Solvers for time-harmonic wave propagation problems

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3 June 2021

Outline

Motivation and challenges

What is the best two-level method for Helmholtz?

The grid coarse space: assessment on a geophysical benchmark problem

General conclusions

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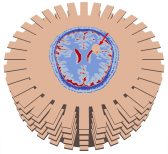
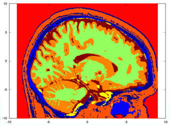
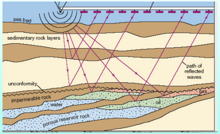
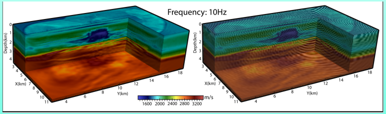








Maxwell's equations

Medical imaging: reconstruct the permittivity ε

$$\nabla \times (\mu^{-1} \nabla \times \mathbf{E}) - \omega^2 \varepsilon \mathbf{E} = \mathbf{J}$$

- $\mathbf{E}$ is the electric field
- $\mu > 0$ is the magnetic permeability,
- $\varepsilon > 0$ is the electric permittivity, ω is the frequency.

Helmholtz equations

Seismic imaging: reconstruct material properties of subsurface

$$-\Delta u - (\omega^2 / c^2) u = f,$$

- $c^2 = \rho c_P^2$, ρ is the density,
- c_P is the speed of longitudinal waves.

Helmholtz equation

Figure 1: Hermann von Helmholtz (1821-1894), physicist, physician, philosopher, ...

$$-\Delta u - k^2 u = f$$

a.k.a. the *reduced wave equation* or time-harmonic wave equation.

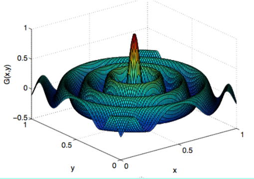
Scalar wave equation ($c(x)$ local speed of propagation

$$\partial_{tt} v - c^2(x)\Delta v = F(x,t)$$

If $F(x,t) = f(x)e^{-i\omega t}$ (mono-chromatic) we can assume

$$v(x,t) = u(x)e^{-i\omega t}$$

which leads to



$$-\Delta u - n(x)^2\omega^2 u = f,$$

where $n(x) = \frac{1}{c(x)}$ is the **index of refraction**, $k^2 = n^2\omega^2$ is called **wave number**.

AIM: After discretisation maximise accuracy and minimise the number of degrees of freedom (#DoF)

FACTS:

If $h\omega$ is kept constant the error increases with $\omega \to$ pollution error [Babuska, Sauter, *SINUM*, 1997]

Other discretisations (FEM) or higher dimensions and for quasi-optimality we need

$$h^p \omega^{p+1} \lesssim 1$$

[Melenk, Sauter, *SINUM*, 2011].

For a bounded error $h \sim \omega^{-1-1/2p}$ [Du, Wu, *SINUM*, 2015].

Consequences

- High-frequency solution u oscillates at a scale $1/\omega \Rightarrow h \sim \frac{1}{\omega} \Rightarrow$ large #DoF.
- Pollution effect requires $h \ll \frac{1}{\omega}$, $h \sim \omega^{-1-1/p}$, with p the finite element order $\Rightarrow$ even larger #DoF.
- Trade-off: number of points per wavelength (*ppwl*) $G = \frac{\lambda}{h} = \frac{2\pi}{\omega h}$ and polynomial degree $\Rightarrow$ dispersion analysis (measuring the ratio between the numerical and physical wave speeds).

Test case: 2 km $\times$ 4 km $\times$ 12 km, $c(x, y, z) = c_0 + \alpha \times z$ with c_0=1 km/s with a source at 1 km depth and frequency is 8 Hz. FD grid (h=31.25 m) corresponds to 4 *ppwl*, adapted tetrahedral FE grid.

			#dof (M)		Error norm	
$\alpha(s^{-1})$	$\lambda_{min}(m)$	$\lambda_{max}(m)$	FD	FE	FD	FE
0.8	125	1200	13	28	0.0079	0.034
2	125	3125	13	16	0.044	0.034

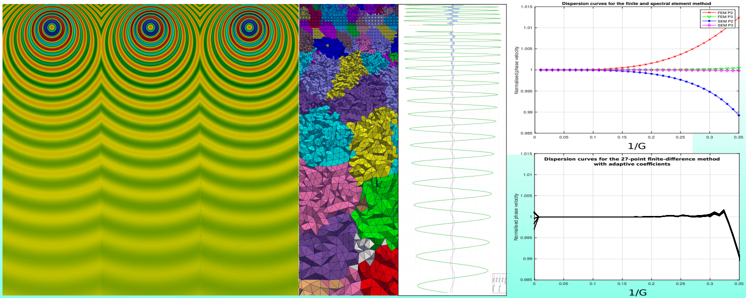


Figure 2: **(a) FD solution. (b) Analytical solution. (c) FE solution. (d) FE mesh. (e) Comparison between (a-c) along a vertical profile cross-cutting source position. (f-g) Phase velocity dispersion curves for (f) FE and (g) FD. G: number of *ppwl*.**

Why the high-frequency forward problem is hard? (Efficiency)

After discretisation we get a large linear system to solve $A\mathbf{u} = \mathbf{b}$.

