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Abstract. Various nonlinear Schwarz domain decomposition methods were proposed to solve the one–dimensional equidistribution principle in SINUM Vol. 50, Issue 4, pp. 2111-2135, 2012, DOI: 10.1137/110849936. A corrected proof of convergence for the linearized Schwarz algorithm presented in Section 3.2, under additional hypotheses, is presented here. An alternative linearized Schwarz algorithm for equidistributed grid generation is also provided.

Key words. domain decomposition, Schwarz methods, moving meshes, equidistribution

AMS subject classifications. 65M55, 65M15, 65Y05, 65M50, 65L50, 65N50

In [1] we analyzed various parallel domain decomposition (DD) methods applied to the nonlinear boundary value problem formulation of the 1D equidistribution principle.

If $u(x)$ is a given function on $\Omega_p = (0,1)$, then an equidistributing mesh transformation is found by solving the nonlinear two–point boundary value problem (BVP)

$$\frac{d}{d\xi}\left(M(x,u)\frac{dx}{d\xi}\right) = 0, \quad x(0) = 0, \quad x(1) = 1, \tag{1}$$

on $\Omega_c := (0,1)$ for the mesh transformation $x(\xi) : \Omega_c \to \Omega_p$. After analyzing nonlinear Schwarz methods to solve this BVP, a Schwarz iteration requiring only a single linear solve during each iteration was proposed in Section 3.2 of [1]. For this iteration, we decompose the domain $\Omega = (0,1)$ into two overlapping subdomains $\Omega_1 = (0,\beta)$ and $\Omega_2 = (\alpha,1)$ with $\alpha < \beta$. The proposed parallel Schwarz iteration then computes: for $n = 1,2,\ldots$

$$\begin{aligned} (M(x_1^{n-1})x_{1,\xi}^n)_\xi &= 0, \quad \xi \in \Omega_1, & (M(x_2^{n-1})x_{2,\xi}^n)_\xi &= 0, \quad \xi \in \Omega_2, \\ x_1^n(0) &= 0, & x_2^n(\alpha) &= x_1^{n-1}(\alpha), \\ x_1^n(\beta) &= x_2^{n-1}(\beta), & x_2^n(1) &= 1. \end{aligned} \tag{2}$$

The nonlinear mesh density function $M(x)$ is frozen at the previous iteration, and hence each Schwarz iteration involves the solution of a single linear BVP on each subdomain.

Felix Kwok pointed out that the inequality used in [1] to majorize the series in the expression for the subdomain solution $x_1^n(\xi)$ does not hold in general, and in fact we do not always have convergence of the sequence $\{x_j^n(\xi)\}$ for $j = 1,2$. In fact, the lack of convergence for some mesh density functions has been pointed out already in [3] for the analogous (linearized) single domain iteration. We show below, however, that the conclusion of Theorem 3.4 of [1] is correct under additional hypotheses on the mesh density function $M(x)$.

Suppose $x(\xi) \in X$, where X is the space of twice continuously differentiable functions satisfying $x(0) = 0, x(1) = 1$ and $0 \le x(\xi) \le 1$ for all $\xi \in [0,1]$. Assume $M(x)$ is a continuously differentiable function of the mesh transformation $x = x(\xi)$ and that M is uniformly bounded away from zero and infinity, that is, there exists constants $\check{m}$ and $\hat{m}$ so that

$$0 < \check{m} \le M(x) \le \hat{m} < \infty, \qquad \text{for all}\, x \in [0,1]. \tag{3}$$

Defining $N(x) := 1/M(x)$, the function $N(x)$ satisfies

$$\frac{1}{\hat{m}} \le N(x) \le \frac{1}{\check{m}}.$$

We now assume, in addition, N is Lipschitz with Lipschitz constant $L_N > 0$,

$$|N(x) - N(y)| \le L_N |x - y|, \qquad \text{for all} \quad x, y \in [0, 1]. \tag{4}$$

Hence for any ξ and $x, y \in X$ we have

$$|N(x(\xi)) - N(y(\xi))| \le L_N \|x - y\|_\infty, \tag{5}$$

where the $\|\cdot\|_\infty$ is the infinity norm over ξ.

