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RESEARCH PAPAER

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A reevaluation of the SIMP method with filtering and an alternative formulation for solid–void topology optimization

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Abstract The most popular way to introduce the notion of topology into the structural analysis of the topology optimization problem is through the Solid Isotropic Material with Penalization (SIMP) method. The fundamental principle behind its use requires a density design variable dependent material constitutive law that penalizes intermediate density material in combination with an active volume constraint. Here, the SIMP method with filtering is reevaluated, and an alternative topology optimization problem formulation, called the SINH (pronounced "cinch") method, is developed that exploits this principle. The main advantages of the SINH method are that the optimization problem is consistently defined, the topology description is unambiguous, and the method leads to predominantly solid–void designs.

Keywords Topology optimization · Restriction methods · Filter · SIMP method · SINH method

1 Introduction

We are interested in topology optimization formulations that lead to solid–void structural designs, well behaved, preferably well posed by mathematical proof, and unambiguous optimization problem definitions, simple algorithm implementation, and computational efficiency. Perhaps the most popular and competitive methods in terms of these goals are topology optimization formulations that combine the Solid Isotropic Material with Penalization (SIMP) scheme with filter techniques to yield a well-behaved optimization problem. For a detailed historical perspective of the SIMP method, refer to Rozvany (2001).

The discrete and penalized, continuous solid–void structural topology problems are ill posed (e.g., see Haber et al. 1996), so the density design field variation is restricted here. A variety of restriction methods have been developed, e.g., by perimeter control (Haber et al. 1996), explicit density slope constraints (Petersson and Sigmund 1998), adaptive density design variable bounds that impose density slope constraints (Zhou et al. 2001), regularized density control (Borrvall and Petersson 2001), MOLE method (Poulsen 2003), or by blurring filters (Sigmund 1994). For a good comparison between various restriction methods, refer to Borrvall (2001). The most prevalent approach in the literature is to filter the sensitivities (Sigmund 1994). Alternatively, the design can be directly filtered (Bruns and Tortorelli, 2001). There are advantages and disadvantages in both approaches. Our goal is to remedy the drawbacks of both filter approaches and propose a new formulation that incorporates the advantages of both.

In section 2, structural topology optimization is briefly reviewed, and density measures are introduced. The SIMP method is reevaluated in terms of these measures in section 3, and the SINH method is formulated in section 4.

2 Topology optimization

The topology optimization problem is stated as

$$\text{minimize} \quad \Theta_0(\mathbf{d}) \tag{1}$$

$$\text{subject to} \quad \Theta_i(\mathbf{d}) \leq 0 \tag{2}$$

$$\underline{d}_j \leq d_j \leq \overline{d}_j \tag{3}$$

where Θ_0 is the objective function, Θ_i (for $i = 1, nc$) are the inequality or equality constraints and d_j (for $j = 1, nd$) are the design variables that are bounded above and below by $\overline{d}_j$ and $\underline{d}_j$.

The structural response is solved via the finite element method (e.g., Bathe 1996), and when the design domain is discretized by linear elastic continuum finite elements, the finite element equation reduces to

$$\mathbf{K}\,\mathbf{U} = \mathbf{P} \tag{4}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{P}$ is the external force vector, and $\mathbf{U}$ is the displacement vector.

Although the density design variables can be assigned to nodes, and the field is subsequently interpolated appropriately, a density design variable d_i is assigned to every

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element i here that ranges between its small lower bound $\underline{d}_i \approx 0$, e.g., $\underline{d}_i = 10^{-6}$, and upper bound $\overline{d}_i = 1$. A first density measure η_{1i} is computed for every finite element i and is defined as a function of the density design variable field $\mathbf{d}$, i.e., $\eta_{1i} = \hat{\eta}_{1i}(\mathbf{d})$. In a manner consistent with the density design variable range, the density measure η_1 ranges from a small value to one representing void and solid material, respectively, and therefore the variation in the first density measure field η_1 depicts the topology of the structure. We assume that the solid material constitutive behavior can be adequately modeled by

$$\sigma = \mathsf{C}\,\mathbf{E} \tag{5}$$

where σ, $\mathbf{E}$, and C are the stress, strain, and elasticity tensors, respectively. The material density is introduced into the structural analysis by the elasticity tensor C_i for every element i via

$$\mathsf{C}_i(\mathbf{d}) = \hat{\eta}_{1i}(\mathbf{d})\,\hat{\mathsf{C}}_i \tag{6}$$

where $\hat{\mathsf{C}}$ is the elasticity tensor of the solid construction material, and the elasticity tensor C is used in the computations. A small, positive, nonzero lower bound $\underline{d}_i$, and therefore the lower bound on η_{1i}, ensures that the stiffness matrix $\mathbf{K}$ remains nonsingular throughout the optimization history.

The usual structural problem is to determine the stiffest structural design or equivalently least compliant structural design where the compliance is defined by

$$\Theta_0(\mathbf{d}) = \mathbf{U}(\mathbf{d})^T\,\mathbf{P} \tag{7}$$

where the structural displacement response $\mathbf{U}$ is implicitly design-dependent through (4) and (6), and we assume here that the external load $\mathbf{P}$ is design-independent. We define the effective volume v, corresponding to the design domain V_o, as

$$v(\mathbf{d}) = \int_{V_o} \hat{\eta}_2(\mathbf{d})dv \tag{8}$$

where the second density measure η_{2i} for every element i is defined as a function of the density design variables $\mathbf{d}$, i.e., $\eta_2 = \hat{\eta}_2(\mathbf{d})$. The effective volume v is constrained by it upper bound $\bar{v}$ defined as a fraction of the maximal volume $\int_{V_o} dv$, i.e.

$$\Theta_1(\mathbf{d}) = v(\mathbf{d}) - \bar{v}. \tag{9}$$

Analytical sensitivities, i.e., $\frac{\mathrm{D}\Theta_0}{\mathrm{D}\mathbf{d}}(\mathbf{d})$ and $\frac{\mathrm{D}\Theta_i}{\mathrm{D}\mathbf{d}}(\mathbf{d})$, are calculated here by the adjoint method, and the Method of Moving Asymptotes (MMA) (Svanberg 1987) is used to solve the large-scale optimization problem.

