On the uniform convergence analysis of the Schwarz alternating method for optimal control problems

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Abstract In this paper, we investigate the uniform convergence of the Schwarz alternating method for unconstrained elliptic optimal control problems in one dimension. We derive the convergence factor of the method and find that the convergence factor of the method can be uniformly bounded by a factor (< 1) associated with that for the state equation. We also observe that the local error propagation operators of the method under a standard choice of energy norms in the robust analysis of optimal control problems are nonexpansive. These observations indicate that the existing convergence analysis frameworks of domain decomposition methods for PDEs based on the standard choice of energy norms are not straightforwardly applicable to that for optimal control problems.

1 Introduction

Domain decomposition methods (DDM for short) have been widely used to solve problems in modern science and engineering. The essential parallel ability makes

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them attractive in applications. They have been successfully used to construct fast solvers for partial differential equations and optimal control problems. For more details on the design and the convergence analysis of DDM for the equations, we refer to the monograph [11], the review papers [12, 14] and the references cited therein. As to the design of DDM for nonselfadjoint or indefinite problems, we refer to [2, 3, 13] and the references therein.

Although numerous numerical experiments in the literature have illustrated the efficiency and robustness of DDM for optimal control problems, there are few theoretical results, especially regarding their uniform convergence. In [7], the author discussed a two-level overlapping method for the optimal control problem following the standard framework for two level methods in [2, 3, 13]. The convergence results were obtained under the condition that the coarse mesh size H is sufficiently small. The analysis indicates $H \leq H_0 = O(\alpha^{\frac{1}{2}})$ under the full regularity assumption of the solutions. However, the numerical results in [7] and [10] indicate that the requirement $H \leq H_0 = O(\alpha^{\frac{1}{2}})$ is unnecessary. The main challenge comes from the saddle point structure of the problem.

In this paper, we will explore the convergence properties of DDM for optimal control problems. We will focus on the Schwarz alternating method (which is the origin of various DDM) and the following one-dimensional unconstrained elliptic distributed optimal control problem

$$\min_{u\in L^2(\Omega)} J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \tag{1}$$

subject to

$$-y'' = f + u \ \text{ in } \Omega \quad \text{and} \quad y = 0 \ \text{ on } \partial\Omega, \tag{2}$$

where $\Omega = (0,1)$, $u \in L^2(\Omega)$ is the control variable, $y_d \in L^2(\Omega)$ is the desired state or observation and $\alpha > 0$ is the regularization parameter.

We will derive the exact convergence factor of the method in this case and highlight the fact that the convergence factor of the method for the state equation gives a uniform upper bound of that for the optimal control problem, which has been further developed in an upcoming paper of ours. We also show that the local error propagation operators of the method is not always nonexpansive under a standard choice of energy norms in robust analysis of optimal control problems (cf. Section 4), which makes it impossible to rely on existing frameworks of DDM based on nonexpansive operators for PDEs directly to analyze the uniform convergence of DDM for optimal control problems.

2 The Schwarz alternating method for unconstrained elliptic optimal control problems

Since the optimal control problem (1)-(2) is a convex optimization problem, the solution can be characterized by the following reduced first-order optimality system

$$\begin{cases} -y'' = f - \alpha^{-1} p & \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \\ -p'' = y - y_d & \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

where p is the adjoint state and we eliminate u by the equation $\alpha u + p = 0$ in the first order optimality system.

Let $\Omega_1 = (0, s), \Omega_2 = (r, 1)$ with $0 < r < s < 1$. The Schwarz alternating method for the state equation and the optimal control problem is given by

Algorithm 1. *1. Initialization: choose $y^{(0)} \in H_0^1(\Omega)$.*
2. For $k = 0, 1, \cdots$, solving the following problem on Ω_i $(i = 1, 2)$ alternatively:

$$\begin{cases} -(y^{(2k+i)})'' = f \quad \text{in } \Omega_i, \quad y^{(2k+i)} = y^{(2k+i-1)} \quad \text{on } \partial\Omega_i, \\ \quad y^{(2k+i)} \quad = y^{(2k+i-1)} \quad \text{in } \overline{\Omega} \setminus \overline{\Omega_i}. \end{cases}$$

and

Algorithm 2. *1. Initialization: choose $y^{(0)},\ p^{(0)} \in H_0^1(\Omega)$.*
2. For $k = 0, 1, \cdots$, solving the following problem on Ω_i $(i = 1, 2)$ alternatively:

