Block preconditioning for multiphysics applications

Coupled poromechanics and contact mechanics

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June 7, 2021

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Outline

To simulate different processes, such as mechanics, fluid flow, thermal equilibrium, etc. implies the solution of a **system of PDEs**, where each equation has its own characteristics, e.g., nature (elliptic, parabolic, hyperbolic), time and space scales, and order of magnitude of the coefficients, and can be discretized with different Galerkin approaches (finite volumes, finite elements, etc.).

From a numerical standpoint, we need to solve a system of discretized **non linear equations**. After the linearization stage (Newton's scheme), a **block linear system** is obtained. Due to the different nature of each block, the solution of this linear system can be extremely difficult to obtain.

Preconditioners for single block matrices

To efficiently solve large and sparse linear systems, we have to use **iterative methods** [Saa03]. Direct methods, indeed, have too high memory requirement and are almost sequential by design. However, iterative methods are effective only if properly preconditioned. For a single physics matrix, such as a stiffness matrix or a mass matrix, the literature offers several **preconditioners**:

- incomplete factorizations [Saa94, LM99, Ben02]
- approximate inverses [BMT96, Tan99, Huc03a, JFSG15]
- domain decomposition methods [DJN15, Zam16, BMP16, LS17]
- multigrid methods [MR82, Stü83, BBKL15, PMFJ19]

If we try to use one of these techniques for a block matrix, they will fail, because they do not consider the multiphysics nature of the block matrix.

Monolithic approaches

Standard approaches, such as incomplete factorizations, approximate inverses, and multigrid usually do not work on block matrices, due to the different natures of the blocks. Just to cite few of them, they can be:

- stiffness matrices from structural mechanics problem;
- mass matrices;
- Laplacian matrices from finite volumes based on TFPA/MPFA;
- not deriving from PDEs (e.g., constraints);
- null blocks.

This huge variety requires the use of **specialized** solvers, based on the underlying physics. Indeed, it is of paramount importance to solve with different tool matrices arising from different problems.
This is why monolithic approaches cannot be used and the block approach is used in multiphysics problems.

Let's start with a simple 2×2 example:

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

that corresponds to:

$$\begin{cases} A\mathrm{x} + B\mathrm{y} = \mathrm{f} \\ C\mathrm{x} + D\mathrm{y} = \mathrm{g} \end{cases}$$

Assuming that A is a regular matrix, from the first equation, we can solve for x, obtaining:

$$x = A^{-1}(\mathrm{f} - B\mathrm{y})$$

Substituting in the second equation, we have:

$$CA^{-1}\mathrm{f} - CA^{-1}B\mathrm{y} + D\mathrm{y} = \mathrm{g}$$

where $S = D - CA^{-1}B$ is the **Schur complement**. The solution is:

$$\mathrm{y} = S^{-1}\left(\mathrm{g} - CA^{-1}\mathrm{f}\right)$$

Block LDU factorization

Another way to solve the same problem is to compute its block LDU factorization:

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix} = \mathcal{LDU}$$

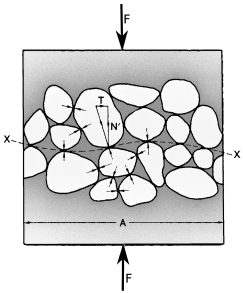
where $S = D - CA^{-1}B$ is *again* the Schur complement. Once the matrix is factorized, the solution is trivial. To apply the inverse of the LDU factorization, the inverse of S is needed.

This is very nice ..., but we **do not know**:

- the inverse of A, to compute the Schur complement
- the inverse of S, the Schur complement itself

According to the different approximation we choose, we will obtain different methods.

Coupled modeling of flow in deformable porous media



Coupled poromechanics is the transient process involving the simultaneous pore fluid flow and solid skeleton deformation in saturated porous media [Bio41, Cou04]. Meaning with σ_{tot} and σ_{eff} the total and effective stress, respectively, and b the Biot coeffient, the relation with the pore pressure p is

$$\sigma_{tot} = \sigma_{eff} + bp.$$

There are several applications in environmental, geotechnical, petroleum and biomedical engineering, such as hydrocarbon recovery, subsurface hydrology, geothermal energy extraction, and geologic carbon storage, but also biomechanical modeling of bone or soft tissue deformations.

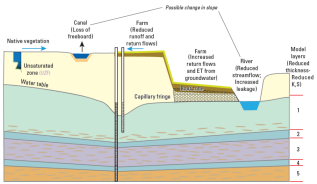


Figure: Subsidence caused by overexploitation of aquifer.

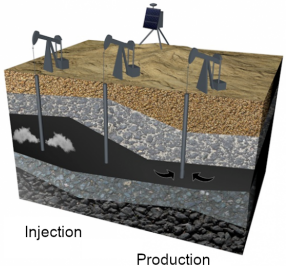


Figure: Injection and production wells in oil field.

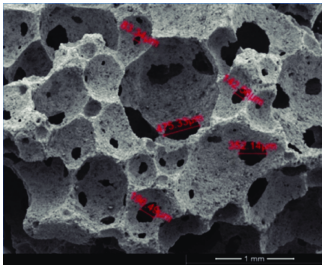


Figure: Microscopic image of a bone.