Filters can be introduced to remedy the ill-posed topology optimization problem. In the most common approach developed and thoroughly demonstrated by Sigmund and coworkers (Sigmund 1994, 1997; Buhl et al. 2000; Pedersen et al. 2001; Bendsøe and Sigmund 2003), all response sensitivities except for the volume sensitivity are smoothed by a blurring filter that acts within a mesh-independent filter radius. The modified objective sensitivities $\frac{\mathrm{D}\tilde{\Theta}_0}{\mathrm{D}d_i}$ for every element i are computed by

$$\frac{\mathrm{D}\tilde{\Theta}_0}{\mathrm{D}d_i}(\mathbf{d}) = \frac{1}{\omega_i\,d_i}\sum_j \omega_j(s_{ij})\,d_j\,\frac{\mathrm{D}\Theta_0}{\mathrm{D}d_j}(\mathbf{d}) \tag{10}$$

for

$$\omega_i = \sum_j \omega_j(s_{ij}) \tag{11}$$

where ω_j is the filter kernel that is based on the distance s_{ij}, i.e.,

$$s_{ij} = ((x_j - x_i)^2 + (y_j - y_i)^2)^{\frac{1}{2}}, \tag{12}$$

of the surrounding element j centroids (x_j, y_j) within a fixed mesh-independent radius r of the element i centroid (x_i, y_i). For example, a Gaussian-weighted kernel is computed as

$$\omega_j(s_{ij}) = \begin{cases} \dfrac{\exp(-\frac{s_{ij}^2}{2(\frac{r}{3})^2})}{2\pi(\frac{r}{3})} & \text{for } s_{ij} \le r \\ 0 & \text{for } s_{ij} > r \end{cases} \tag{13}$$

where only those elements for which $s_{ij} \le r$ affect the sensitivities of element i. In accordance with the formulation of the SIMP method that typically appears in the literature, the first and second density measures for every element i are defined by

$$\hat{\eta}_{1i}(\mathbf{d}) = d_i^p \text{ and} \tag{14}$$

$$\hat{\eta}_{2i}(\mathbf{d}) = d_i. \tag{15}$$

A less-cited, second approach is based on filtering the density design variable field $\mathbf{d}$ and evaluating all response sensitivities in a consistent manner (Bruns and Tortorelli, 2001). The filtered density design variable field Φ is computed for every element i as

$$\phi_i = \hat{\phi}_i(\mathbf{d}) = \sum_j \frac{\omega_j(s_{ij})}{\omega_i}\,d_j \tag{16}$$

where where ω_i and s_{ij} are computed by (11) and (12), respectively, and ω_j is computed by (14) for a Gaussian-weighted kernel. Unlike the previous approach, the first and second density measures for every element i are defined by

$$\hat{\eta}_{1i}(\mathbf{d}) = \hat{\phi}_i^p(\mathbf{d}) \tag{17}$$

$$\hat{\eta}_{2i}(\mathbf{d}) = \hat{\phi}_i(\mathbf{d}) \tag{18}$$

in a SIMP-like manner.

The advantages of filtering the sensitivities are that the design can be directly interpreted from the density design field, i.e., through $\hat{\eta}_{1i}(\mathbf{d})$ in (15), and relatively quick computation. The main disadvantage is that the approach is heuristic because the sensitivities are not consistent with the primal analysis. Therefore, the optimization problem is not well posed in a rigorous sense. However, the approach leads to practical, useful topology designs and a well-behaved optimization problem. The advantage of filtering the design is that the optimization problem is consistent and regularized (Bourdin 2001). However, since the design is defined by the density measure field through the filtered density design field, i.e., by (18), the main disadvantage of the approach is that the topologies are somewhat less distinct or more diffuse compared to the first approach.

3 SIMP method

The most popular way to introduce the notion of topology into the structural analysis is through the Solid Isotropic Material with Penalization (SIMP) method or its variant forms. The SIMP approach was first considered by Bendsøe (1989) and developed independently by Zhou and Rozvany (1991). For a detailed history of the SIMP method, refer to Rozvany (2001). The term "SIMP" was proposed by Rozvany et al. (1992), and the method is also referred to as the power law or penalized, proportional stiffness model, and in the past, it has been referred to as an artificial interpolation or fictitious material model. Bendsøe and Sigmund (1999) have shown that composite materials from intermediate densities are physically realizable, and therefore the method is not artificial nor fictitious from a design point of view. However, this issue may not be particularly useful for designers who are interested in solid–void topology designs. Therefore, we desire a method that leads primarily to solid–void designs.

The fundamental principle of the SIMP method is that the load capacity of the structure is progressively taxed more for the intermediate densities than solid and void densities, e.g., as penalty parameter $p > 1$ is increased, for an equal effective volume v. The synergy between the penalization of intermediate densities and the resource constraint leads to solid–void structural designs. Although this seemingly trivial interaction has been acknowledged from its inception (Bendsøe 1989), it is sometimes overlooked. The recent work by Martínez (2005) punctuates this interaction.