$$\begin{cases} -(y^{(2k+i)})'' = f - \alpha^{-1} p^{(2k+i)} \quad \text{in } \Omega_i, \quad y^{(2k+i)} = y^{(2k+i-1)} \quad \text{on } \partial\Omega_i, \\ -(p^{(2k+i)})'' = y^{(2k+i)} - y_d \quad \text{in } \Omega_i, \quad p^{(2k+i)} = p^{(2k+i-1)} \quad \text{on } \partial\Omega_i, \\ \quad y^{(2k+i)} \quad = y^{(2k+i-1)}, \quad p^{(2k+i)} = p^{(2k+i-1)} \quad \text{in } \overline{\Omega} \setminus \overline{\Omega_i}. \end{cases}$$

3 Convergence analysis

In this part, we give an explicit formulation of the convergence rate of the Schwarz alternating method in the one-dimensional case. One can also refer to [9] for details of the calculation. The main technique lies in solving fourth order ODEs in the variable p after elimination of y from the homogeneous version of (3).

It is easy to derive that the convergence factor of that for Algorithm 1 is $\rho_e = \frac{r(1-s)}{s(1-r)} < 1$, since $0 < r < s < 1$. Now we focus on Algorithm 2.

Let $e_y^{(k)} = y^{(k)} - y$ and $e_p^{(k)} = p^{(k)} - p$ for $k = 0, 1, \ldots$. The error systems of the method are given by

$$\begin{cases} -(e_y^{(2k+1)})'' = -\alpha^{-1} e_p^{(2k+1)} & \text{in } (0, s),\ e_y^{(2k+1)}(0) = 0,\ e_y^{(2k+1)}(s) = e_y^{(2k)}(s), \\ -(e_p^{(2k+1)})'' = e_y^{(2k+1)} & \text{in } (0, s),\ e_p^{(2k+1)}(0) = 0,\ e_p^{(2k+1)}(s) = e_p^{(2k)}(s) \end{cases} \tag{4}$$

and

$$\begin{cases} -(e_y^{(2k+2)})'' = -\alpha^{-1} e_p^{(2k+2)} & \text{in } (r, 1),\ e_y^{(2k+2)}(r) = e_y^{(2k+1)}(r),\ e_y^{(2k+2)}(1) = 0, \\ -(e_p^{(2k+2)})'' = e_y^{(2k+2)} & \text{in } (r, 1),\ e_p^{(2k+2)}(r) = e_p^{(2k+1)}(r),\ e_p^{(2k+2)}(1) = 0. \end{cases} \tag{5}$$

A direct calculation shows that the solutions of (4) and (5) satisfy

$$\begin{cases} (e_y^{(2k+1)}(x))^2 + \alpha^{-1}(e_p^{(2k+1)}(x))^2 = L(x,s)\left((e_y^{(2k)}(s))^2 + \alpha^{-1}(e_p^{(2k)}(s))^2\right), \ x \in [0,s], \\ (e_y^{(2k+2)}(x))^2 + \alpha^{-1}(e_p^{(2k+2)}(x))^2 = R(r,x)\left((e_y^{(2k+1)}(r))^2 + \alpha^{-1}(e_p^{(2k+1)}(r))^2\right), \ x \in [r,1], \end{cases}$$

where

$$\begin{cases} L(x,s) = \dfrac{\sinh^2(\gamma x) + \sin^2(\gamma x)}{\sinh^2(\gamma s) + \sin^2(\gamma s)} \quad x \in [0,s], \\ R(r,x) = \dfrac{\sinh^2(\gamma(1-x)) + \sin^2(\gamma(1-x))}{\sinh^2(\gamma(1-r)) + \sin^2(\gamma(1-r))} \quad x \in [r,1], \end{cases} \tag{6}$$

and $\gamma = \frac{\sqrt{2}}{2}\alpha^{-\frac{1}{4}}$. Since

$$g(z) = \sinh^2(z) + \sin^2(z) = \frac{1}{2}(\cosh(2z) - \cos(2z))$$

is positive, strictly convex, strictly increasing in $(0,\infty)$ and $\lim_{z\to+\infty} g(z) = +\infty$, we have

$$\max_{x\in[0,1]}\left((e_y^{(2k+1)}(x))^2 + \alpha^{-1}(e_p^{(2k+1)}(x))^2\right) = (e_y^{(2k)}(s))^2 + \alpha^{-1}(e_p^{(2k)}(s))^2,$$
$$\max_{x\in[0,1]}\left((e_y^{(2k+2)}(x))^2 + \alpha^{-1}(e_p^{(2k+2)}(x))^2\right) = (e_y^{(2k+1)}(r))^2 + \alpha^{-1}(e_p^{(2k+1)}(r))^2.$$