How bad things can be?

- A is symmetric but non-hermitian for the damped equations and/or when Robin BC are used.
- A is getting larger with ω: its size n is increasing like $N^d \sim \omega^{(1+1/p)d}$ where $N \sim 1/h$.
- A can become arbitrarily ill-conditioned

... conventional iterative methods fail

[Ernst, Gander, *Numerical analysis of multiscale problems*, 2012], [Gander, Zhang, *SIREV*, 2019],

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The holy grail ...

- Solution of the discretised PDEs in *optimal time* for large ω
- Solvers should have good parallel properties.
- Solvers should be robust w.r.t heterogeneities.

$A\mathrm{u} = \mathrm{b}$? Landscape of linear solvers for large linear systems

- Direct solvers: MUMPS, SuperLU (Demmel, ...), PastiX, UMFPACK, PARDISO (O. Schenk)
- Iterative methods (Krylov): CG (Stiefel-Hestenes), GMRES (Saad), BiCGSTAB (van der Vorst)

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How large is truly large (for heterogeneous Helmholtz, e.g. geophysics) to justify the use of DD?

- Problems (in applications) do not need to be overesolved. (to much precision not necessary in FWI)
- Use direct methods when this is possible.

[Patrick Amestoy; Romain Brossier; Alfredo Buttari; Jean-Yves L'Excellent; Theo Mary; Ludovic Métivier; Alain Miniussi; Stephane Operto, *Geophysics*, 2016] (3d problem from FWI with 50 Million dofs solved with MUMPS)

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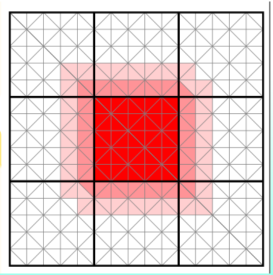
Solve the preconditionned $A\mathbf{u} = \mathbf{b}$, i.e. $M^{-1}A\mathbf{u} = M^{-1}\mathbf{b}$ by GMRES

The one-level preconditioner

$$\boxed{M^{-1} = \sum_{j=1}^{N} R_j^T D_j A_j^{-1} R_j}, \text{ where}$$

R_j — $\Omega \to \Omega_j$ is restriction operator
R_j^T — $\Omega_j \to \Omega$ is prolongation operator
D_j — corresponds to the partition of unity.

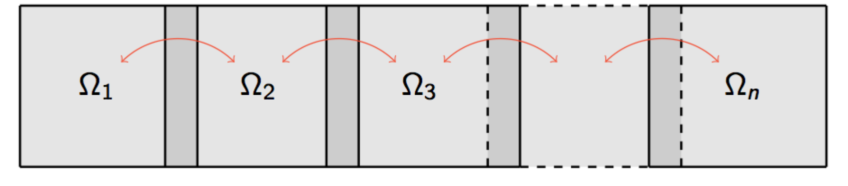
$\Omega = \cup_{j=1}^{N} \Omega_j$



Definition of the local matrices A_j
A_j is the stiffness matrix of the local Robin problem

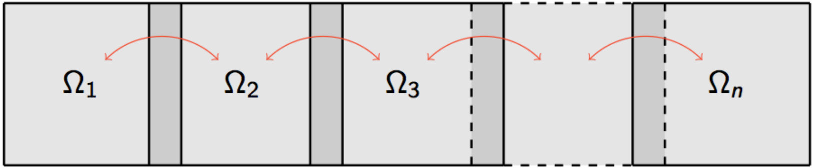
$$\begin{aligned} (-\Delta - k^2)(u_j) &= f && \text{in } \Omega_j \\ \left(\frac{\partial}{\partial n_j} + ik\right)(u_j) &= 0 && \text{on } \partial\Omega_j \setminus \partial\Omega. \end{aligned}$$

One level is not enough (only neighbouring subdomains communicate)



The two-level method

One level is not enough (only neighbouring subdomains communicate)