As is commonly illustrated in the literature, the interpolation scheme of the traditional SIMP method is depicted in Fig. 1a, and to emphasize the role of volume constraint, its interpolation is also presented. To generalize its role, the density function ρ is either defined directly by its corresponding density design variable d or by the filtered density design variable field ϕ, i.e., $\rho = \hat{\rho}(\mathbf{d}) = d$ or $\rho = \hat{\rho}(\mathbf{d}) = \phi$, respectively. Therefore, Fig. 1a depicts the interpolation scheme for the traditional SIMP-power law method where the first η_1 and second η_2 density measures are defined by

$$\eta_1 = \hat{\eta}_1(\rho) = \rho^p \text{ and} \tag{19}$$

$$\eta_2 = \hat{\eta}_2(\rho) = \rho. \tag{20}$$

As the penalty parameter p is increased in Fig. 1, the structural stiffness is progressively penalized, and for a given volume, the intermediate density material is structurally less effective. Therefore, the topology optimization algorithm will redistribute, i.e., within the constraints of the mesh discretization, the material of given volume more effectively. Penalty parameter p is generally increased via a continuation method (Rozvany et al. 1994) from lower bound $\underline{p}$ to upper bound $\overline{p}$ in increments Δp after convergence at each step to hopefully avoid premature convergence to local minima. However, a global optimum cannot be guaranteed, as noted by Stolpe and Svanberg (2001a).

We define the SIMP method by its underlying principle rather than the particular interpolation scheme. There are

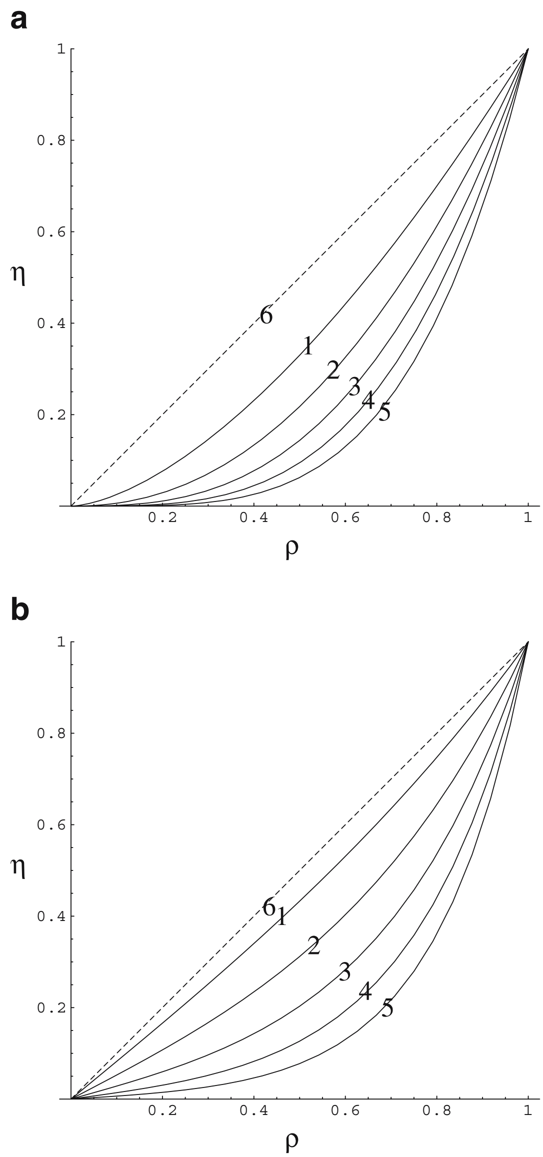

Fig. 1 a SIMP method with power law first density measure $\eta_1 = \hat{\eta}_1(\rho) = \rho^p$ where penalty parameter (1)$p = 1.6$, (2)$p = 2.2$, (3)$p = 2.8$, (4)$p = 3.4$, and (5)$p = 4$ and (6) linear second density measure $\eta_2 = \hat{\eta}_2(\rho) = \rho$. **b** SIMP method with hyperbolic sine first density measure $\eta_1 = \hat{\eta}_1(\rho) = \frac{\sinh(p\,\rho)}{\sinh(p)}$ where penalty parameter (1)$p = 1.1$, (2)$p = 2.1$, (3)$p = 3.1$, (4)$p = 4.1$, and (5)$p = 5.1$ and (6) linear second density measure $\eta_2 = \hat{\eta}_2(\rho) = \rho$.

several interpolation scheme variants that improve the SIMP method with a power law, e.g., the rational function of the Rational Approximation of Material Properties (RAMP) (Stolpe and Svanberg 2001b) or more implementation-intensive, spline-based interpolation schemes (Pedersen 2002). The SIMP method with hyperbolic sine first density measure η_1, i.e.

$$\eta_1 = \hat{\eta}_1(\rho) = \frac{\sinh(p\,\rho)}{\sinh(p)} \tag{21}$$

and linear second density measure η_2, i.e., (20), is used here and is depicted in Fig. 1b. This interpolation scheme has a simple form (and sensitivity computation) similar to the power law. Similar to the RAMP scheme, it is advantageous that the density measure sensitivity $\frac{D\eta}{D\mathbf{d}}(\rho)$ does not vanish as any ρ approaches zero for many problems, e.g., eigenfunction/vibration problems (Hansen 2005). Also, the Hashin–Shtrikman upper and lower bounds can be satisfied for a wide range of penalty $p > 3$ approximately for solid–void material with regard to Bendsøe and Sigmund (1999), and consequently a microstructural geometry can be realized (Sigmund 1995).