Suppose X_1 is the space of twice continuously differentiable functions on $(0, \beta)$ satisfying $x(0) = 0$ and X_2 is the space of twice continuously differentiable functions on $(\alpha, 1)$ satisfying $x(1) = 1$.

The following lemma, from [1], providing a representation of the subdomain solutions in (2), is needed in the argument below.

Lemma 1. *Assume M is continuously differentiable and satisfies (3) and (4). The subdomain solutions, $x_1^n(\xi) \in X_1$ and $x_2^n(\xi) \in X_2$ of (2) are unique and given by*

$$x_1^n(\xi) = x_2^{n-1}(\beta)\tilde{G}_1(\xi, x_1^{n-1}) \tag{6}$$

and

$$x_2^n(\xi) = x_1^{n-1}(\alpha) + (1 - x_1^{n-1}(\alpha))\tilde{G}_2(\xi, x_2^{n-1}), \tag{7}$$

where

$$\tilde{G}_1(\xi, x) := \frac{G_1(\xi, x)}{G_1(\beta, x)} \qquad \text{and} \qquad \tilde{G}_2(\xi, x) := \frac{G_2(\xi, x)}{G_2(1, x)},$$

with

$$G_1(\xi, x) := \int_0^\xi \frac{d\tilde{\xi}}{M(x(\tilde{\xi}))} \qquad \text{and} \qquad G_2(\xi, x) := \int_\alpha^\xi \frac{d\tilde{\xi}}{M(x(\tilde{\xi}))}. \tag{8}$$

The following fact, also from [1], will be useful in our analysis.

Lemma 2. *For any initial guesses $x_1^0(\beta)$ and $x_2^0(\alpha)$ satisfying $0 \le x_1^0(\beta) \le 1$ and $0 \le x_2^0(\alpha) \le 1$, the subdomain solutions $x_1^n(\xi)$ and $x_2^n(\xi)$ defined in (6) and (7) are members of X_1 and X_2 respectively. Furthermore, $0 \le \tilde{G}_1(\xi, x) \le 1$ for $\xi \in (0, \beta)$ and $0 \le \tilde{G}_2(\xi, x) \le 1$ for $\xi \in (\alpha, 1)$ for functions $x \in X_1$ and $x \in X_2$ respectively.*

To analyze the iteration we consider the double step as described in Lemma 3.

Lemma 3. *Using the notation above, a double step of the iteration (2) may be written as the mapping*

$$\begin{pmatrix} x_1^n(\xi) \\ x_2^n(\xi) \end{pmatrix} = \begin{pmatrix} F_1(x_1^{n-2}(\xi), x_2^{n-2}(\xi)) \\ F_2(x_1^{n-2}(\xi), x_2^{n-2}(\xi)) \end{pmatrix}, \tag{9}$$

where

$$F_1(x_1^{n-2}, x_2^{n-2}) := \left(x_1^{n-2}(\alpha) + (1 - x_1^{n-2}(\alpha))\tilde{G}_2(\beta, x_2^{n-2})\right)\tilde{G}_1\left(\xi, x_2^{n-2}(\beta)\tilde{G}_1\left(\xi, x_1^{n-2}\right)\right) \tag{10}$$

$$F_2(x_1^{n-2},x_2^{n-2}) := \Big(1-x_2^{n-2}(\beta)\tilde G_1(\alpha,x_1^{n-2})\Big)\tilde G_2\left(\xi, x_1^{n-2}(\alpha)+(1-x_1^{n-2}(\alpha))\tilde G_2(\xi,x_2^{n-2})\right) + x_2^{n-2}(\beta)\tilde G_1(\alpha,x_1^{n-2}). \tag{11}$$

Proof. Substituting the expression for $x_2^{n-1}(\beta)$ from (7) at iteration $n-1$ into (6) and substituting the expression for $x_1^{n-1}(\alpha)$ from (6) at iteration $n-1$ into (7), we obtain

$$\begin{aligned} x_1^n(\xi) &= x_2^{n-1}(\beta)\tilde G_1(\xi,x_1^{n-1}) = [x_1^{n-2}(\alpha)+(1-x_1^{n-2}(\alpha))\tilde G_2(\beta,x_2^{n-2})]\tilde G_1(\xi,x_1^{n-1}) \\ x_2^n(\xi) &= x_1^{n-1}(\alpha)+(1-x_1^{n-1}(\alpha))\tilde G_2(\xi,x_2^{n-1}) \\ &= x_2^{n-2}(\beta)\tilde G_1(\alpha,x_1^{n-2})[1-\tilde G_2(\xi,x_2^{n-1})]+\tilde G_2(\xi,x_2^{n-1}). \end{aligned}$$