"Coarse" information: Z or how to add a second level

$$\boxed{M_2 = QM^{-1}P + H}, \text{ where}$$

A	is the stiffness matrix,
M^{-1}	is the one-level ORAS preconditioner,
H	$= ZE^{-1}Z^*$ is the coarse matrix, where
Z	is the matrix whose columns span the coarse space and $E = Z^*AZ$
P = Q = I	additive two-level preconditioner (I is the identity matrix),
P = I - AH,Q = I - HA	hybrid two-level preconditioner

Adding a coarse space: the grid coarse space

The "grid" coarse space is based on a **geometrical** coarse mesh correction of diameter H_{coarse}

Definition of Z

R_0	the interpolation matrix from the fine to the coarse FEM grid
$Z = R_0^T$	coarse space matrix
$E = Z^T A Z$	is the stiffness matrix of the problem discretised on the coarse mesh

For scalability and robustness w.r.t to the frequency we need $H_{\text{coarse}} \sim k^{-\alpha'}$, $0 < \alpha' <= 1$.

Remark: The definition of the coarse space doesn't have to be geometrical $\Rightarrow$ spectral coarse spaces (based on local eigenvalue problems)

The theory of the grid coarse space is based on the problem with absorption in the Dirichlet case (i.e. replace the $-\Delta - k^2$ operators with $-\Delta - (k^2 + i\xi)$) [Graham, Spence, Vainikko, *Math. Comp.*, 2017] and extended to Maxwell in [Bonazzoli, Dolean Graham, Spence, Tournier, *Math. Comp.*, 2019]

Theorem. When $|\xi| \sim k^2$ (max absorption) and $\delta \sim H_{\text{coarse}}$ (generous overlap), $H \sim H_{\text{coarse}} \sim k^{-1}$, then weighted GMRES will converge with the number of iterations independent of the wavenumber.

Residual $\mathbf{r}_m$ is minimized in the norm induced by (D $= \nabla$(Helmholtz) and D $= \nabla \times$ (Maxwell))

$$\langle \mathbf{V}, \mathbf{W} \rangle_{D_k} = (\mathbf{v}_h, \mathbf{w}_h)_{\mathsf{D},k} \quad (\mathbf{v}_h, \mathbf{w}_h \in V_h \text{ with coefficient vectors } \mathbf{V}, \mathbf{W}),$$
$$(\mathbf{v}_h, \mathbf{w}_h)_{\mathsf{D},k} = (\mathsf{D}\mathbf{v}_h, \mathsf{D}\mathbf{w}_h)_{L^2(\Omega)} + k^2 (\mathbf{v}_h, \mathbf{w}_h)_{L^2(\Omega)}$$

Q1: Is the grid coarse space optimal for heterogeneous problems?

Spectral coarse spaces

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Q2: Can we extend the idea of spectral coarse spaces to Helmholtz ? How do we choose the 'modes' which go into the space?

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A: There are several spectral versions: DtN, H-GenEO (high-frequencies, no theory), Δ-GenEO (low-frequencies, theory for general non-symmetric problems)

Q1: Is the grid coarse space optimal for heterogeneous problems?

Q2: Can we extend the idea of spectral coarse spaces to Helmholtz ? How do we choose the 'modes' which go into the space?

A: There are several spectral versions: DtN, H-GenEO (high-frequencies, no theory), Δ-GenEO (low-frequencies, theory for general non-symmetric problems)

Aim: theory and limitations for Δ-GenEO and numerical comparison of spectral coarse spaces (DtN, H-GenEO) with the grid coarse space for a few benchmark problems

Definition (DtN eigenproblem)
Find $(u_{\Gamma_i}, \lambda) \in V(\Gamma_i) \times \mathbb{C}$ such that

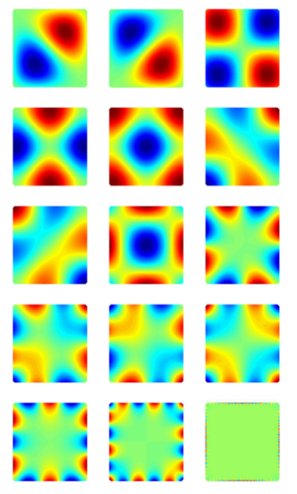
$$\mathrm{DtN}_{\Omega_i}(u_{\Gamma_i}) = \lambda u_{\Gamma_i}$$

Definition (DtN operator)
Let $D \subset \Omega$, let $\Gamma_D = \partial D \setminus \partial\Omega$. Let $v_{\Gamma_D} : \Gamma_D \to \mathbb{C}$. Then

$$\mathrm{DtN}_D(v_{\Gamma_D}) = \frac{\partial v}{\partial n}|_{\Gamma_D}$$

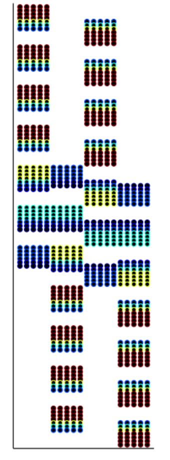
where $v : D \to \mathbb{C}$ is the extension of v_{Γ_D}.