Hence

$$\max_{x\in[0,1]}\left((e_y^{(2k+2)}(x))^2 + \alpha^{-1}(e_p^{(2k+2)}(x))^2\right) = \rho_c \max_{x\in[0,1]}\left((e_y^{(2k)}(x))^2 + \alpha^{-1}(e_p^{(2k)}(x))^2\right),$$

where $k = 0,1,2,\ldots$, $\rho_c = L(r,s)R(r,s)$ is the convergence factor. By the convexity of $g(z)$, $g(0) = 0$ and the fact $0 < r < s$, we have

$$\cosh(2\gamma r) - \cos(2\gamma r) = \cosh(2\gamma s\frac{r}{s}) - \cos(2\gamma s\frac{r}{s}) < \frac{r}{s}(\cosh(2\gamma s) - \cos(2\gamma s)),$$

which implies $\rho_c < \rho_e < 1$.

Furthermore, for given $0 < r < s < 1$, ρ_c is strictly increasing with respect to α, which implies that $\rho_c(\alpha) < \lim_{\alpha\to+\infty}\rho_c(\alpha) = \frac{r^2(1-s)^2}{s^2(1-r)^2} = \rho_e^2$ for any $\alpha > 0$.

4 Convergence analysis based on energy norms

We provide some observations on the convergence analysis based on energy norms. These observations indicate that the existing convergence analysis frameworks of the domain decomposition methods for PDEs based on a standard choice of energy norms in the robust analysis of optimal control problems are not straightforwardly applicable to optimal control problems. It is worth noting that numerical results show the uniform convergence of the method with a convergence factor under this standard energy norm.

Let V be a Hilbert space and $s(\cdot,\cdot)$ be a bilinear form on V. Denote by $(\cdot,\cdot)$ the inner product of V and $\|\cdot\|$ be the induced norm. We consider the following problem:

Find $\bar{z} \in V$ such that

$$s(\bar{z}, w) = (f, w) \quad \forall\, w \in V$$

for $f \in V'$ with V' being the dual space of V. We also assume that

$$s(z, w) \le C_s \|z\|\|w\| \;\text{ and }\; \inf_{z\in V}\sup_{w\in V}\frac{s(z,w)}{\|z\|\|w\|} = \inf_{w\in V}\sup_{z\in V}\frac{s(z,w)}{\|z\|\|w\|} > 0.$$

Let $V = V_1 + V_2$ and V_i $(i = 1, 2)$ be two closed subspaces of V. We define $T_i : V \to V_i$ by

$$s(T_i z, w_i) = s(z, w_i) \quad \forall\, z \in V,\ \forall\, w_i \in V_i.$$

Then the local error propagation operators of the Schwarz alternating method based on the decomposition $V = V_1 + V_2$ are given by $I - T_i$ $(i = 1, 2)$ and the global error propagation operator is given by

$$E = (I - T_2)(I - T_1).$$

One crucial assumption in the convergence analysis of the method under the norm $\|\cdot\|$ is that $I - T_i$ $(i = 1, 2)$ are nonexpansive (which was also used originally in P.-L. Lions [6]; see also [4] for further details.) or more generally (cf. [15, Assumption (A2)])

$$\|T_i v\|^2 \le \omega(T_i v, v),\ \forall v \in V \;\text{ for some constant }\; \omega \in (0, 2). \tag{7}$$

Now, we investigate the local error propagation operators of the Schwarz alternating method for optimal control problems. We will illustrate that the assumption (7) fails in the OCP case for the simple one dimensional problem. This indicates that the local error propagation operators are expansive in this case.