Then substituting the expressions for x_1^{n-1} and x_2^{n-1} from (6) and (7) gives (9) with F_1 and F_2 as in (10) and (11). ▯

We prove that iteration (2) converges by showing that the mapping $F=(F_1,F_2)$ defined in (10) and (11) is a contraction in $\|\cdot\|_\infty$ under suitable restrictions on M.

We begin with some preliminary results.

Lemma 4. *For any functions $x\in X_1$ and $y \in X_2$ we have*

$$|G_1(\beta,x)-G_1(\beta,y)| \le \beta L_N\|x-y\|_\infty$$

and

$$|G_2(1,x)-G_2(1,y)| \le (1-\alpha)L_N\|x-y\|_\infty.$$

Proof. We give the proof of the first inequality, the proof of the second is similar. Basic inequalities and the Lipschitz condition (4) give

$$\begin{aligned} |G_1(\beta,x)-G_1(\beta,y)| &= \left|\int_0^\beta N(x(\tilde\xi))-N(y(\tilde\xi))\,d\tilde\xi\right| \le \int_0^\beta \left|N(x(\tilde\xi))-N(y(\tilde\xi))\right| d\tilde\xi \\ &\le \int_0^\beta L_N|x(\tilde\xi)-y(\tilde\xi)|\,d\tilde\xi \le \beta L_N\|x-y\|_\infty. \end{aligned}$$

▯

Using the bounds on M and properties of definite integrals, the following bounds on $G_1(\beta,x)$ and $G_2(1,x)$ hold.

Lemma 5. *The quantities $G_1(\beta,x_1)$ and $G_2(1,x_2)$ satisfy*

$$\frac{\beta}{\hat m} \le G_1(\beta,x_1)\le\frac{\beta}{\check m},\quad \frac{\check m}{\beta}\le\frac{1}{G_1(\beta,x_1)}\le\frac{\hat m}{\beta}$$

for any function $x_1\in X_1$ and

$$\frac{1-\alpha}{\hat m}\le G_2(1,x_2)\le\frac{1-\alpha}{\check m},\quad \frac{\check m}{1-\alpha}\le\frac{1}{G_2(1,x_2)}\le\frac{\hat m}{1-\alpha},$$

for any function $x_2\in X_2$,respectively.

The results in Lemma 4 and 5 allow us to prove the following pointwise bounds.

Lemma 6. *For any $x,y\in X$ the operators $\tilde G_1$ and $\tilde G_2$ satisfy*

$$|\tilde G_1(\xi,x)-\tilde G_1(\xi,y)|\le L_{\tilde G_1}(\xi)\|x-y\|_\infty,$$

where

$$L_{\tilde{G}_1}(\xi) = \frac{2\hat{m}^2 L_N \xi}{\check{m}\beta},$$

for any functions $x, y \in X_1$*, and*

$$|\tilde{G}_2(\xi, x) - \tilde{G}_2(\xi, y)| \leq L_{\tilde{G}_2}(\xi)\|x - y\|_\infty,$$

where

$$L_{\tilde{G}_2}(\xi) = \frac{2\hat{m}^2 L_N(\xi - \alpha)}{\check{m}(1 - \alpha)},$$

for any functions $x, y \in X_2$*.*

Proof. We show the proof for $\tilde{G}_1$, the proof for $\tilde{G}_2$ is similar. The result is obtained via the sequence of inequalities