To provide the modes in the coarse space we use the Helmholtz extension v. How many modes do we need ?



> Choose only eigenfunctions with eigenvalue λ such that
>
> $$\mathrm{Re}(\lambda) < k_i \qquad \text{with} \qquad k_i = \max_{x \in \Omega_i} k(x)$$
>
> Notice this criteria depends on the heterogeneity

[Dolean, Nataf, Scheichl, Spillane, *CMAM*, 2012] (Laplace) & [Conen, Dolean, Krause and Nataf, 14] (Helmholtz)



Notations

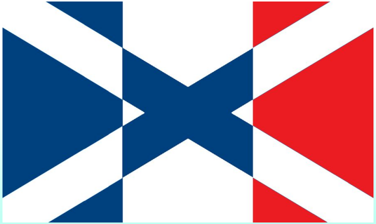
- Γ_i — degrees of freedom on the boundary of Ω_i
- I — degrees of freedom in the interior of Ω_i
- $\widetilde{A}^{(i)}$ — local Neumman matrix on Ω_i
- M_{Γ_i} — local mass matrix on the boundary of Ω_i

The discrete eigenproblem for (u_{Γ_i}, λ) is then

$$\left(\widetilde{A}^{(i)} - A_{\Gamma_i I} A_{II}^{-1} A_{I\Gamma_i}\right) u_{\Gamma_i} = \lambda M_{\Gamma_i} u_{\Gamma_i}$$

Construct the matrix W_i of size (#dof on subdomain Ω_i)× (# coarse modes on subdomain Ω_i) using the eigenvectors and partition of unity.

SPD problems with heterogeneous coefficients



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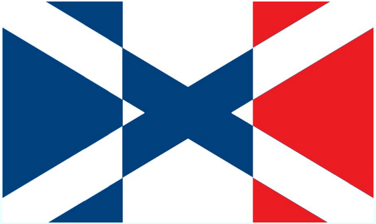
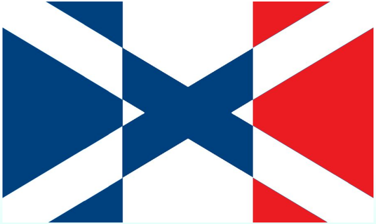


[Spillane, Dolean, Hauret, Nataf, Pechstein, Scheichl, *Numer. Math*, 2014] & [Haferssas, Jolivet, Nataf, *SISC*, 2017]

The replica GenEO coarse space for the Helmholtz problem fails.

SPD problems with heterogeneous coefficients



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The replica GenEO coarse space for the Helmholtz problem fails.

GenEO (Generalised Eigenproblems in the Overlap): in each Ω_i solve the discrete eigenproblem

$$D_i A_i D_i u = \lambda \widetilde{A}^{(i)} u$$

where D_i are diagonal matrices corresponding to a partition of unity Choose only eigenfunctions with eigenvalue λ such that

$$\lambda > \lambda_{min}$$

Theory and limitations of Δ - GenEO

Designed for a more general BVP

$$-\nabla \cdot (A(\mathbf{x})\nabla u) - \kappa u = f \quad \text{in } \Omega, \tag{1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1b}$$

with A an SPD matrix-valued function, $a_{\min}|\xi|^2 \leq A(\mathbf{x})\xi \cdot \xi \leq a_{\max}|\xi|^2, \mathbf{x} \in \Omega, \xi \in \mathbb{R}^d$, $\kappa \in L^\infty(\Omega)$.
Helmholtz problem with $A = I$ and $\kappa = \omega^2 n$ + absorbing BC appears often in geophysics.
The finite element solution $u_h \in V^h$ satisfies the weak formulation $b(u_h, v_h) = F(v_h)$

$$b(u,v) = \int_\Omega (A(\mathbf{x})\nabla u \cdot \nabla v - \kappa uv)\, \mathrm{d}\mathbf{x} \quad \text{and} \quad F(v) = \int_\Omega fv\, \mathrm{d}\mathbf{x}. \tag{2}$$

After discretisation by FEM we get the symmetric linear system

$$B\mathbf{u} = \mathbf{f} \tag{3}$$

where B is symmetric but generally indefinite.

