtained with uniform energy density. (69)

Now for the other strength criteria the strongest design will be different from the stiffest design. We especially note for the Tsai-Wu criterion that the sign of stresses changes the criterion but not the energy densities. Even for this most complicated criterion sensitivity analysis can be performed analytically, as shown by Hammer (1994). See also the paper by Pedersen and Hammer (1994) for specific laminate results. However, no general result such as (67) can be proven.

In the papers by Selyugin (1992, 1994, 1995a,b) the von Mises stress is applied and then the optimal design will depend on the power of the nonlinearity, except for the specific case of isotropic, incompressible material or for the 1-D case of truss optimization. The contradiction to the result (56) is illustrated by Selyugin (1995a) for an isotropic, but compressible material with a rather coarse finite element model.

It is soon realized that to solve the problem of optimal design with strength constraints, we must use mathematical programming. Then we can include in the formulation, say for a laminate, multiple layers, multiple loads and multiple strength constraints. The optimization problem may then be formulated as by Pedersen and Hammer (1994)

maximize the load factor λ,

with equilibriums $[\mathbf{S}]\{\mathbf{D}\}_\ell = \lambda\{\mathbf{A}\}_\ell$, for $\ell = 1, 2, \ldots, L$,

subject to the strength constraints

$(F/F_0)_{jk\ell n} \leq 1$,

with index j for plate position, index k for layer, index ℓ for load case and index n for strength constraint. In the secant formulation of the nonlinear problems the numerical procedure will be almost as for the linear problems, except for an inner iteration loop.

6 Shape design

Shape design for maximum strength is important for fillets in plates and shells, for holes in plates and shells, for cavities in 3-D continua, and for shapes more generally. The objective is mostly to minimize stress concentration, but more generally it may be the tool to control evolution of plasticity and damage.

Most of the problems solved relate to linear, isotropic elasticity, see the review of Ding (1986). For 3D-problems with these most simple materials, we still need a number of research activities. The design parameterization is of vital importance, as discussed by Pedersen and Laursen (1982-83) and by Pedersen *et al.* (1992) dealing with 2D-anisotropic, but linear elastic materials.

The paper by Petersen and Frederiksen (1994) shows the plasticity evolutions of an initial and an optimized shape. However, much more can be done in relation to the control of these evolutions. See, as an example, the work by Morthland *et al.* (1995).

In the present paper we just want to show some initial calculations to obtain an idea of the influence of elastic nonlinearities. The actual problem is shown in Fig. 1, i.e. a biaxially loaded hole for which we know that in linear elasticity the optimal hole shape is an ellipse with ratio of axes equal to the ratio of the two far-away stresses. We want to analyse this shape with the four different isotropic material models, also shown in Fig. 1.

In Fig. 2 we show the resulting stress fields close to the quarter hole shape. Lines show the principal stress directions with line intensity indicating stress level. Theoretically, with $\sigma_1 = 3$ and $\sigma_2 = 2$, we should have $\sigma_{\max} = 5$, but due to the fact that the model in Fig. 1 is finite we obtain a higher value $\sigma_{\max} = 5.14$ for the linear case of $p = 1.0$, This proven optimal solution has uniform energy density along the boundary of the hole. Now increasing the nonlinearity $p = 0.75$, $p = 0.50$ and $p = 0.25$, the shape may no longer be optimal, but we actually find rather uniform energy density along the boundary of the hole.

It should be noted that the maximum stress decreases with increasing nonlinearity. In fact, this is the explanation

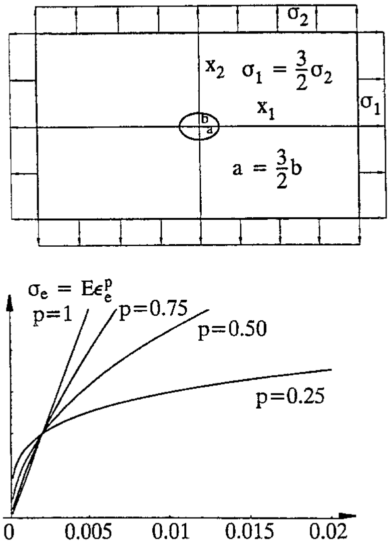

Fig. 1. Hole problem and applied materials

for the numerical stability and fast convergence of the solution by secant formulation. From the optimization point of view, the rather unchanged stress field indicates that the optimal shape will be rather independent of the nonlinearity. Further studies are needed before more strict conclusions can be drawn.

Lastly, we show in Fig. 3 the strain fields corresponding to the four material cases. The maximum strain is doubled for the case of $p = 0.25$ compared with the linear case. Although the level is changed, we again see a rather uniform distribution along the boundary of the hole.