$$\begin{aligned}
|\tilde{G}_1(\xi, x) - \tilde{G}_1(\xi, y)| &= \left| \frac{G_1(\xi, x)}{G_1(\beta, x)} - \frac{G_1(\xi, y)}{G_1(\beta, y)} \right| \\
&= \left| \frac{\int_0^\xi (N(x)G_1(\beta, y) - N(y)G_1(\beta, x))\, d\tilde{\xi}}{G_1(\beta, x)G_1(\beta, y)} \right| \\
&\leq \left(\frac{\hat{m}}{\beta}\right)^2 \int_0^\xi |N(x)G_1(\beta, y) - N(y)G_1(\beta, x)|\, d\tilde{\xi} \\
&\leq \left(\frac{\hat{m}}{\beta}\right)^2 \int_0^\xi N(x)|G_1(\beta, y) - G_1(\beta, x)| + G_1(\beta, x)|N(x) - N(y)| d\tilde{\xi} \\
&\leq \left(\frac{\hat{m}}{\beta}\right)^2 \frac{2\beta L_N \xi}{\check{m}} \|x - y\|_\infty \\
&= \frac{2\hat{m}^2 L_N \xi}{\check{m}\beta} \|x - y\|_\infty.
\end{aligned}$$

□

We now state and prove the corrected convergence result for the linearized iteration (2).

THEOREM 7. *Assume M is continuously differentiable and satisfies (3) and (4). For any continuously differentiable initial guesses $x_1^0(\xi) \in X_1$ and $x_2^0(\xi) \in X_2$, the linearized, parallel Schwarz iteration (2) will converge if, in addition, M is such that the maximum of*

$$L_{\tilde{G}_1}(\beta)(1 + L_{\tilde{G}_1}(\beta)) + \frac{1}{1 + \frac{\beta - \alpha}{1 - \beta}\frac{\check{m}}{\hat{m}}} + L_{\tilde{G}_2}(\beta), \tag{12}$$

and

$$L_{\tilde{G}_1}(\alpha) + \frac{1}{1 + \frac{\beta - \alpha}{\alpha}\frac{\check{m}}{\hat{m}}} + L_{\tilde{G}_2}(1)\left(1 + L_{\tilde{G}_2}(1)\right) \tag{13}$$

is strictly less than one, where $L_{\tilde{G}_1}$ and $L_{\tilde{G}_2}$ are defined in Lemma 6.

Proof. We first note that in this linearized iteration, one gains regularity: if the initial guess is continuously differentiable, the explicit solution formulas (6) and (7) show that after one iteration the iterates are already twice continuously differentiable, and hence the algorithm produces a unique sequence of classical solutions.

We now show that $F = (F_1, F_2)^T$ is a contraction, component–wise in norm $\|\cdot\|_\infty$ in the space $X \times X$. Assume $x_j(\xi)$ and $y_j(\xi)$ are elements of X_j for $j = 1, 2$. For brevity we introduce

$$G_x = \tilde{G}_1(\xi, x_2(\beta)\tilde{G}_1(\xi, x_1)) \qquad \text{and} \qquad G_y = \tilde{G}_1(\xi, y_2(\beta)\tilde{G}_1(\xi, y_1)).$$

Using (10), adding and subtracting $(x_1(\alpha)+(1-x_1(\alpha))\tilde{G}_2(\beta, y_2))G_y$, and applying the triangle inequality gives

$$\begin{aligned} |F_1(x_1, x_2) - F_1(y_1, y_2)| &= \left|\left(x_1(\alpha) + (1 - x_1(\alpha))\tilde{G}_2(\beta, x_2)\right) G_x \right. \\ &- \left.\left(y_1(\alpha) + (1 - y_1(\alpha))\tilde{G}_2(\beta, y_2)\right) G_y\right| \\ &\leq \left|\left(x_1(\alpha) + (1 - x_1(\alpha))\tilde{G}_2(\beta, x_2)\right) (G_x - G_y)\right| \\ &+ \left|\left(\left(x_1(\alpha) + (1 - x_1(\alpha))\tilde{G}_2(\beta, x_2)\right) - \left(y_1(\alpha) + (1 - y_1(\alpha))\tilde{G}_2(\beta, y_2)\right)\right) G_y\right|. \end{aligned}$$

We analyze each term in the sum above separately. Lemma 2 ensures $0 \leq \tilde{G}_2 \leq 1$ and since $0 \leq x_{1,2}, y_{1,2} \leq 1$ we have

$$x_1(\alpha) + (1 - x_1(\alpha))\tilde{G}_2(\beta, x_2) \leq x_1(\alpha) + (1 - x_1(\alpha)) = 1.$$

Therefore, the first term in the sum above can be bounded by $|G_x - G_y|$.