Overlapping partition $\{\Omega_j\}_{1\leq j\leq N}$ of Ω, with Ω_j of diameter H_j and H the maximal diameter of the subdomains.
Define $\widetilde{V}^j = \{v|_{\Omega_j} : v \in V_h\}$, $V^j = \{v \in \widetilde{V}^j : \operatorname{supp}(v) \subset \Omega_j\}$, and for u, $v \in \widetilde{V}^j$

$$b_j(u,v) := \int_{\Omega_j} \left(A(\mathbf{x})\nabla u \cdot \nabla v - \kappa uv\right) \mathrm{d}\mathbf{x} \qquad \text{and} \qquad a_j(u,v) := \int_{\Omega_j} A(\mathbf{x})\nabla u \cdot \nabla v \mathrm{d}\mathbf{x}.$$

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Extension operators: $\mathcal{R}_j^T : V^j \to V_h$, $1 \le j \le N$ and R_j^T denote its matrix representation with respect to the nodal basis and set $R_j = (R_j^T)^T$.

One-level additive Schwarz preconditioner

$$M_{\mathsf{AS}}^{-1} = \sum_{j=1}^{N} R_j^T (R_j B R_j^T)^{-1} R_j.$$

POU operators $\Xi_j : \widetilde{V}^j \to V^j$ $(\sum_{j=1}^N R_j^T \Xi_j (v|_{\Omega_j}) = v,\ \forall v \in V_h)$ and the generalised eigenvalue problems

$$\text{find } p \in \widetilde{V}_j \backslash \{0\},\ \lambda \in \mathbb{R} : \qquad a_j(p, v) = \lambda\, a_j(\Xi_j(p), \Xi_j(v)), \qquad \text{for all } v \in \widetilde{V}^j, \tag{4}$$

$$\text{find } q \in \widetilde{V}_j \backslash \{0\},\ \lambda \in \mathbb{R} : \qquad b_j(q, v) = \lambda\, a_j(\Xi_j(q), \Xi_j(v)), \qquad \text{for all } v \in \widetilde{V}^j, \tag{5}$$

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Definition (Δ-GenEO and $\mathcal{H}$-GenEO coarse spaces)

For each j, $1 \le j \le N$, let $(p_l^j)_{l=1}^{m_j}$ and $(q_l^j)_{l=1}^{m_j}$ be the eigenfunctions of the eigenproblems (4) and (5) corresponding to the m_j smallest eigenvalues, respectively. Then we define the Δ-**GenEO** and $\mathcal{H}$-**GenEO** coarse spaces, respectively, by

$$V_\Delta^0 := \text{span}\{\mathcal{R}_j^T \Xi_j(p_l^j) :\ l = 1,\ldots,m_j;\ j = 1,\ldots,N\} \text{ and} \tag{6}$$

$$V_\mathcal{H}^0 := \text{span}\{\mathcal{R}_j^T \Xi_j(q_l^j) :\ l = 1,\ldots,m_j;\ j = 1,\ldots,N\}. \tag{7}$$

Two-level extensions of Schwarz methods:
$M_{\text{AS},\Delta}^{-1} = M_{AS}^{-1} + R_{0,\Delta}^T B_{0,\Delta}^{-1} R_{0,\Delta}$ and $M_{\text{AS},\mathcal{H}}^{-1} = M_{AS}^{-1} + R_{0,\mathcal{H}}^T B_{0,\mathcal{H}}^{-1} R_{0,\mathcal{H}}$ where
$B_{0,\Delta} := R_{0,\Delta} B R_{0,\Delta}^T$ and $B_{0,\mathcal{H}} := R_{0,\mathcal{H}} B R_{0,\mathcal{H}}^T$

Theorem (Robustness w.r.t. heterogeneities and "mild" indefiniteness)

Let the fine-mesh diameter h be sufficiently small. Then there exist thresholds $H_0 > 0$ and $\Theta_0 > 0$ such that, for all $H \leq H_0$ and

$$\Theta := \max_{1 \leq j \leq N} \left(\lambda^j_{m_j+1}\right)^{-1} \leq \Theta_0$$

the following statements hold:

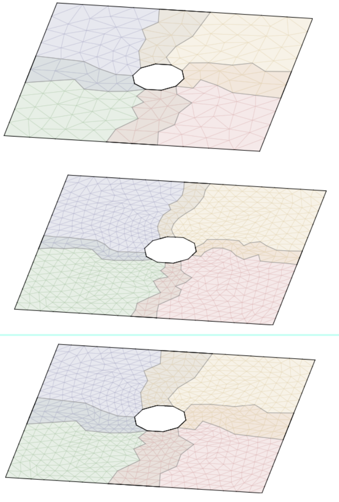
- *Local matrices B_j and the coarse matrix $B_{0,\Delta}$ are non-singular.*
- *If the problem $B\mathbf{u} = \mathbf{f}$ is solved by GMRES with left preconditioner $M^{-1}_{\mathrm{AS},\Delta}$ and residual minimisation in the energy norm $\|u\|_a := \left(\int_\Omega \nabla u \cdot A \nabla u\right)^{1/2}$, then there exist a constant $c \in (0,1)$, which depends on H_0 and Θ_0 but is independent of all other parameters, such that we have the robust GMRES convergence estimate*