Numerical models

The numerical solution of Biot's models is still challenging because of two main issues:

1. the instabilities that may arise at the boundary of heterogeneous media as spurious oscillations in the pore pressure solution
2. the efficient numerical solution of the large-size ill-conditioned discretized system of equations

We use a three-field formulation $\boldsymbol{u} - \boldsymbol{v} - p$ combining Galerkin Finite Elements for the equilibrium and Mixed Finite Elements for the flow with the aim at developing a mass conservative approach. Other possible discretizations are hybridized formulations and finite volume techniques, with the common objective of using low-order approximation spaces.

The governing equations are:

$$\begin{cases} -\nabla\boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{f} & \rightarrow \text{linear momentum balance} \\ \mu\boldsymbol{\kappa}^{-1}\boldsymbol{q} + \nabla p = 0 & \rightarrow \text{Darcy's law} \\ b\nabla\dot{\boldsymbol{u}} + M^{-1}\dot{p} + \nabla\boldsymbol{q} = g & \rightarrow \text{mass balance} \end{cases}$$

where $\boldsymbol{\sigma} = \mathbf{C}_{dr} : \nabla^s \boldsymbol{u} - bp\mathbf{1}$ is the total stress, $\mathbf{C}_{dr}$ the rank-four elasticity tensor, μ the fluid viscosity, $\boldsymbol{\kappa}$ the rank-two permeability tensor, M Biot's modulus and $\boldsymbol{f}$ and g external forcing terms.
The primary unknowns are: displacement $\boldsymbol{u}$, Darcy's flux $\boldsymbol{q}$ and pressure p.
Appropriate initial and boundary conditions, such as prescribed tractions $\bar{\boldsymbol{t}}$ and pressures $\bar{p}$, close the problem.

With the spaces:

$$\boldsymbol{\mathcal{U}} = \{\boldsymbol{H}^1(\Omega)\} \qquad \boldsymbol{\mathcal{Q}} = \{\boldsymbol{H}(div;\Omega)\} \qquad \mathcal{P} = \{L^2(\Omega)\}$$

the semidiscrete weak from is: find $\{\boldsymbol{u}, \boldsymbol{q}, p\} \in \boldsymbol{\mathcal{U}} \times \boldsymbol{\mathcal{Q}} \times \mathcal{P}$ such that:

$$\begin{aligned}
&(\nabla^s \boldsymbol{\eta}, \mathbf{C} : \nabla^s \boldsymbol{u})_\Omega - (\nabla \boldsymbol{\eta}, bp)_\Omega = (\boldsymbol{\eta}, \bar{\boldsymbol{t}})_\Gamma + (\boldsymbol{\eta}, \boldsymbol{f})_\Omega && \forall \boldsymbol{\eta} \in \boldsymbol{\mathcal{U}} \\
&\left(\boldsymbol{\psi}, \mu \boldsymbol{\kappa}^{-1} \boldsymbol{q}\right)_\Omega - (\nabla \boldsymbol{\psi}, p)_\Omega = \left(\boldsymbol{\psi} \cdot \boldsymbol{n}, \bar{b}\right)_\Gamma && \forall \boldsymbol{\psi} \in \boldsymbol{\mathcal{Q}} \\
&(\chi, b \nabla \dot{\boldsymbol{u}})_\Omega + \left(\chi, M^{-1} \dot{p}\right)_\Omega + (\chi, \nabla \boldsymbol{q})_\Omega = (\chi, g)_\Omega && \forall \chi \in \mathcal{P}
\end{aligned}$$

This is equivalent to the matrix form:

$$\begin{bmatrix} K & 0 & -Q \\ 0 & A & -B \\ 0 & B^T & 0 \end{bmatrix} \begin{bmatrix} \mathrm{u} \\ \mathrm{q} \\ \mathrm{p} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q^T & 0 & P \end{bmatrix} \begin{bmatrix} \dot{\mathrm{u}} \\ \dot{\mathrm{q}} \\ \dot{\mathrm{p}} \end{bmatrix} = \begin{bmatrix} \mathrm{f} \\ \mathrm{g} \\ \mathrm{h} \end{bmatrix}$$

A popular discrete finite element space for displacement, flux and pressure is $\mathbb{Q}_1 - \mathbb{RT}_0 - \mathbb{P}_0$.

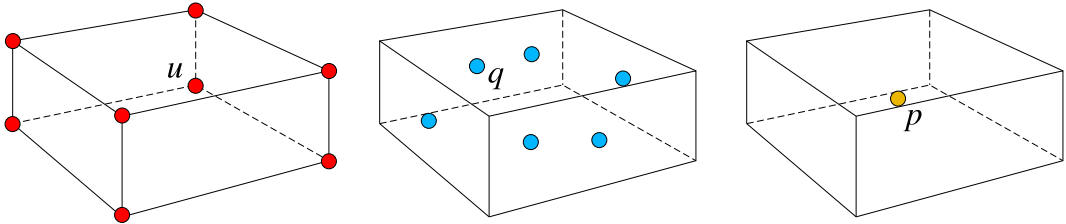


Figure: Location of degrees of freedom for $\mathbb{Q}_1 - \mathbb{RT}_0 - \mathbb{P}_0$ space.