The convergence of the Schwarz alternating method for the optimal control problem is equivalent to the convergence for the following coupled equation (after proper scaling or changing of variables)

$$\begin{cases} -\alpha^{1/2}\tilde{p}'' - \tilde{y} = 0 \text{ in } \Omega, & \tilde{y} = 0 \text{ on } \partial\Omega, \\ -\alpha^{1/2}\tilde{y}'' + \tilde{p} = 0 \text{ in } \Omega, & \tilde{p} = 0 \text{ on } \partial\Omega, \end{cases} \tag{8}$$

which is equivalent to (cf. [1, 5])

$$s((\tilde{y}, \tilde{p}), (\varphi, \phi)) = 0 \quad \forall\, (\varphi, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega),$$

where $s((y, p), (\varphi, \phi)) = \alpha^{1/2}\int_\Omega p'\phi' dx - \alpha^{1/2}\int_\Omega y'\varphi' dx - \int_\Omega y\phi dx - \int_\Omega \varphi p dx$.

For the bilinear form $s(\cdot,\cdot)$ defined above, we have (cf. [8, 10, 1, 5])

$$\beta_0\|(y, p)\|_\alpha \le \sup_{0\ne(\varphi,\phi)\in H_0^1(\Omega)\times H_0^1(\Omega)} \frac{s((y, p), (\varphi, \phi))}{\|(\varphi, \phi)\|_\alpha} \le \beta_1\|(y, p)\|_\alpha, \tag{9}$$

for any $(y, p) \in H_0^1(\Omega) \times H_0^1(\Omega)$, where $\beta_0 > 0$, $\beta_1 > 0$ are independent of α and

$$\|(\varphi, \phi)\|_\alpha^2 = \alpha^{1/2}|\varphi|^2_{H_0^1(\Omega)} + \|\varphi\|^2_{L^2(\Omega)} + \alpha^{1/2}|\phi|^2_{H_0^1(\Omega)} + \|\phi\|^2_{L^2(\Omega)}. \tag{10}$$

It is worth noting that the norm $\|\cdot\|_\alpha$ has been successfully used to construct robust methods for optimal control problems, such as multigrid methods (e.g. [8, 1]), preconditioned MINRES methods (e.g. [10]) and adaptive finite element methods (e.g. [5]). In order to obtain robust convergence results of the Schwarz alternating method, which has been observed numerically, we also use this norm here. However, our examples below show that the local error propagation operators are not nonexpansive under this norm.

Remark 1 One can certainly consider another (possibly more sophisticated) norm for the space of solutions, but whether such a norm would lead to a satisfactory convergence analysis remains an open question.

Let $\mathcal{T}_h$ be a uniform partition of $\Omega = (0,1)$ and the mesh size is denoted by h. In the following, we consider the cases where the problem is discretized by finite difference methods and finite element methods. We first give a nonoverlapping decomposition of Ω and then add several layers to each subdomain to generate the overlapping decomposition. We consider the case where one layer is added (cf. Figure 1).

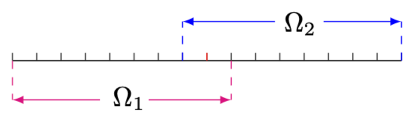

Fig. 1 Mesh and decomposition of $\Omega = (0,1)$.

We continue now for the case of two different discretization techniques. Without loss of generality, we only focus on the three nodes related to the overlapping part in each case.

4.1 The finite difference case

We discretize the problem (8) by the central difference method. The matrix related to the discrete problem is given by

$$C_1 = \begin{pmatrix} tA_1 & -B_1 \\ -B_1 & -tA_1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $t = \alpha^{1/2}h^{-2} > 0$. The proper inner product related to C_1 on $\mathbb{R}^3 \times \mathbb{R}^3$ is

$$((v,w),(u,r))_t = tv^T A_1 u + v^T B_1 u + tw^T A_1 r + w^T B_1 r \quad \forall\, (v,w),\ (u,r) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Therefore, we have $\beta_0\|Y\|_t \le \sup_{0\neq Z\in\mathbb{R}^3\times\mathbb{R}^3} \frac{Y^T C_1 Z}{\|Z\|_t} \le \beta_1\|Y\|_t, \ \forall\, Y \in \mathbb{R}^3\times\mathbb{R}^3$, where $\|\cdot\|_t$ is the norm induced by the inner product $(\cdot,\cdot)_t$ given above and β_0, β_1 are the same constants as in (9).

For given $(v,w)\in\mathbb{R}^3\times\mathbb{R}^3$, $(v_1,w_1) = T_1(v,w)$ can be computed as follows.

(1) Compute $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T$ by $F = C_1\begin{pmatrix} v \\ w\end{pmatrix}$.