Next, the topology design of a cantilever beam depicted in Fig. 2 is investigated. Unless otherwise noted, the following parameters are used for the topology optimization. A dead load $P = 0.001N$ is applied to the beam tip in a downward, vertical direction. The $30mm \times 20mm$ domain (with a $1mm$ thickness) is discretized by four-node 60×40 quadrilateral element mesh. The linear elastic material response is computed with Young's modulus $E = 1N/mm^2$ and Poisson's ratio $\nu = 0.3$. The density design variables are initially set to $d_j = 0.3$, and their lower and upper bounds are set to $\underline{d}_j = 10^{-3}$ and $\overline{d}_j = 1$, respectively. The filter length r is set to $1.0mm$. Here, the penalty parameter p is increased from $\underline{p} = 0.1$ to $\overline{p} = 5.1$ in steps of $\Delta p = 0.1$. The upper bound on the total volume $\overline{v}$ is set to $\sim$40% of the maximal volume. The objective is to design the stiffest structure by minimizing its compliance, i.e., by (1) and (7). Refer to Table 1 for the ρ, η_1, η_2, and filter approach definitions for several cases. For case 1, checkerboard patterns appear in the topology plot of Fig. 3a because no filter is implemented. Figure 3b,c depicts the optimal topologies due to the filtered SIMP methods with a hyperbolic sine first density measure.

Next, we qualitatively compare the topology designs due to various topology optimization formulations with the SIMP method. For each case, the first density measure η_1 depicts the topology of the structure since the first density measure is used in the structural analyses, and the second density measure η_2 is used to compute the volume. Refer to Table 1 for the ρ, η_1, η_2, and filter approach definitions for several cases. For case 1, checkerboard patterns appear in the topology plot of Fig. 3a because no filter is implemented. Figure 3b,c depicts the optimal topologies due to the well-established approaches

Fig. 2 Design domain of tip-loaded cantilever beam

Table 1 Topology optimization with SIMP method

Case	$\rho = \hat{\rho}(\mathbf{d})$	$\eta_1 = \hat{\eta}_1(\rho)$	$\eta_2 = \hat{\eta}_2(\rho)$	Filter approach
1	d	$\frac{\sinh(p\rho)}{\sinh(p)}$	ρ	N/A
2	d	$\frac{\sinh(p\rho)}{\sinh(p)}$	ρ	Filter sensitivities
3	ϕ	$\frac{\sinh(p\rho)}{\sinh(p)}$	ρ	Filter design

N/A Not applicable, *SIMP* Solid Isotropic Material with Penalization

of filtering the sensitivities and of filtering the design, respectively, with a hyperbolic sine first density measure.

4 SINH method

A related but fundamentally different problem is termed the SINH (pronounced "cinch") method. Unlike SIMP, SINH is not an acronym; instead, it merely references the use of the hyperbolic sine function here. In the same manner that the SIMP method does not rely on the power law interpolation scheme, the SINH method does not rely on a hyperbolic sine interpolation scheme. The basis of the SINH method is that intermediate density material is made less volumetrically effective than solid or void material.

There has been some previous research that is related to the SINH method developed here. Rietz (2001) defines an optimization problem that mirrors the SINH method as a substitute problem for the topology optimization problem with the SIMP method through a change of variables. The purpose of introducing the change of variables is merely as a means to mathematically prove that the SIMP method yields a discrete solution under some conditions. The change of variables, i.e., $y = x^p$ therefore $x = y^{\frac{1}{p}}$, provides a convenient way to associate the behavior of the SIMP method to that of the SINH method, which he calls the SIMP- $\frac{1}{p}$ problem, and vice versa. However, he does not acknowledge nor investigate the viability of the SIMP- $\frac{1}{p}$ problem in its own right. Guedes and Taylor (1997) present a topology optimization formulation in which the stiffness is linearly scaled by the density design variables, i.e., $\eta_{1i} = d_i$, and the volume is computed by a linear scaling of the elemental density design variables by elemental weight w_i, i.e., by $\eta_{2i} = w_i\, d_i$ for element i. Initially, $w_i = 1$ until convergence, and then w_i is uniformly inflated for density design variable d_i values below a threshold density value after each convergence of the optimization problem. In this way, low intermediate density material, i.e., below the threshold value, is progressively made less volumetrically effective via this continuation method. They demonstrate that this form of filtering over the entire optimization history yields predominantly solid–void designs. However, they do not adequately address numerical instability nor mesh dependency problems. Furthermore, the method appears to be heuristic since the filter is design-dependent. Zhou and Rozvany (1991) and later

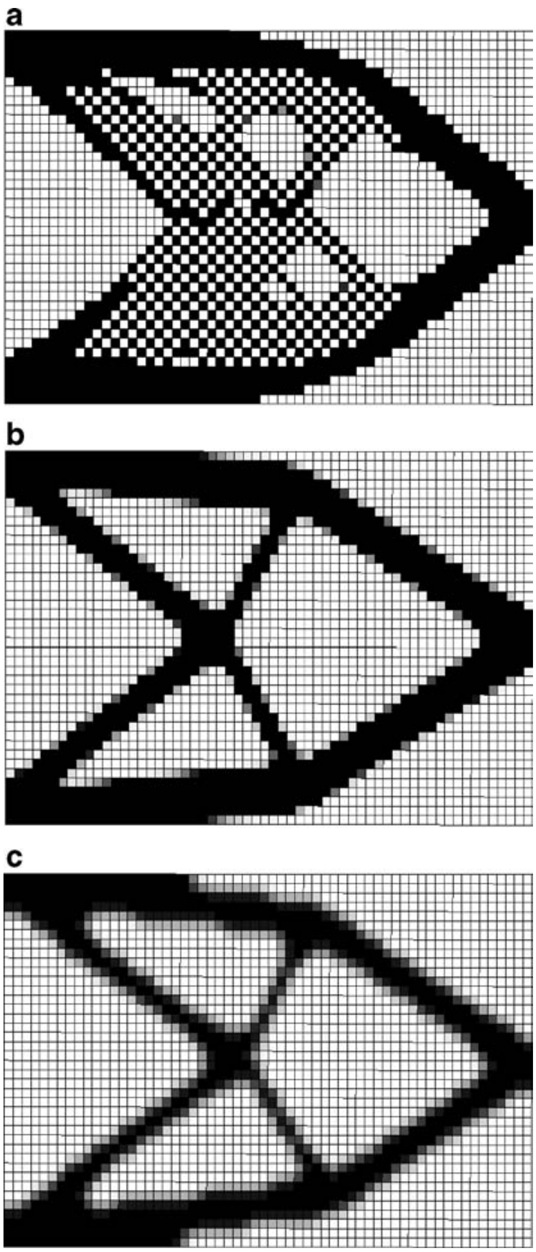

Fig. 3 Topology plots of first density measures η_1 for cases **a** 1, **b** 2, and **c** 3 of Table 1 with 60×40 mesh discretizations