Using implicit Euler as time scheme, with $\gamma = \Delta t$, the linear system is:

$$\mathcal{A}\mathrm{x} = \mathrm{b} \rightarrow \mathcal{A} = \begin{bmatrix} K & 0 & -Q \\ 0 & A & -B \\ \gamma^{-1}Q^T & B^T & \gamma^{-1}P \end{bmatrix}$$

where:

- K and A are symmetric and positive definite matrices (SPD)
- P is a symmetric and positive semidefinite matrix (SPSD)
- Q and B are rectangular coupling blocks

To solve this block linear system, specific techniques are needed.

Stabilization

The presented framework can be unstable in the LBB sense [Woh11]. This happens in the presence of incompressible fluid and solid constituents and undrained conditions. There are different approaches to overcome this limitation (e.g. [SK90]). Common techniques are bubble stabilization [RHO+18] and jump stabilization [FCFW20]. Usually, these approaches do not change the global structure of the block linear system, since they just provide a correction matrix to be summed to one of the already present blocks. Following [FCFW20], we have:

$$\mathcal{A} = \begin{bmatrix} K & 0 & -Q \\ 0 & A & -B \\ \gamma^{-1}Q^T & B^T & \gamma^{-1}P + A_{stab} \end{bmatrix}$$

In other cases, the stabilized system can be reduced, through static condensation, to another with the original block patter.

Pressure-displacement formulations

In the literature there exists several preconditioners for the two-field (pressure-displacement) formulation of the poromechanics problem. The block matrix is:

$$\mathcal{A} = \begin{bmatrix} K & C \\ C^T & P \end{bmatrix}$$

Available approaches are:

- Polynomial-pressure-projection [WB08, HMFJ18, Cho19]
- Block preconditioning [BFG08, WCT16, AGH$^+$20]
- Multigrid methods [GR17, LRGO17]

For the three-field formulation (pressure-velocity-displacement) available works are:

- Bubble functions [RHO$^+$18, NRH21]
- Sequential approaches (fixed stress splitting) [KTJ11, BBN$^+$17, DW18]
- Spectrally-equivalent block diagonal preconditioners [LMW17, HKLW20]
- Block approaches [CWF16, FCF19, FFJ$^+$19]

A general multiphysics problem can be defined as:

$$\mathcal{A} = \begin{bmatrix} A_1^{(1)} & B_2^{(1)} & B_3^{(1)} & \cdots & B_n^{(1)} \\ C_2^{(1)} & A_2^{(2)} & B_3^{(2)} & & B_n^{(2)} \\ C_3^{(1)} & C_3^{(2)} & A_3^{(3)} & & B_n^{(3)} \\ \vdots & & & \ddots & \vdots \\ C_n^{(1)} & C_n^{(2)} & C_n^{(3)} & \cdots & A_n^{(n)} \end{bmatrix}$$

where:

- $A_i^{(i)} \in \mathbb{R}^{n_i \times n_i}$ is the square diagonal block referring to the i-th process described by n_i inner variables, with $i = 1, \ldots, n$;
- $B_i^{(j)} \in \mathbb{R}^{n_j \times n_i}$ and $C_i^{(j)} \in \mathbb{R}^{n_i \times n_j}$, with $i = 2, \ldots, n$ and $j = 1, \ldots, i-1$, are the off-diagonal rectangular blocks coupling the variables associated to the i-th and j-th processes.

Decoupling strategy

The basic idea is to decouple the different processes, so that each single-physics process can be addressed independently of the others. In particular, we have:

$$\mathcal{S} = \mathcal{G}\mathcal{A}\mathcal{F}$$

with no coupling blocks, i.e.:

$$\mathcal{S} = \begin{bmatrix} S_1 & & & & \\ & S_2 & & & \\ & & S_3 & & \\ & & & \ddots & \\ & & & & S_n \end{bmatrix}$$

where $S_i \in \mathbb{R}^{n_i \times n_i}$, $i = 1, \ldots, n$.

Decoupling strategy

Block matrices $\mathcal{G}$ and $\mathcal{F}$ are:

$$\mathcal{G} = \begin{bmatrix} I & & & & \\ G_2^{(1)} & I & & & \\ G_3^{(1)} & G_3^{(2)} & I & & \\ \vdots & & & \ddots & \\ G_n^{(1)} & G_n^{(2)} & G_n^{(3)} & \cdots & I \end{bmatrix}, \qquad \mathcal{F} = \begin{bmatrix} I & F_2^{(1)} & F_3^{(2)} & \cdots & F_n^{(1)} \\ & I & F_3^{(2)} & & F_n^{(2)} \\ & & I & & F_n^{(3)} \\ & & & \ddots & \\ & & & & I \end{bmatrix}$$

Then, $\mathcal{G}\mathcal{A}\mathcal{F} = \mathcal{S}$ if and only if:

$$\begin{cases} A_i F_{i+1} = -B_{i+1} \\ G_{i+1} A_i = -C_{i+1} \end{cases}, \qquad i = 1, \dots, n-1$$

Schur complements and observations

- The matrix $\mathcal{S}$ contains the exact Schur complements, defined as:

$$S_i = A_i^{(i)} - C_i A_{i-1}^{-1} B_i$$

- If the coupled problem is well-posed, $\mathcal{S}$ is non-singular and each Schur complement S_i can be inverted, with the inverse of $\mathcal{A}$ reading:

$$\mathcal{A}^{-1} = \mathcal{F}^{-1} \mathcal{S}^{-1} \mathcal{G}^{-1}$$

- As $\mathcal{F}$, $\mathcal{S}$ and $\mathcal{G}$ are dense, they cannot be generally computed.
- A general preconditioning framework can be obtained by approximating $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{S}^{-1}$, e.g., by solving inexactly the set of multiple right-hand side systems defining F_i and G_i, and the local single-physics systems with S_i. Different local approximations will produce different schemes. The key is to keep each approximation as **sparse** as possible.

