CRM Summer School on Multiphyscis Solvers: Mini-course on Stationary Iterative Methods

Felix Kwok

May 31 – June 1, 2021

The module serves as an introduction to multiphysics solvers and basic notions in iterative methods. The following outline describes the content of the first three lectures, scheduled for Monday May 31 (morning and afternoon) and Tuesday June 1 (morning only). Each lecture is 1.5 hours long. The fourth lecture of this mini-course will take place the following Monday (June 7) and will be given by Hamdi Tchelepi. I may not show all the proofs in detail, but major theorems will be illustrated by numerical examples.

1 Examples of multiphysics problems

Goal: Show examples of multiphysics problems. We illustrate how for some single-physics problems can be solved efficiently by exploiting the underlying physics, but this cannot be applied blindly to multiphysics problems.

1.1 Two-phase flow in porous media

We derive the discrete model using mass conservation. If the fluid pressures are known, then the saturation (i.e., fraction of volume occupied by one of the phases) can be calculated by forward substitution. If the saturation is known, then fluid pressure at a fixed time step satisfies an elliptic equation, which can be solved by multigrid. But the two problems are coupled.

1.2 Different physics in different parts of the domain

We consider the problem of cooling a hot metal object by a cold fluid. The fluid flow is governed by the Navier-Stokes equation; the temperature in the solid is governed by conduction only, whereas in the fluid, both convective and diffusive effects are present. Here, the two subdomains have different physics, and an interface condition couples the two. It is therefore natural to use specialized solvers in the two subdomains and iterate to convergence, but the convergence rate depends on "coupling strength". Other examples of this type (mentioned but not done in detail):

- coupling due to geometry (matrix-fracture coupling)
- coupling due to simplifying assumptions (Navier-Stokes near the airfoil, Stokes far away)

2 Fixed point methods

Goal: Introduce basic notions of iterative methods, as well as some classical iterative methods for linear problems.

2.1 Fundamentals of fixed point methods

We use Newton's method as a first example of fixed point methods. We derive Newton's method for $f(x) = 0$ by doing a Taylor expansion about the initial guess, then write it in fixed point form $x_{k+1} = G(x_k)$. For a general fixed point map G, we deduce the behaviour of the error $e_k = x_k - x^*$ in terms $G(x^*)$ and its derivatives.

2.2 Classical iterative methods for linear problems

We show the classical Jacobi and Gauss-Seidel methods as examples of fixed point methods linear problems. We mention block variants, which can be applied to coupled problems. The error of such methods behaves like $e_k = G^k e_0$ for some G. We show the necessary and sufficient conditions for convergence, as well as the convergence rate, in terms of the spectrum of G. We illustrate the theory by showing actual convergence curves together with the numerically calculated spectrum in a few examples.

3 Krylov methods

Goal: Introduce Krylov methods and their role in accelerating stationary methods.

3.1 Conjugate gradient

We formulate $Ax = b$ as a minimization problem when A is symmetric positive definite. We show visually with a 2×2 problem that steepest descent converges, but it is suboptimal because the search directions are almost linearly dependent. We take conjugate directions to remove this redundancy and obtain CG. We state the optimality property and the classical convergence result in terms of the condition number of A. (The proof may be omitted depending on time and preparation of the participants.) We mention other Krylov methods that can be used for non-SPD problems, without going into detail (MINRES for indefinite problems, GMRES/BiCG/QMR for non-symmetric problems).

3.2 Preconditioning

We explain the basic idea of preconditioning and how stationary iterative methods induce a preconditioned problem. We show how the optimality property implies that CG and GMRES always converge in fewer iterations than the underlying stationary method. Several numerical examples illustrate this.