From the definition of G_x and G_y and Lemma 6 we have

$$\begin{aligned} |G_x - G_y| &= \left|\tilde{G}_1(\xi, x_2(\beta)\tilde{G}_1(\xi, x_1)) - \tilde{G}_1(\xi, y_2(\beta)\tilde{G}_1(\xi, y_1))\right| \\ &\leq L_{\tilde{G}_1}(\xi)\left|x_2(\beta)\tilde{G}_1(\xi, x_1) - y_2(\beta)\tilde{G}_1(\xi, y_1)\right|. \end{aligned}$$

Adding and subtracting $x_2(\beta)\tilde{G}_1(\xi, y_1)$ inside the absolute value, using the triangle inequality, Lemma 6, and the fact that $|x_2(\beta)| \leq 1$ and $0 \leq \tilde{G}_1 \leq 1$, we have

$$\begin{aligned} \left|x_2(\beta)\tilde{G}_1(\xi, x_1) - y_2(\beta)\tilde{G}_1(\xi, y_1)\right| &\leq |x_2(\beta)|\left|\tilde{G}_1(\xi, x_1) - \tilde{G}_1(\xi, y_1)\right| + |\tilde{G}_1(\xi, y_1)| \cdot |x_2(\beta) - y_2(\beta)| \\ &\leq L_{\tilde{G}_1}(\xi)\|x_1 - y_1\|_\infty + |\tilde{G}_1(\xi, y_1)|\|x_2 - y_2\|_\infty, \\ &\leq L_{\tilde{G}_1}(\beta)\|x - y\|_\infty + |\tilde{G}_1(\beta, y_1)|\|x - y\|_\infty, \end{aligned} \tag{14}$$

where $L_{\tilde{G}_1}$ is defined in the statement of Lemma 6 and we have now defined $x, y \in X_1 \times X_2$ as $x := (x_1, x_2)$ and $y := (y_1, y_2)$. Hence

$$|G_x - G_y| \leq L_{\tilde{G}_1}(\beta)\left(L_{\tilde{G}_1}(\beta)\|x - y\|_\infty + |\tilde{G}_1(\beta, y_1)|\|x - y\|_\infty\right).$$

We now turn to the second term in the sum. Since $|G_y| \leq 1$ we have

$$\begin{aligned} &\left|\left(\left(x_1(\alpha) + (1 - x_1(\alpha))\tilde{G}_2(\beta, x_2)\right) - \left(y_1(\alpha) + (1 - y_1(\alpha))\tilde{G}_2(\beta, y_2)\right)\right) G_y\right| \\ \leq &\left|\left(\left(x_1(\alpha) + (1 - x_1(\alpha))\tilde{G}_2(\beta, x_2)\right) - \left(y_1(\alpha) + (1 - y_1(\alpha))\tilde{G}_2(\beta, y_2)\right)\right)\right|. \end{aligned}$$

Adding and subtracting $\tilde{G}_2(\beta, y_2)x_1(\alpha)$ we can rewrite this last expression as

$$\begin{aligned}
& \Big| x_1(\alpha) - \tilde{G}_2(\beta, x_2)x_1(\alpha) + \tilde{G}_2(\beta, y_2)x_1(\alpha) - \tilde{G}_2(\beta, y_2)x_1(\alpha) \\
+ \quad & \tilde{G}_2(\beta, x_2) - y_1(\alpha) + \tilde{G}_2(\beta, y_2)y_1(\alpha) - \tilde{G}_2(\beta, y_2) \Big| \\
= \quad & \Big| (1 - \tilde{G}_2(\beta, y_2))(x_1(\alpha) - y_1(\alpha)) + (1 - x_1(\alpha))(\tilde{G}_2(\beta, x_2) - \tilde{G}_2(\beta, y_2)) \Big| \\
\leq \quad & (1 - \tilde{G}_2(\beta, y_2))|x_1 - y_1| + |\tilde{G}_2(\beta, y_2) - \tilde{G}_2(\beta, x_2)|.
\end{aligned}$$