Application of the general block preconditioning framework to the three-field poromechanical problem:

$$\mathcal{A} = \begin{bmatrix} A_1^{(1)} & B_2^{(1)} & B_3^{(1)} \\ C_2^{(1)} & A_2^{(2)} & B_3^{(2)} \\ C_3^{(1)} & C_3^{(2)} & A_3^{(3)} \end{bmatrix} = \begin{bmatrix} K & 0 & -Q \\ 0 & A & -B \\ Q^T & \gamma B^T & P \end{bmatrix}$$

Mechanics and Darcy's flow are already decoupled and because of symmetry only one of the two decoupling factors can be computed, e.g., $\mathcal{F}$:

$$\begin{cases} F_3^{(1)} = K^{-1}Q \\ F_3^{(2)} = A^{-1}B \end{cases} \quad \text{and} \quad \begin{cases} G_3^{(1)} = -F_3^{(1),T} \\ G_3^{(2)} = -\gamma F_3^{(2),T} \end{cases}$$

The diagonal blocks of $\mathcal{S}$ are:

$$S_1 = K \qquad S_2 = A \qquad S_3 = P + F_3^{(1),T} K F_3^{(1)} + \gamma F_3^{(2),T} A F_3^{(2)}$$

Approximate inverses

The blocks of the decoupling factor $\mathcal{F}$ can be computed explicitly and inexactly using sparse approximate inverse techniques, such as block FSAI and SPAI, where the sparsity pattern for the blocks is enforced [BKT01, Huc03b, CS98, JF11, JFSG15].

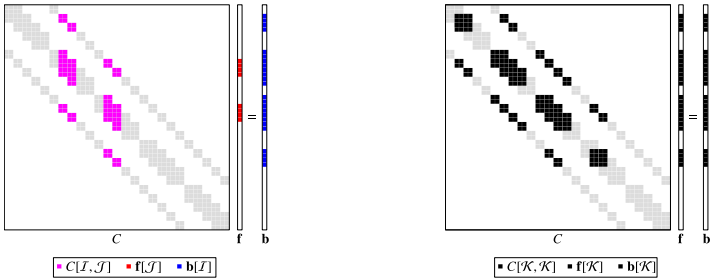


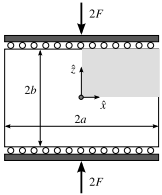
Figure: Schematic representation of a linear system solution subject to sparsity constraints, before (a) and after (b) pattern symmetrization, i.e., forming $\mathcal{K} = \mathcal{I} \cup \mathcal{J}$. Here $\mathcal{I}$ and $\mathcal{J}$ are the sets for rows and columns, respectively. The pattern of matrix C is shown in gray.

By enforcing a sparse structure to the columns of $F_3^{(1)}$ and $F_3^{(2)}$, the size of the multiple right-hand sides systems are reduced. Moreover, they become dense and can be solve in parallel.

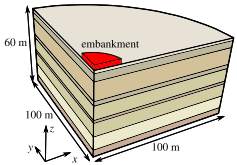
$$\begin{cases} K[\mathcal{K}_1^{(k)}, \mathcal{K}_1^{(k)}]\ F_1[\mathcal{K}_1^{(k)}, k] = Q[\mathcal{K}_1^{(k)}, k] \\ A[\mathcal{K}_2^{(k)}, \mathcal{K}_2^{(k)}]\ F_2[\mathcal{K}_2^{(k)}, k] = B[\mathcal{K}_2^{(k)}, k] \end{cases} , \quad k = 1, \ldots, n_p,$$

To select the subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ is not trivial and an effective choice is based on the combination of static and adaptive approaches [NFA20, FCF21]. The strategy is:

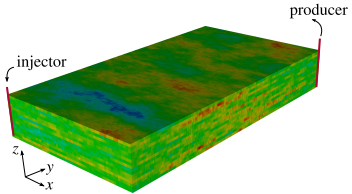
- start from the non-zero pattern of the right-hand side
- add to $\mathcal{K}$ the position corresponding to the largest components of the residual vector
- drop small entries



(a) `Mandel`



(b) `Treporti`



(c) `Reservoir`

Figure: Proposed test cases: sketches.

case	n_u	n_q	n_p	total
`Mandel`	177,147	161,280	51,200	389,627
`Treporti`	178,923	170,257	55,368	404,548
`Reservoir`	687,531	651,280	211,200	1,550,011

Table: Proposed test cases: sizes.

Numerical results

The setup cost is split into two phases: $T_p^{(1)}$ and $T_p^{(2)}$, i.e., a γ-independent and a γ-dependent phase. $T_p^{(1)}$ is a one-time cost, while $T_p^{(2)}$ will be spent any time Δt changes.