(2) Set $F_1 = (f_1, f_2, f_4, f_5)^T$ and solve

$$C_2\begin{pmatrix} v_1 \\ w_1\end{pmatrix} = \begin{pmatrix} tA_2 & -B_2 \\ -B_2 & -tA_2\end{pmatrix}\begin{pmatrix} v_1 \\ w_1\end{pmatrix} = F_1, \ \text{where } A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2\end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}.$$

If we take $(v,w) = (0,0,v^{(3)},0,0,w^{(3)})$, we have

$$F = (0, -tv^{(3)}, 2tv^{(3)} - w^{(3)}, 0, tw^{(3)}, -v^{(3)} - 2tw^{(3)})^T,$$

$$\|T_1(v,w)\|_t^2 = \frac{t^2(6t^3+5t^2+2t+1)}{(9t^4+10t^2+1)}[(v^{(3)})^2 + (w^{(3)})^2]$$

and

$$(T_1(v,w),(v,w))_t = \frac{2t^3(3t^2+1)}{(9t^4+10t^2+1)}[(v^{(3)})^2 + (w^{(3)})^2].$$

Therefore, for $t > 0$ and $(v,w)\neq 0$, we have

$$\eta = \frac{\|T_1(v,w)\|_t^2}{(T_1(v,w),(v,w))_t} = \frac{6t^3+5t^2+2t+1}{2t(3t^2+1)}.$$

This means that the assumption (7) does not hold uniformly for all α and h.

4.2 The finite element case

As in the finite difference case, we can do a similar analysis for the finite element case. We use the linear Lagrange finite element method to discretize the problem. In this setting, we have

$$\tilde{B}_1 = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4\end{pmatrix}, \quad \tilde{B}_2 = \begin{pmatrix} 4 & 1 \\ 1 & 4\end{pmatrix}, \quad \tilde{C}_1 = \begin{pmatrix} \tilde{t}A_1 & -\tilde{B}_1 \\ -\tilde{B}_1 & -\tilde{t}A_1\end{pmatrix}, \quad \tilde{C}_2 = \begin{pmatrix} \tilde{t}A_2 & -\tilde{B}_2 \\ -\tilde{B}_2 & -\tilde{t}A_2\end{pmatrix}$$

with $\tilde{t} = 6\alpha^{1/2}h^{-2} > 0$ and the proper inner product related to $\tilde{C}_1$ on $\mathbb{R}^3\times\mathbb{R}^3$ is

$$((v,w),(u,r))_{\tilde{t}} = \tilde{t}v^T A_1 u + v^T\tilde{B}_1 u + \tilde{t}w^T A_1 r + w^T\tilde{B}_1 r \quad \forall\,(v,w),\ (u,r)\in\mathbb{R}^3\times\mathbb{R}^3.$$

As in the previous part, we take $(v,w) = (0,0,v^{(3)},0,0,w^{(3)})$, we have

$$F = (0, -\tilde{t}v^{(3)} - w^{(3)}, 2\tilde{t}v^{(3)} - 4w^{(3)}, 0, -v^{(3)} + \tilde{t}w^{(3)}, -4v^{(3)} - 2\tilde{t}w^{(3)})^T,$$

$$\|T_1(v,w)\|_{\tilde{t}}^2 = \frac{2(\tilde{t}^3+4\tilde{t}^2+7\tilde{t}+10)}{3(\tilde{t}^2+25)}[(v^{(3)})^2+(w^{(3)})^2]$$

and

$$(T_1(v,w),(v,w))_{\tilde{t}} = \frac{2(\tilde{t}-1)(\tilde{t}^2-2)(\tilde{t}^2+5)}{3(\tilde{t}^2+1)(\tilde{t}^2+25)}[(v^{(3)})^2+(w^{(3)})^2].$$

Especially, for $1 < \tilde{t} < \sqrt{2}$, we have $(T_1(v,w),(v,w))_{\tilde{t}} < 0$.

For $(\tilde{t}-1)(\tilde{t}^2-2) \neq 0$ and $(v,w) \neq 0$, we have

$$\tilde{\eta} = \frac{\|T_1(v,w)\|_{\tilde{t}}^2}{(T_1(v,w),(v,w))_{\tilde{t}}} = \frac{(\tilde{t}^3+4\tilde{t}^2+7\tilde{t}+10)(\tilde{t}^2+1)}{(\tilde{t}-1)(\tilde{t}^2-2)(\tilde{t}^2+5)}.$$

This also means that the assumption (7) does not hold uniformly for all α and h.

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