Hence, using Lemma 6, we have

$$\begin{aligned}
|F_1(x_1, x_2) - F_1(y_1, y_2)| &\leq |G_x - G_y| + (1 - \tilde{G}_2(\beta, y_2))|x_1 - y_1| + |\tilde{G}_2(\beta, y_2) - \tilde{G}_2(\beta, x_2)| \\
&\leq L_{\tilde{G}_1}(\beta) \Big(L_{\tilde{G}_1}(\beta)\|x - y\|_\infty + |\tilde{G}_1(\beta, y_1)| \|x - y\|_\infty \Big) \\
&+ (1 - \tilde{G}_2(\beta, y_2))\|x - y\|_\infty + L_{\tilde{G}_2}(\beta)\|x - y\|_\infty \\
&\leq \Big(L_{\tilde{G}_1}(\beta)(1 + L_{\tilde{G}_1}(\beta)) + (1 - \tilde{G}_2(\beta, y_2)) + L_{\tilde{G}_2}(\beta) \Big) \cdot \|x - y\|_\infty,
\end{aligned}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Finally, the quantity $1 - \tilde{G}_2(\beta, w)$ can be rewritten as

$$1 - \tilde{G}_2(\beta, w) = 1 - \frac{\int_\alpha^\beta N(w(s))ds}{\int_\alpha^1 N(w(s))ds} = \frac{\int_\beta^1 N(w(s))ds}{\int_\alpha^1 N(w(s))ds} = \frac{1}{1 + \frac{\int_\alpha^\beta N(w(s))ds}{\int_\beta^1 N(w(s))ds}}. \tag{15}$$

Now

$$\frac{\int_\alpha^\beta N(w(s))ds}{\int_\beta^1 N(w(s))ds} \geq \frac{(\beta - \alpha)\frac{1}{\check{m}}}{(1 - \beta)\frac{1}{\hat{m}}}.$$

Hence

$$1 - \tilde{G}_2(\beta, x) \leq \frac{1}{1 + \frac{\beta - \alpha}{1 - \beta}\frac{\check{m}}{\hat{m}}}.$$

Therefore, for all ξ, and functions $x_1(\xi), x_2(\xi), y_1(\xi)$ and $y_2(\xi)$, we have

$$|F_1(x_1, x_2) - F_1(y_1, y_2)| \leq \left(L_{\tilde{G}_1}(\beta)(1 + L_{\tilde{G}_1}(\beta)) + \frac{1}{1 + \frac{\beta - \alpha}{1 - \beta}\frac{\check{m}}{\hat{m}}}. + L_{\tilde{G}_2}(\beta) \right) \cdot \|x - y\|_\infty.$$

The result for the second component of F follows in a similar fashion and the proof is complete. □

The sufficient condition that the expressions in (12) and (13) be less than one imposes a rather stringent requirement on the function M; for example the range of M values, $[\check{m}, \hat{m}]$, and the derivative of M. And although the requirement is not necessary in practice, there are indeed difficulties with convergence for highly nonlinear mesh density functions.

An alternating Schwarz version of this linearized algorithm is presented in [2]. This alternating algorithm obtains faster convergence by using the most recently computed boundary information at the subdomain interfaces. The second transmission condition in (2) would be changed to $x_2^n(\alpha) = x_1^n(\alpha)$ to obtain the alternating

Schwarz iteration. The improved convergence is obtained at the cost of losing obvious parallelization. The technique in the proof above can be easily modified to handle the alternating case and hence provide a corrigendum of the result in [2].

We conclude by pointing out that there is, in fact, a simpler linear Schwarz iteration for the solution of the equidistributing mesh transformation which is convergent without the restrictions on M stated in Theorem 7. This linear iteration, however, gives the inverse mesh transformation, $\xi(x)$, as its limit. The iteration is obtained by rearranging the nonlinear BVP for $x(\xi)$ to give the linear BVP

$$\frac{d}{dx}\left(\frac{1}{M(x)}\frac{d\xi}{dx}\right) = 0, \qquad \xi(0) = 0, \xi(1) = 1,$$

for $\xi(x)$. As is well–known, for sufficiently smooth $\check{m} \leq M(x) \leq \hat{m}$ the maximum principle gives the convergence result of the parallel or alternating Schwarz methods applied to this linear problem. The drawback of this approach is that the limiting solution $\xi(x)$ then needs to be inverted numerically if the mesh locations in the physical domain are required.

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