	γ	μ	n_{it}	$T_p^{(1)}$ [s]	$T_p^{(2)}$ [s]	T_s [s]
`Mandel`	0.15	1.579	883	19.224	2.035	25.049
	1	1.567	337	25.561	5.702	10.003
	100	1.568	210	24.011	3.866	5.927
`Treporti`	0.01	1.794	272	28.712	9.788	7.992
	1	1.662	183	31.706	10.829	4.989
	100	1.719	91	31.727	7.456	3.011
`Reservoir`	$8.64 \cdot 10^2$	1.144	299	16.805	5.068	28.435
	$8.64 \cdot 10^4$	1.139	182	22.456	9.179	17.740
	$8.64 \cdot 10^6$	1.130	182	16.582	4.926	19.708

Table: Numerical results for the test cases.

Limits of the aFSAI-based approach:

- **robustness**: depending on the problem features, an explicit sparse approximation of the decoupling blocks might not exist, so that large densities could be necessary to obtain convergence;
- **scalability**: the iteration count to converge depends on the size of the spatial and temporal discretizations.

An alternative idea relies on using an implicit approximation of the decoupling block. The application of $F_3^{(1)}$ and $F_3^{(2)}$ is obtained by a matrix-vector product with Q and B, and the application of an inner preconditioner for K and A, respectively ... but we loose the possibility to explicitly compute the Schur complement S_3:

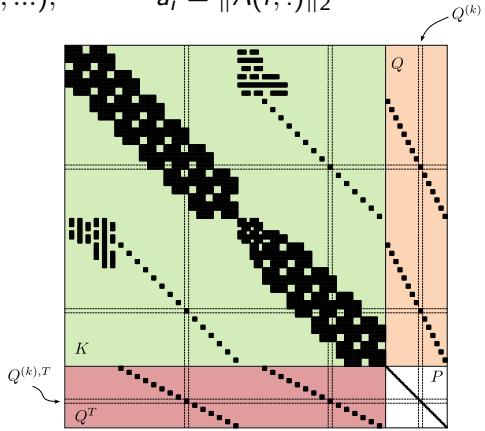
$$S_3 = P + F_3^{(1),T} K F_3^{(1)} + \gamma F_3^{(2),T} A F_3^{(2)} = P + S_K + \gamma S_A$$

We try to use physically-based approximations of S_K and S_A:

- The contribution S_A preserves the structure of a scaled Laplacian [BMS94]:

$$S_A = B^T \widetilde{A}^{-1} B \qquad \widetilde{A} = \text{diag}(a_1, a_2, ...), \qquad a_i = \|A(i,:)\|_2$$

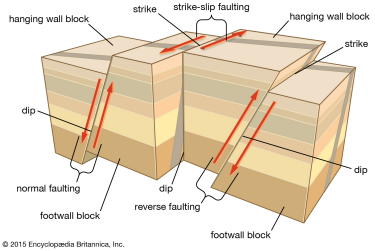
- The contribution S_K can be computed using the classical *uncoupled* solution to precondition the fully *coupled* model, i.e., the so-called fixed-stress approximation [CWT15]. This procedure can be generalized in a purely algebraic way [FCF19].



Contact mechanics problem for hydraulically active fractures

Faults and fractures in geomechanics

Faults and fractures are discontinuities in the subsurface that can slide and/or open. As a consequence of geological movements or human activities (e.g. fracking), they can activate/propagate and cause several phenomena, such as micro-seismicity and fluid leakage, with a huge impact on both the environment and economy of the area. Usually, in the fractures there is fluid, which behavior is strongly coupled with the mechanics response and can be described as a **coupled fluid-structure interaction**.



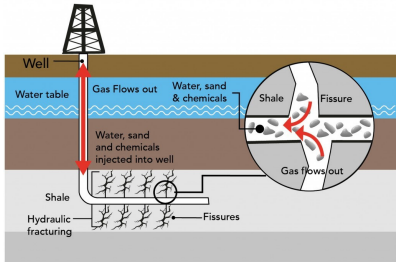


Figure: Earth fissure in Arizona.

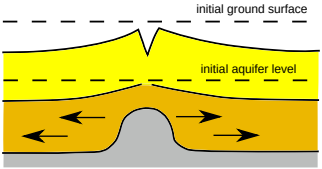


Figure: Scheme of subsidence with fissures.

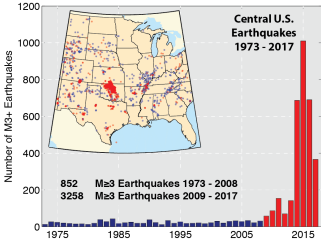


Figure: Number of earthquakes in the Central US.

Numerical models

To model faults and fractures there are several options available, such as:

- explicit discretization of the discontinuity
  - ⋆ a thin layer of finite element with specific properties [RBT08, PRFY14, LKMR19]
  - ⋆ discrete fracture model (DFM) with zero-thickness special interface elements [GTB68, FGJT08, GKFT16, SFW+17]
- implicit discretization of the discontinuity
  - ⋆ embedded discrete fracture model (EDFM) [SS15, RJY16, WDG+19, WYM19]
  - ⋆ extended finite elements methods (XFEM) [FFS16, VK17, KVH18]
- regularized (smooth) field in a continuum discretization, e.g., phase field and damage-mechanics-based approaches [VdB13, AMS+14, WWW14, GLH+19]

We use **DFM** with **Lagrange multipliers** to impose the constraints [HR10, JJ14, FFJT16, BBK+20, KMR19], instead of the penalty method [PO92, ZGL11, GKFT16, SFW+17].