$$\|r_\ell\|_a^2 \leq \left(1 - c^2\right)^\ell \|r_0\|_a^2 \,, \tag{8}$$

for $\ell = 0, 1, \ldots$, where r_ℓ denotes the residual after ℓ iterations of GMRES.

Conclusions: It works if the indefinitness is "mild" (e.g. low frequency Helmholtz in heterogeneous media)
[Bootland, Dolean, Graham, Ma, Scheichl, *arXiv*, 2021]

Comparison and implementation of the different coarse spaces



Helmholtz benchmark test cases

- 3 methods: Grid coarse space, DtN, H-Geneo.
- test cases: Marmousi, Cobra cavity and Overthrust)
- low and high frequency
- precision: *underesolved* (5 ppwl), and with conventional resolution (10 ppwl).

Open source libraries and DSL

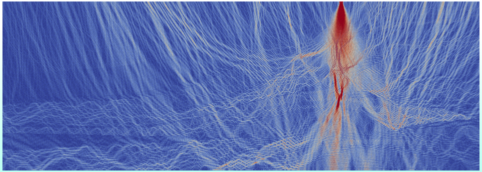
- Discretisation: FreeFEM `https://freefem.org`
- Solvers:

ffddm `FreeFem-sources/examples/ffddm`

HPDDM `https://github.com/hpddm/hpddm`

PETSc `https://www.mcs.anl.gov/petsc`

			Coarse grid		H-GenEO		DtN	
Problem	d	freq	5 ppwl	10 ppwl	5 ppwl	10 ppwl	5 ppwl	10 ppwl
Marmousi	2D	low	✓	✓	✓	✓	✓✓	✓✓
		high	✓✓	✓	✗	✓✓	✓	✓



Results with 10 points per wavelength

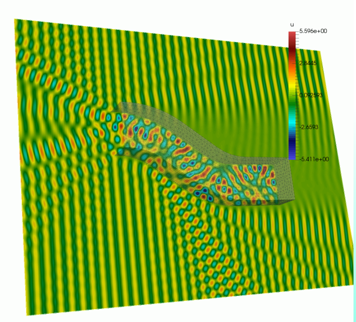
- Grid coarse space vs. one-level

	One-level (min. overlap)					One-level (coarse overlap)					Coarse grid				
$f \setminus N$	10	20	40	80	160	10	20	40	80	160	10	20	40	80	160
1	34	49	72	143	–	30	43	63	97	–	16	18	19	21	–
5	62	94	137	191	268	58	87	126	175	246	29	29	34	34	36
10	85	136	185	272	371	78	124	172	251	346	35	41	43	46	45
20	101	152	213	299	419	92	142	198	272	389	39	47	48	49	49

- H-Geneo coarse space with $\nu = 160$ modes

1	7	8	8	13	–
5	10	9	10	10	12
10	20	16	14	13	13
20	45	40	34	25	19

			Coarse grid		H-GenEO		DtN	
Problem	d	freq	5 ppwl	10 ppwl	5 ppwl	10 ppwl	5 ppwl	10 ppwl
COBRA Cavity	2D	low	✓	✓✓	✗	✗	✓✓	✓
		high	✗	✓✓	✗	✗	✓✓	✓
COBRA Cavity	3D	low	✓	✓✓	✗	✗	✓✓	✓
		high	✗	✓✓	✗	✗	✓✓	✓



Results with 10 points per wavelength

- Grid coarse space vs. one-level

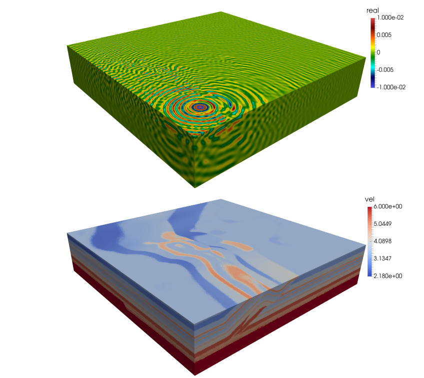
	One-level				Coarse grid			
$k \setminus N$	20	40	80	160	20	40	80	160
50	27	38	47	52	8	8	8	9
100	80	103	115	147	11	23	11	11
150	143	181	235	292	16	16	17	17
200	192	268	308	427	15	15	16	16