Cardoso and Fonseca (2003) present the complementary optimization problem to the topology optimization problem with the SIMP method that is reminiscent of the SINH method presented here. However, the objective of minimal penalized volume limits the problem formulation, and therefore, this necessitates the reformulation to a topology optimization problem with the SINH method to have general applicability to more diverse problems. In summary, elements of the SINH method appear in previous topology optimization work, but they are combined here in a way that leads to a more general and effective formulation.

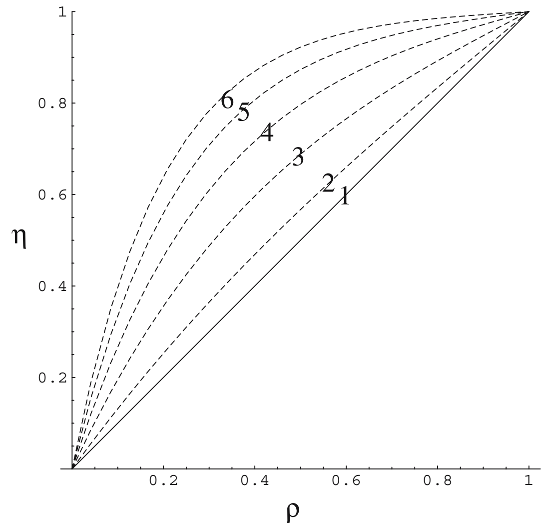

Fig. 4 SINH method with (1) linear first density measure $\eta_1 = \hat{\eta}_1(\rho) = \rho$ and hyperbolic sine second density measure $\eta_2 = \hat{\eta}_2(\rho) = 1 - \frac{\sinh(p\,(1-\rho))}{\sinh(p)}$ where penalty parameter (2)$p = 1.1$, (3)$p = 2.1$, (4)$p = 3.1$, (5)$p = 4.1$, and (6)$p = 5.1$

Figure 4 depicts the interpolation scheme for the basic SINH-hyperbolic sine method where the linear first η_1 and hyperbolic sine second η_2 density measures are defined by

$$\eta_1 = \hat{\eta}_1(\rho) = \rho \text{ and} \tag{22}$$

$$\eta_2 = \hat{\eta}_2(\rho) = 1 - \frac{\sinh(p\,(1-\rho))}{\sinh(p)}. \tag{23}$$

As the penalty parameter p is increased, the volume is progressively penalized, and therefore, the intermediate density material is volumetrically less effective. Since intermediate density material consumes more volume with respect to its load-carrying capability than solid or void material, the topology optimization algorithm will redistribute, i.e., within the constraints of the mesh discretization, the intermediate density material of given volume more effectively.

One potential drawback of this approach is that the effective volume can inaccurately reflect the true volume, i.e., particularly at intermediate optimization iterations, since the intermediate density volumes are penalized. However, this should not be a concern because the design is predominantly solid and void at the end of the optimization, and therefore the effective volume will approximate the true volume. Also, from a practical design point of view, the desired volume $\overline{v}$ is generally an approximate, i.e., not rigidly assigned, design goal. Furthermore, a linear first density measure does not satisfy the Hashin–Shtrikman bounds, but this issue should not be a concern because the final design is predominantly solid and void (or it can be addressed in the hybrid SINH method).

Again, the topology design of a cantilever beam depicted in Fig. 2 is investigated. Refer to Table 2 for the ρ_1, η_1, ρ_2, η_2, and filter approach definitions for several cases. For case 4, checkerboard patterns appear in the topology plot of Fig. 5a because no filter is implemented. Analogous to case 3, the first and second density functions are filtered in case 5, and it

Table 2 Topology optimization with basic SINH method

Case	$\rho_1 = \hat{\rho}_1(\mathbf{d})$	$\eta_1 = \hat{\eta}_1(\rho_1)$	$\rho_2 = \hat{\rho}_2(\mathbf{d})$	$\eta_2 = \hat{\eta}_2(\rho_2)$	Filter approach
4	d	ρ_1	d	$1-\frac{\sinh(p(1-\rho_2))}{\sinh(p)}$	N/A
5	ϕ	ρ_1	ϕ	$1-\frac{\sinh(p(1-\rho_2))}{\sinh(p)}$	Filter design
6	d	ρ_1	ϕ	$1-\frac{\sinh(p(1-\rho_2))}{\sinh(p)}$	Filter design

N/A Not applicable

Table 3 Topology optimization with hybrid SINH method

Case	$\rho_1 = \hat{\rho}_1(\mathbf{d})$	$\eta_1 = \hat{\eta}_1(\rho_1)$	$\rho_2 = \hat{\rho}_2(\mathbf{d})$	$\eta_2 = \hat{\eta}_2(\rho_2)$	Filter approach
7	d	$\frac{\sinh(p_1\rho_1)}{\sinh(p_1)}$	ϕ	$1-\frac{\sinh(p_2(1-\rho_2))}{\sinh(p_2)}$	Filter design
8	ϕ	$\frac{\sinh(p_1\rho_1)}{\sinh(p_1)}$	ϕ	$1-\frac{\sinh(p_2(1-\rho_2))}{\sinh(p_2)}$	Filter design

is noteworthy that this leads to a very similar blurred optimal topology, cf. Figs. 3c, 5b. However, recall that case 3 is based on the SIMP method and case 5 is based on the SINH method, and furthermore the blurred topology of case 3 is due to the filtered density design field. Consequently, we define the first density measure η_1 by density design field $\mathbf{d}$ directly in case 6, and its more clearly delineated optimal topology is shown in Fig. 5c. The formulation of case 6 represents the basic SINH method.