The governing equations are:

$$\begin{cases} -\nabla\boldsymbol{\sigma}(\boldsymbol{u}, p) = 0 & \rightarrow \text{linear momentum balance} \\ \dot{g}_N(\boldsymbol{u}) + \nabla\boldsymbol{q}(\boldsymbol{u}, p) = q_s & \rightarrow \text{mass balance} \\ \boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n}_f - p\boldsymbol{n}_f = 0 & \rightarrow \text{traction balance on the fracture} \end{cases}$$

with the normal and tangential constraints:

$$t_N = \boldsymbol{t} \cdot \boldsymbol{n}_f \leq 0 \qquad g_N = [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}_f \geq 0 \qquad t_N g_N = 0$$
$$\|\boldsymbol{t}_T\|_2 - \tau_{\max}(t_N) \leq 0 \qquad \dot{\boldsymbol{g}}_T \cdot \boldsymbol{t}_T - \tau_{\max}(t_N)\,\|\dot{\boldsymbol{g}}_T\|_2 = 0$$

where $g_N = [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}_f$ is the normal component of the jump, $\boldsymbol{q}(\boldsymbol{u}, p) = -\dfrac{C_f(\boldsymbol{u})}{\mu}\nabla p$ represents Darcy's flux and $\boldsymbol{t}$ are the Lagrangian multipliers, i.e., the tractions.

Discrete weak form

With the spaces:

$$\boldsymbol{\mathcal{U}} = \{\boldsymbol{H}^1(\Omega)\} \quad \boldsymbol{\mathcal{M}}(\boldsymbol{t}_N) = \{\mu \in L(\Gamma_f)^3, \mu_N \leq 0, \|\boldsymbol{\mu}_T\|_2 \leq \tau_{\max}(t_N)\} \quad \mathcal{P} = \{L^2(\Gamma_f)\}$$

the discrete weak from is: find $\{\boldsymbol{u}, \boldsymbol{t}, p\} \in \boldsymbol{\mathcal{U}} \times \boldsymbol{\mathcal{M}}(t_N) \times \mathcal{P}$ such that:

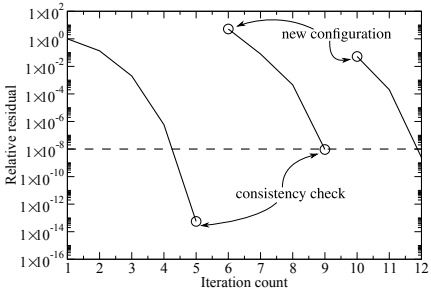
$$\begin{aligned}
&(\nabla^s \boldsymbol{\eta}, \boldsymbol{\sigma})_\Omega + (\llbracket \boldsymbol{\eta} \rrbracket, \boldsymbol{t} - p\boldsymbol{n}_f)_{\Gamma_f} - (\boldsymbol{\eta}, \bar{\boldsymbol{t}})_{\partial\Omega_\sigma} = 0 && \forall \boldsymbol{\eta} \in \boldsymbol{\mathcal{U}} \\
&(t_N - \mu_N, g_N)_{\Gamma_f} + (\boldsymbol{t}_T - \boldsymbol{\mu}_T, \Delta_n \boldsymbol{g}_T)_{\Gamma_f} \geq 0 && \forall \boldsymbol{\mu} \in \boldsymbol{\mathcal{M}}(t_N) \\
&\left(\chi, \frac{\Delta_n g_N}{\Delta_n t}\right)_{\Gamma_f} + [\chi, p]_{\mathcal{F}_f} - F_{\mathcal{F}_f}(\chi) + G_{\mathcal{F}_f}(\chi) - (\chi, q_s)_{\Gamma_f} = 0 && \forall \chi \in \mathcal{P}
\end{aligned}$$

This is equivalent to the matrix form:

$$\begin{bmatrix} K & C_1 & Q_1 \\ C_2 & -H_t & 0 \\ Q_2 & 0 & T + H_p \end{bmatrix} \begin{bmatrix} \mathrm{u} \\ \mathrm{t} \\ \mathrm{p} \end{bmatrix} = \begin{bmatrix} \mathrm{f} \\ \mathrm{g} \\ \mathrm{h} \end{bmatrix}$$

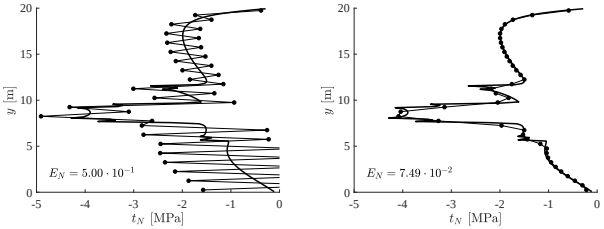
Active set strategy

The contact constraints produce a variational inequality. To solve this, we use the *active-set* strategy, a numerical optimization technique employed in quadratic programming [NW06, AKLR18].
In simple words, we assign an initial status to all the Lagrange multipliers (*active* or *inactive*) and solve the discrete nonlinear problem with Newton's method. Then we check if out hypothesis was correct and, in case, update the element subdivision.