- H-Geneo and DtN coarse space with $\nu = 320$ modes

50	23	30	40	44	8	8	11	11
100	63	85	107	133	15	16	18	16
150	119	158	219	279	33	23	26	26
200	182	246	311	430	57	64	54	49

Overthrust

Problem	d	freq	Coarse grid 5 ppwl	Coarse grid 10 ppwl	H-GenEO 5 ppwl	H-GenEO 10 ppwl	DtN 5 ppwl	DtN 10 ppwl
Overthrust	3D	low	✓✓	✓✓	✗	✓	✓	✗
		high	✓✓	✓✓	✗	✗	✗	✗



- Grid coarse space vs. one-level 5 ppwl

$f \setminus N$	One-level (min. overlap) 10	20	40	80	160	One-level (coarse overlap) 10	20	40	80	160	Coarse grid 10	20	40	80	160
0.5	16	21	28	35	–	14	17	20	19	–	9	10	10	10	–
1	27	36	55	62	78	24	30	48	50	62	17	19	23	23	25
2	32	47	64	81	104	27	40	53	68	86	18	22	25	27	28

$f \setminus N$	One-level (min. overlap) 320	640	1280	One-level (coarse overlap) 320	640	1280	Coarse grid 320	640	1280
5	170	209	256	139	173	213	28(7)	30(8)	30(11)

- Grid coarse space vs. one-level 10 ppwl

$f \setminus N$	One-level (min. overlap) 320	640	1280	One-level (coarse overlap) 320	640	1280	Coarse grid 320	640	1280
2	149	185	226	125	156	189	24(7)	24(9)	23(13)

$f \setminus N$	One-level (min. overlap) 2560	One-level (coarse overlap) 2560	Coarse grid 2560
5	365	338	30(15)

Conclusions on coarse spaces

No clear advantage of one method over another (depending on the frequency or precision)

			Coarse grid		H-GenEO		DtN	
Problem	d	freq	5 ppwl	10 ppwl	5 ppwl	10 ppwl	5 ppwl	10 ppwl
Marmousi	2D	low	✓	✓	✓	✓	✓✓	✓✓
		high	✓✓	✓	✗	✓✓	✓	✓
COBRA Cavity	2D	low	✓	✓✓	✗	✗	✓✓	✓
		high	✗	✓✓	✗	✗	✓✓	✓
COBRA Cavity	3D	low	✓	✓✓	✗	✗	✓✓	✓
		high	✗	✓✓	✗	✗	✓✓	✓
Overthrust	3D	low	✓✓	✓✓	✗	✓	✓	✗
		high	✓✓	✓✓	✗	✗	✗	✗

• Parameters can be tuned in both cases (refinement level for the grid CS and # of modes for the spectral CS)
• In the case of multiple RHS the additional precomputation required by the spectral CS becomes less of a burden compared to the inner iterations required to solve the coarse problems when using the grid CS.

[Bootland, Dolean, Jolivet, Tournier, *arXiv*, 2020]

Outline

Motivation and challenges

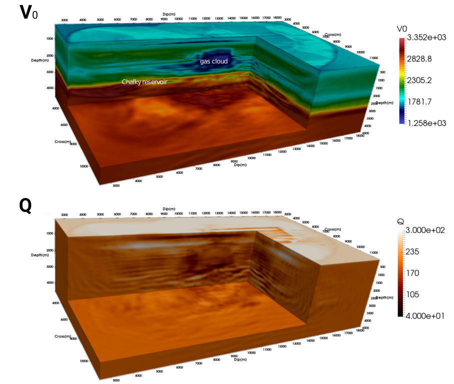
What is the best two-level method for Helmholtz?

The grid coarse space: assessment on a geophysical benchmark problem

General conclusions

Frequency-domain FWI

- Suitable for long-offset acquisitions (few freqs. enough)
- Cheap & straightforward implementation of Q

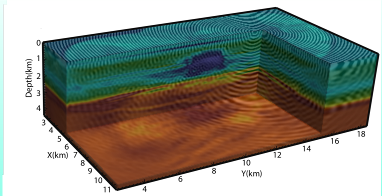


Forward problem & Helmholtz equation

$$(\omega^2/\kappa)u + \Delta u = f$$

- $\kappa = 1/\rho c_P^2$
- ρ is the density
- c_P is the complex-valued P wavespeed

10 Hz monochromatic wavefield



Regular mesh set on smallest wavelength vs. h-adapted unstructured mesh
⇒ # elements decreased by a factor of 2.07