By combining the SINH method with a penalized first density measure η_1 in the same manner as the SIMP method, we can conveniently further penalize intermediate density material. Figure 6 depicts the interpolation scheme for a hybrid SINH method where the hyperbolic sine first η_1 and hyperbolic sine second η_2 density measures are defined by

$$\eta_1 = \hat{\eta}_1(\rho) = \frac{\sinh(p_1\,\rho)}{\sinh(p_1)} \quad \text{and} \tag{24}$$

$$\eta_2 = \hat{\eta}_2(\rho) = 1 - \frac{\sinh(p_2\,(1-\rho))}{\sinh(p_2)}. \tag{25}$$

Refer to Table 3 for the ρ_1, η_1, ρ_2, η_2, and filter approach definitions for select cases. Note that case 7 leads to a primarily solid–void design in Fig. 7 that better approximates what might be expected of the original integer topology design problem than all of the previous formulations.

For the hybrid SINH method here, we increase the penalty parameter p_1 corresponding to the first density measure η_1 from $\underline{p}_1 = 0.1$ to $\overline{p}_1 = 5.1$ in small steps of $\Delta p_1 = 0.05$ and the penalty parameter p_2 corresponding to the second density measure η_2 over the same range from $\underline{p}_2 = 0.1$ to $\overline{p}_2 = 5.1$ in steps of $\Delta p_2 = 0.1$ to avoid potential numerical instability problems. Increasing the penalty parameter p of the first and second density measures in unison over the same range does not pose problems for the discretization of Fig. 7, but in general, if the stiffness, i.e., through η_1, is penalized "too much" compared to the volume, i.e., through η_2, we may encounter numerical instabilities denoted by checkerboard patterns or prematurely converge to local minima. This suggests different interpolation schemes for the density measures and/or continuation strategies whereby the penalty parameters p_1 and p_2 for each density measure be increased independently and/or over different ranges. Alternatively, if the first and second density functions are consistently defined by the filtered density design field in case 8 (in a manner that mirrors case

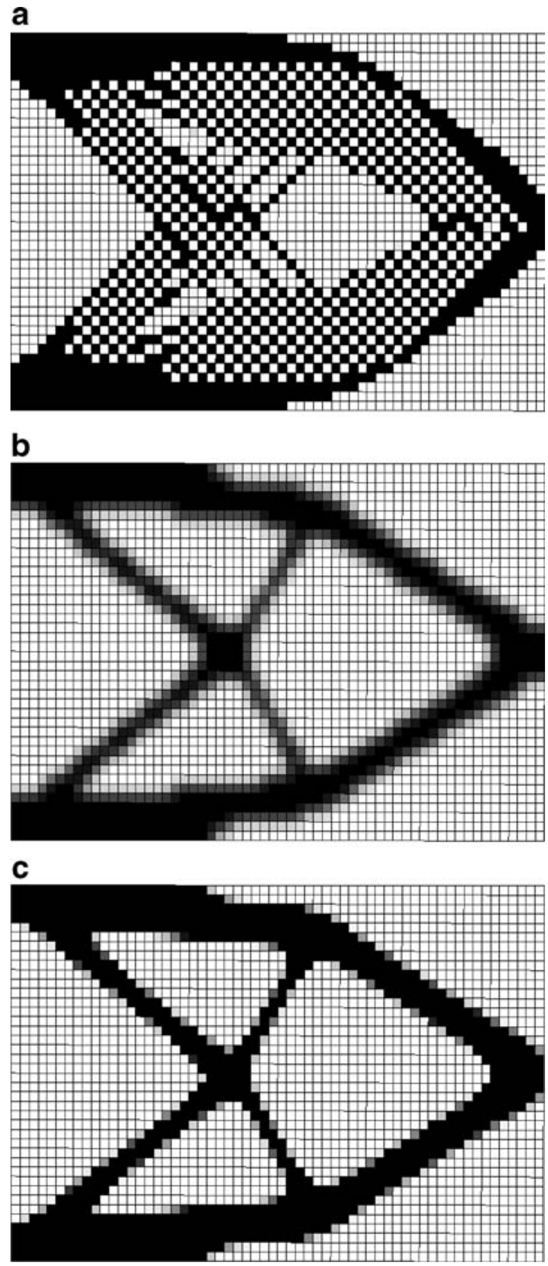

Fig. 5 Topology plots of first density measures η_1 for cases **a** 4, **b** 5, and **c** 6 of Table 2 with 60 × 40 mesh discretizations