Discrete spaces and stabilization

To avoid interpolations between traction and pressure fields, both defined on the fracture, we use the same discrete space for them, i.e., a cell-centered (face-centered) piece-wise constant representation. For the displacement, a standard linear finite element space is employed. For $\boldsymbol{u} - \boldsymbol{t} - p$ the space is $\mathbb{Q}_1 - \mathbb{P}_0 - \mathbb{P}_0$.
This mixed finite element-finite volume space is not uniformly stable [Woh11] and requires a stabilization. Different approaches are available (macroelement [ESW14], algebraic approaches [FCWT20]), but the underlying idea is the same: to fix the Schur complement with a correction matrix (H_t and H_p in the previous Jacobian).



Pure contact mechanics problem

Let's start analyzing the pure contact mechanics problem, i.e.:

$$\mathcal{A} = \begin{bmatrix} K & C_1 \\ C_2 & -H_t \end{bmatrix}$$

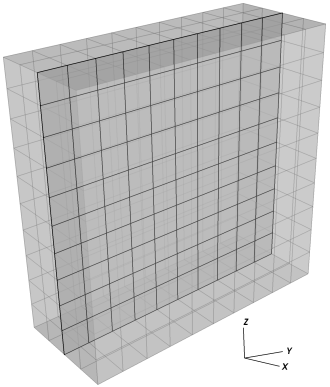
where H_t, the stabilization matrix, is usually singular. As seen before, a block LDU factorization is a possible strategy to build a preconditioner:

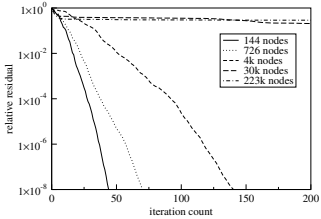
$$\mathcal{A} = \begin{bmatrix} K & C_1 \\ C_2 & -H_t \end{bmatrix} = \begin{bmatrix} I & 0 \\ C_2 K^{-1} & I \end{bmatrix} \begin{bmatrix} K & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & K^{-1} C_1 \\ 0 & I \end{bmatrix} = \mathcal{L}\mathcal{D}\mathcal{U}$$

where the Schur complement is $S = -H_t - C_2 K^{-1} C_1$. Available preconditioners for this problem are [AKW13, FCF19]:

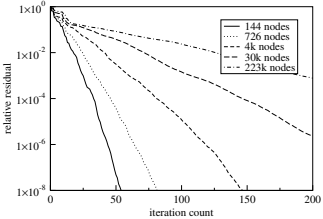
- additive Schwarz preconditioner
- additive/multiplicative field split
- approximate inverses
- block diagonal

All these approaches are based on an approximation of the leading diagonal block K. We compare them on a simple case with 5 refinement levels.

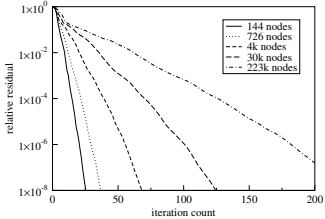




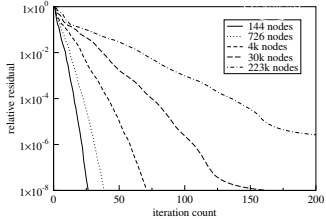
(a) Additive Schwarz method.



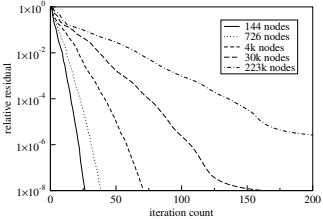
(b) Additive field split.



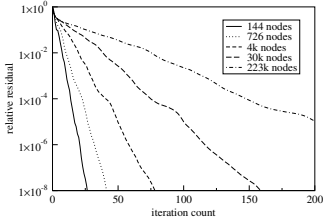
(c) Multiplicative field split.



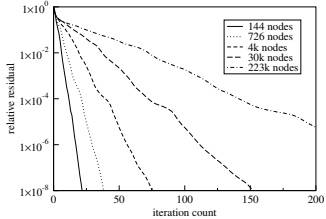
(d) Field split + upper Schur complement.



(e) Field split + full Schur complement.



(f) FSAI + FSAI.



(g) FSAI + block diagonal.

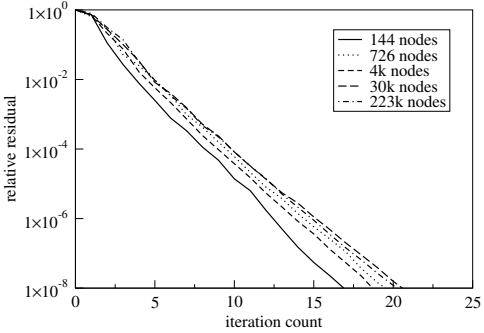
Another Schur complement

According to the different comparisons, it is clear that the the weak scalability is missing, since the number of iterations increases at each refinement level. To improve this aspect, i.e., to build a scalable preconditioner, a better Schur complement is needed.
Another idea is to change the order:

$$\mathcal{A}^\star = \begin{bmatrix} -H_t & C_2 \\ C_1 & K \end{bmatrix}$$

with the Schur complement $S^\star = K + C_1 H_t^{-1} C_2$, but ... H_t is a singular matrix. We can approximate H_t with a regular matrix, e.g., a diagonal matrix. This approach can be extended to other discretizations, also when no stabilization is required.