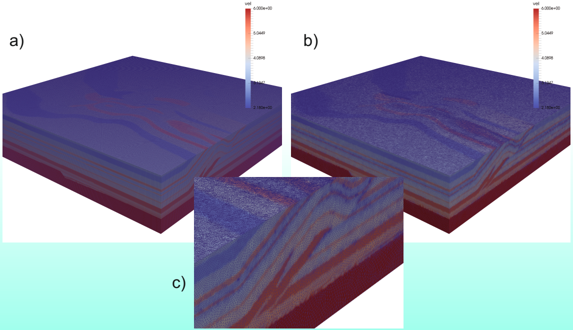


Figure 3: Meshing of Overthrust model, (a) regular and (b,c) adaptive tetrahedral meshes

Test case: P3 FE, 5 points per wavelength, GMRES tolerance $\varepsilon = 10^{-3}$

Single precision arithmetic and approximate factorizations $f = 5$Hz, 74 million dofs, 1060 cores

Cartesian grid				
precision	fine local solver	#it	setup	GMRES
double	Cholesky	10	92.5s	15.5s
double	ICC	10	30.2s	8.9s
single	Cholesky	10	50.3s	10.3s
single	ICC	10	25.8s	6.3s

Table 1: *

Cholesky vs. incomplete Cholesky factorization of local matrices at the fine level
single vs. double precision arithmetic for the whole computation

+ significant memory savings

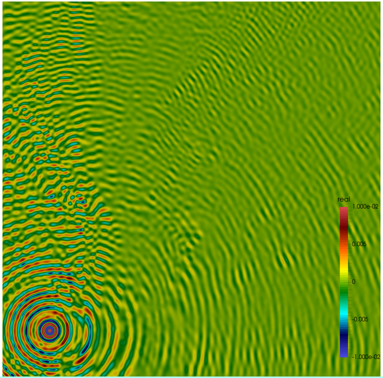
Test case: P3 FE, 5 points per wavelength, GMRES tolerance $\varepsilon = 10^{-3}$

Regular vs. adapted mesh

Regular mesh					
Freq (Hz)	#cores	#elts (M)	#dofs (M)	#it	GMRES
5	265	16	74	7	16s
10	2,120	131	575	15	33s
Adapted mesh					
Freq (Hz)	#cores	#elts (M)	#dofs (M)	#it	GMRES
10	2,120	63	286	14	15s
20	16,960	506	2,285	30	37s

SEG/EAGE Overthrust

Regular vs. adapted mesh



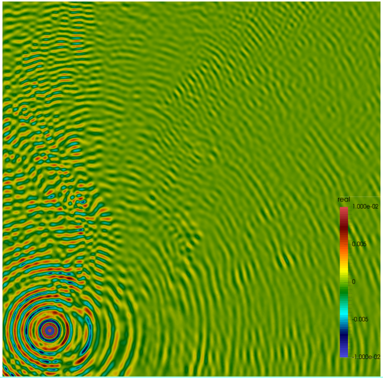


Figure 4: Solution on regular mesh(left) vs. adapted mesh(right) at $f = 10\text{Hz}$

Regular vs. adapted mesh

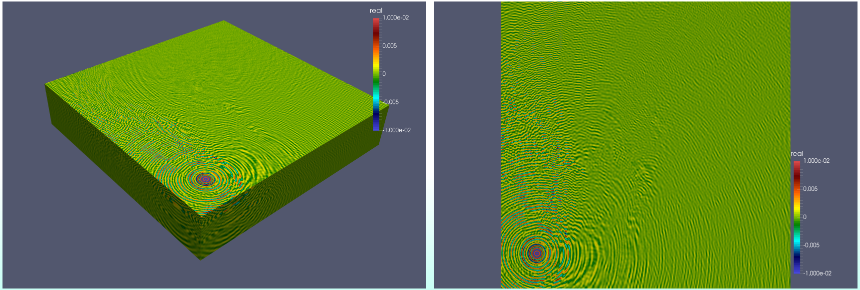


Figure 4: Solution at $f = 20\text{Hz}$

[Dolean, Jolivet, Operto, Tournier, *arXiv*, 2020]

Outline

Motivation and challenges

What is the best two-level method for Helmholtz?

The grid coarse space: assessment on a geophysical benchmark problem

General conclusions

General conclusions

Time harmonic wave problems display difficulties at different levels...

- Theoretical challenges: behaviour of a few methods is not completely understood. (e.g. heterogeneities, spectral coarse spaces..).

- Practical challenges: some applications require the exploitation of specific features of these methods, often not covered by theory (convince a new community to adopt these different methods is not easy!)

- Computational: interplay between precision and parallel performance (not to increase complexity beyond necessity !)

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