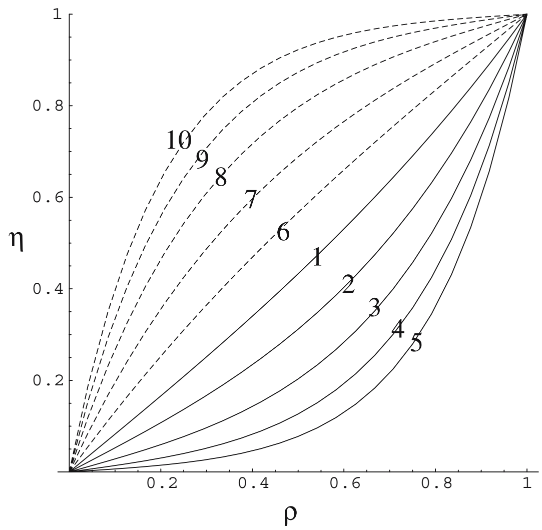

Fig. 6 Hybrid SINH method with hyperbolic sine density measure $\eta_1 = \hat{\eta}_1(\rho) = \frac{\sinh(p_1\,\rho)}{\sinh(p_1)}$ where penalty parameter (1)$p_1 = 1.1$, (2)$p_1 = 2.1$, (3)$p_1 = 3.1$, (4)$p_1 = 4.1$, and (5)$p_1 = 5.1$, and hyperbolic sine second density measure $\eta_2 = \hat{\eta}_2(\rho) = 1 - \frac{\sinh(p_2\,(1-\rho))}{\sinh(p_2)}$ where penalty parameter (6)$p_2 = 1.1$, (7)$p_2 = 2.1$, (8)$p_2 = 3.1$, (9)$p_2 = 4.1$, and (10)$p_2 = 5.1$

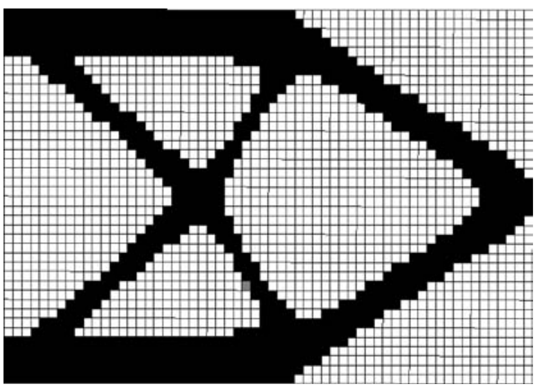

Fig. 7 Topology plot of first density measure $\boldsymbol{\eta_1}$ for case 7 of Table 3 with 60×40 mesh discretization

3), i.e., $\rho_1 = \rho_2 = \rho = \hat{\rho}(\mathbf{d}) = \phi$, then the filter radius r of ρ_1 should be allowed to slowly decrease from a multiple of an element characteristic length, e.g., an element width on a regular mesh, to a size less than this characteristic length to remove the filter effect and to better remedy this potential problem.

Until now, the structural analyses have been performed on a fixed finite element mesh discretization. To show that the structural topology optimization problem is well behaved, we show by experiment that the same optimal topology with refinement of details results from mesh refinement. Figure 8 compares the topology plots with the basic and hybrid SINH methods with mesh refinement.

Next, we show common design examples of topology optimization by using the SINH approach. The design domain of a roller-supported beam vertically loaded along its lower

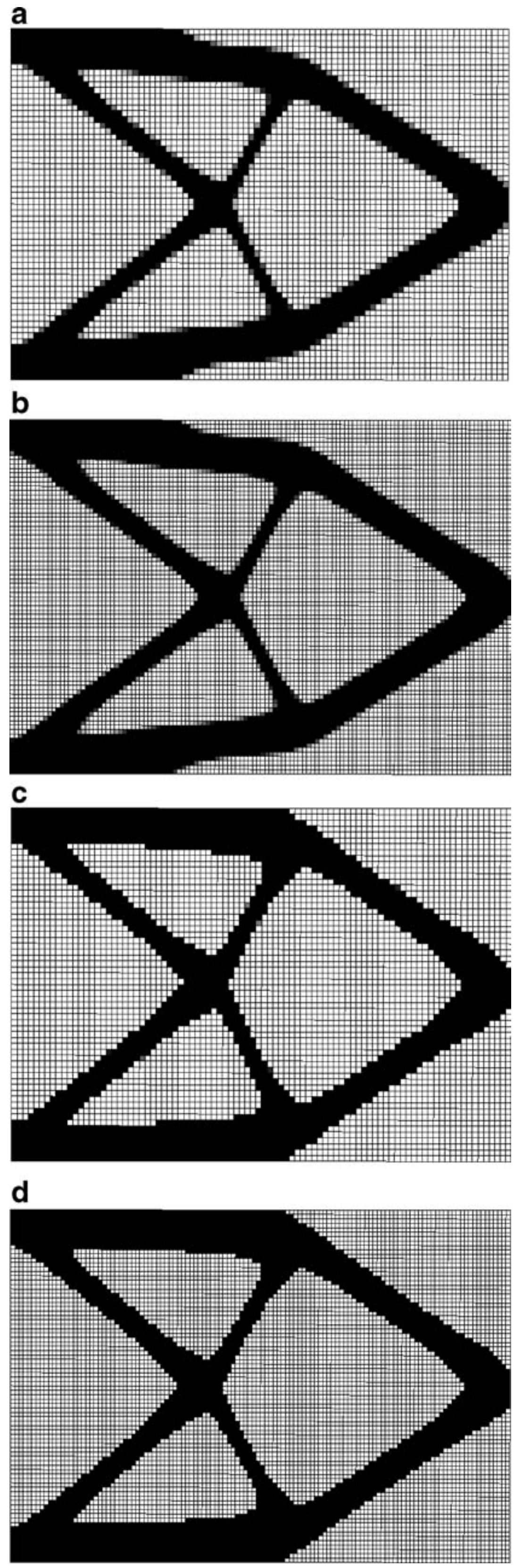

Fig. 8 Topology plot of first density measure $\boldsymbol{\eta_1}$ for case 6 of Table 2 with **a** 90×60 and **b** 120×80 mesh discretizations, and for case 7 of Table 3 with **c** 90×60 and **d** 120×80 mesh discretizations