7 Conclusions

With numerical procedures based on effective sensitivity analysis there seem to be no limitations for optimal design; what we can analyse, we can optimize. Analytical results, on the other hand, can only be obtained for idealized formulations. However, these general results are valuable as a reference for numerical procedures and for a basic understanding.

In this paper we have taken the most simple extension to linear elasticity, i.e. the nonlinear, power-law elasticity. Much criticism of this material law can be found. On the other hand, many practical cases can in the active region at least be curve-fitted to such behaviour, and then the assumption is fully justified. Without further complications we have included the possibility for an anisotropic material.

With energy or potential as objective or constraint, it is important to set up the relations between energies/potentials. Then follows the localized determination of the sensitivity for these global quantities. It is important to note that the sensitivity is not physically localized, but it can be calculated using only localized quantities. Note also that this is an exact result and not an approximation.

For a large class of practical problems based on homogeneous relations, we then prove in relation to stiffness optimization that, for the optimal design, each design subdomain has the same ratio between mean energy density and mean mass density. For point-wise design and given mass density, this gives the uniform energy density, well-known from problems with linear elasticity. From this it follows that, for stiffness optimization, the optimal mass density distribution is independent of the nonlinearity.

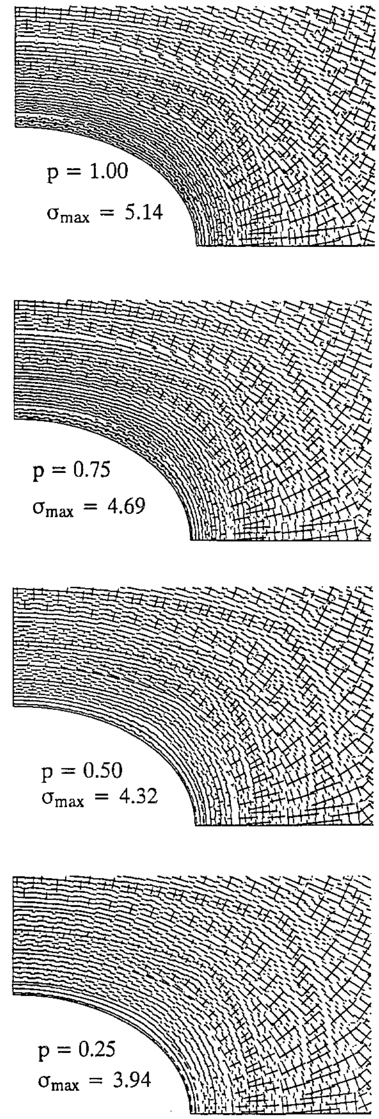

Fig. 2. Stress field with an elliptic hole, corresponding to different degrees of material nonlinearity

Results for linear elasticity on coalignment of principal stresses, principal strains and orthotropic material directions are extended to the nonlinear case. Lastly, for the stiffness optimization, the ultimate optimal material is proven for the general 3D-case and based on cost measured by the Frobenius norm.

For strength optimization we concentrated on a discussion of the possible strength measures; energy density - von Mises measure - Hill measure - Tsai-Wu measure. Different measures normally give different optimal designs, and only with the energy density can we obtain analytical results. For a class of problems with a single load case we can prove that the stiffest design is also the strongest for the power law material.

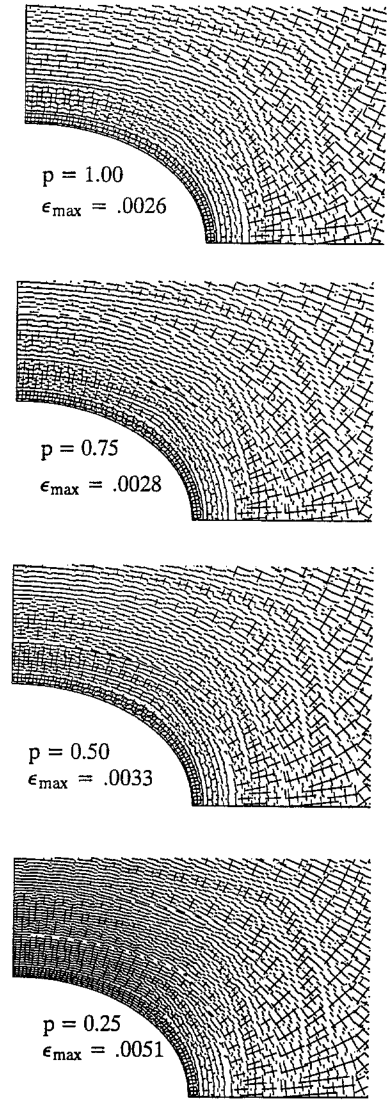

Fig. 3. Strain field with an elliptic hole, corresponding to different degrees of material nonlinearity

The last section on shape optimization shows some numerical calculations that indicate a very weak design dependence on nonlinearity.

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