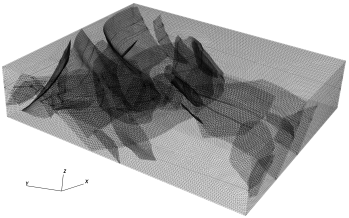
The new Schur complement $S^\star$ is very similar to K, i.e., a structural matrix, and a multigrid preconditioner works properly. For S this was the opposite, since the matrix properties are not suited for a multigrid solver. Moreover, it can be proved that $S^\star$ scales almost perfectly with h. Using a multigrid to solve $S^\star$, the approach is scalable.

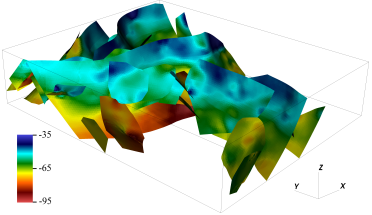


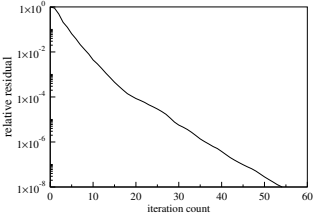
High Island 24L (HI24l) is an hydrocarbon field located in offshore Texas State Waters [Rui19]. It is characterized by 32 faults and the computational grid has 4,848,384 elements, 853,006 nodes and 63,678 interface elements.

n_u	n_t	total
2,559,018	191,034	2,750,052

Table: HI24L: sizes.







Other discretizations are instrisically stable [FFJT16] and provide a Jacobian that is a pure saddle point matrix:

$$\mathcal{A} = \begin{bmatrix} K & C_1 \\ C_2 & 0 \end{bmatrix}$$

For this kind of matrix, a possible approach is to add, for preconditioning purposes only, a diagonal matrix, such as:

$$\mathcal{A}^* = \begin{bmatrix} K & C_1 \\ C_2 & -D \end{bmatrix}$$

where D is the inverse of the **augmentation matrix** [BO06, FGG07]. Then the same preconditioner as before is built on $\mathcal{A}^*$ and applied to $\mathcal{A}$.

The 3×3 Jacobian matrix is:

$$\mathcal{A} = \begin{bmatrix} K & C_1 & Q_1 \\ C_2 & -H_t & 0 \\ Q_2 & 0 & T + H_p \end{bmatrix}$$

Exploiting what we know about the pure contact problem, we derive a strategy for this block linear system:

$$\mathcal{A} = \left[\begin{array}{cc|c} K & C_1 & Q_1 \\ C_2 & -H_t & 0 \\ \hline Q_2 & 0 & T + H_p \end{array}\right] \rightarrow \begin{bmatrix} \mathcal{A}_{cm} & \mathcal{Q}_1 \\ \mathcal{Q}_2 & T + H_p \end{bmatrix}$$

and from here, we compute an approximated *second* Schur complement using the diagonal for the first Schur complement, i.e., the one arising from the pure contact mechanics.

To test this approach, a test case with 320,000 elements, 342,642 nodes and 5,184 interface elements is analyzed. A linearly increasing flow rate is injected in a central well intersecting 9 fractures. During the simulation all fracture states are encountered.

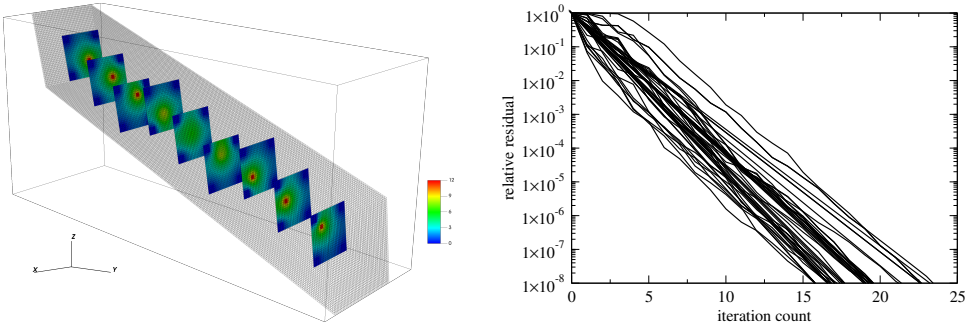


Figure: Computational grid and right-preconditioned GMRES convergence profiles for all the linear systems in one time step.

Conclusions

We have investigated numerical techniques for multiphysics problems, in particular how to build effective preconditioners for block linear systems. Moreover, we have analyzed two different multiphysics problem:

- poroelasticity;
- fluid flow in fractured porous media.

For both of them, specialized block preconditioners based on the problem have been derived. The key is to choose a Schur complement with a physical meaning and approximate it in a cheap way (sparse). This can be done for each multiphysics problem, but usually with different strategies.
To easy this stage, a general framework has been presented and can be used to design a new preconditioner for a novel application.

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