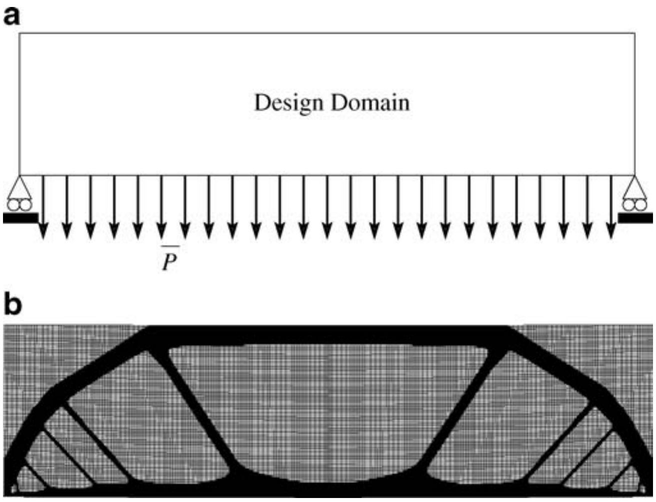

Fig. 9 **a** Design domain and **b** topology plot of first density measure η_1 of roller-supported beam

edge is depicted in Fig. 9a, and its clearly delineated optimal topology is shown in Fig. 9b. Topology optimization with the SINH method is equally applicable to compliant mechanism design which requires nonlinear finite element analyses (Bruns and Tortorelli 2001; Pedersen et al. 2001). The design domain of inverter and gripper mechanisms are depicted in Fig. 10a and b, respectively, and their optimal topologies using the element removal and reintroduction strategy of Bruns and Tortorelli (2003) are shown in Fig. 10c and d, respectively.

A potential advantage of the SINH method is that the second density measure η_2 need not be bounded above by $\eta_2 \leq 1$, e.g., $\eta_2 = \hat{\eta}_2(\rho) = \rho + p\sin(\pi\,\rho)$. This may be useful for design problems where the SIMP method is not effective, i.e., design problems in which less structurally effective intermediate density material performs better than solid or void material.

5 Conclusion

Although structural topology optimization has matured as a research field, this paper readdresses one of its fundamental issues, i.e., a methodology for introducing the notion of topology into the structural analysis and consequently the topology optimization problem. The SINH method addresses the disadvantages of the current topology optimization formulations that implement the popular SIMP method and filter techniques while retaining their advantages. There are several points worth noting:

1. The original topology original problem has been generalized by the independent definitions of the first and second density measures, i.e., η_1 and η_2, respectively.
2. The method penalizes less volumetrically effective intermediate density material.

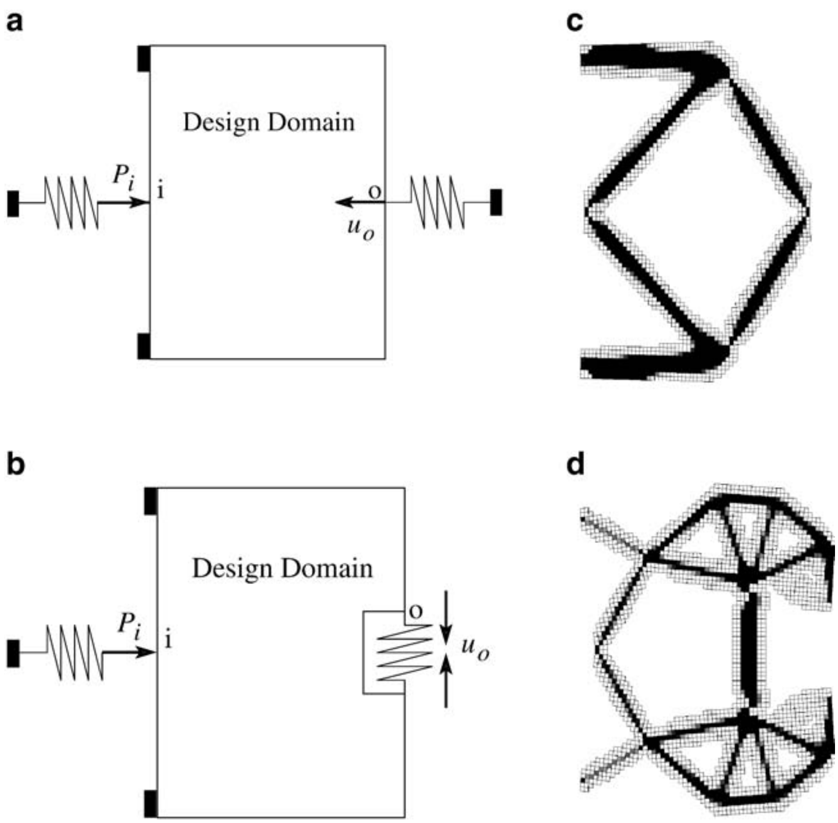

Fig. 10 Design domain of **a** inverter mechanism and **b** gripper mechanism. Topology plot of first density measure η_1 for **c** inverter mechanism and **d** gripper mechanism

3. The method is simple to implement and has quick computation because the filter is not integrated into the structural analysis.
4. Compared to previous filtering techniques, the "regularization" is moved from the structural analysis, i.e., through η_1, to the resource constraint, i.e., through η_2.
5. The method leads to a consistently defined optimization problem, unambiguous topology description, and predominantly solid–void designs for structural problems.
6. The method offers the opportunity to conveniently further penalize toward solid–void designs through its hybrid formulation,
7. The method is applicable to other design problems, e.g., mechanism design.
8. Its extension to three-dimensional problems is trivial.

Here, we have concentrated on the performance and shown that the SINH method is well behaved. Because the formulation is a derivative of what constitutes a well-posed, filter-based topology optimization problem (solution existence proofs provided by Bourdin 2001, where the design is filtered in a similar manner, and by Guo and Gu 2004, where the filter influence progressively decreases for lower densities) and we can demonstrate mesh-independent designs, we conjecture, and leave for future work, that the SINH method leads to a well-posed optimization problem.

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