Optimal Interface Conditions for an Arbitrary Decomposition into Subdomains

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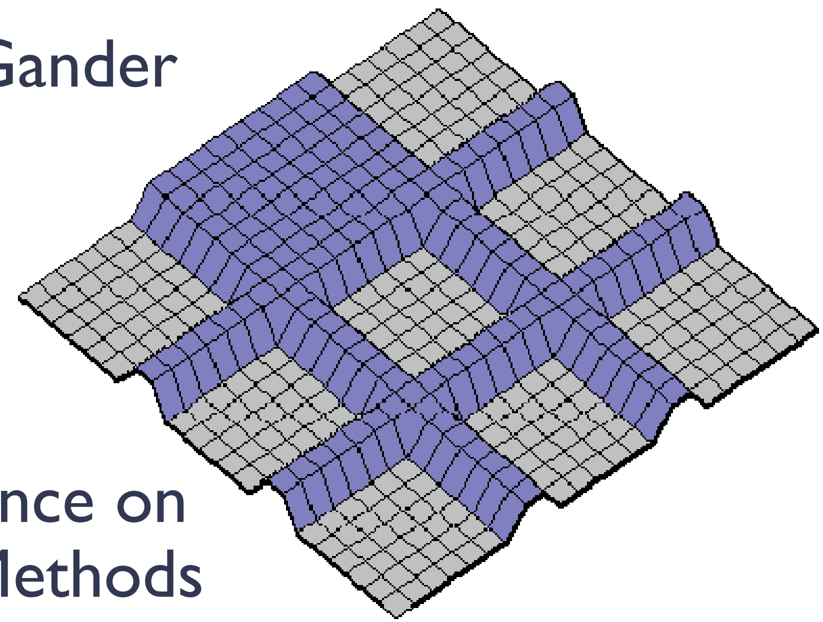


19th International Conference on Domain Decomposition Methods

Zhangjiajie, China, August 18th, 2009

Classical Schwarz Method

- Schwarz (1869), Lions (1988):

$$-\Delta u_j^{k+1} = f \qquad \text{on } \Omega_j$$

$$u_j^{k+1} = g \qquad \text{on } \partial\Omega \cap \overline{\Omega}_j$$

$$u_j^{k+1} = \quad u_i^k \quad \text{on } \Gamma_{ij}$$

Optimal Schwarz Methods

- Change boundary conditions:

$$-\Delta u_j^{k+1} = f \qquad \text{on } \Omega_j$$

$$u_j^{k+1} = g \qquad \text{on } \partial\Omega \cap \overline{\Omega}_j$$

$$B_{ij} u_j^{k+1} = B_{ij} u_i^k \quad \text{on } \Gamma_{ij}$$

- B_{ij} = linear operators *acting on u* along Γ_{ij}
- B_{ij} can be:
  - *Local:* differential operators (compact stencil), e.g. Dirichlet, Neumann, Robin, etc.
  - *Nonlocal:* convolutions (dense matrix blocks), e.g. Steklov-Poincaré, Dirichlet-to-Neumann, etc.

Optimal Schwarz Methods

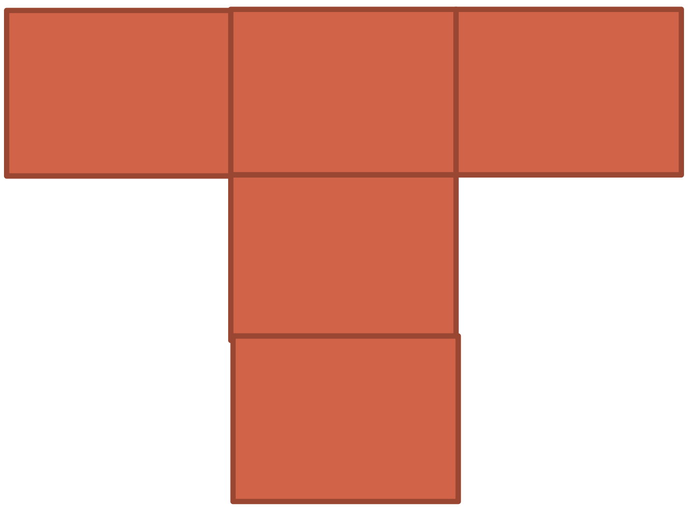
- Optimal operator for convergence is generally nonlocal:
  - Optimal means $\rho = 0$, or convergence in a finite number of iterations
  - Decomposition into strips : Use

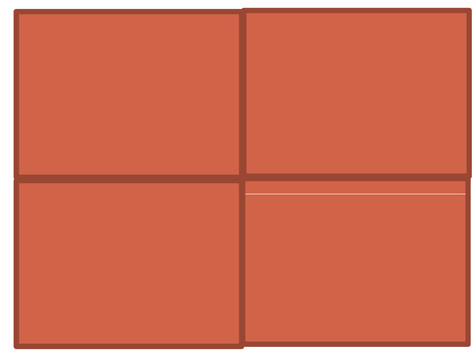
    $$\frac{\partial}{\partial \vec{n}_i} - \Lambda_i$$

    where Λ_i is the Dirichlet-to-Neumann operator (Nataf et al., 1994)
  - Corresponds to Schur complements in the discrete case

Optimal Schwarz Method

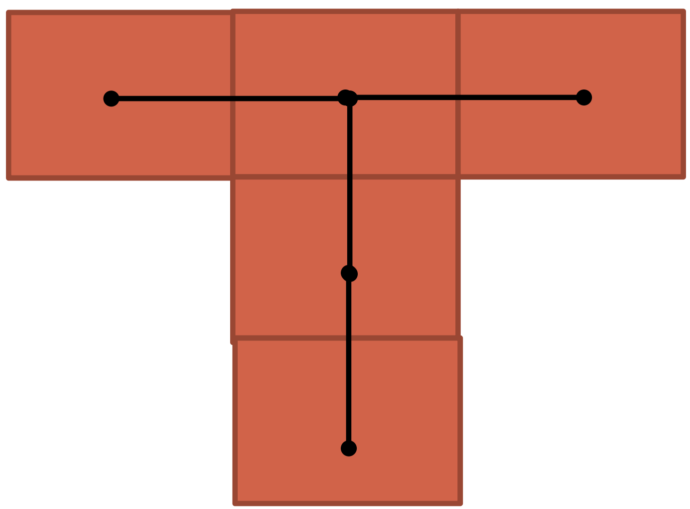
- Optimal Schwarz methods exist when the decomposition has no cycles (Nier 1995)



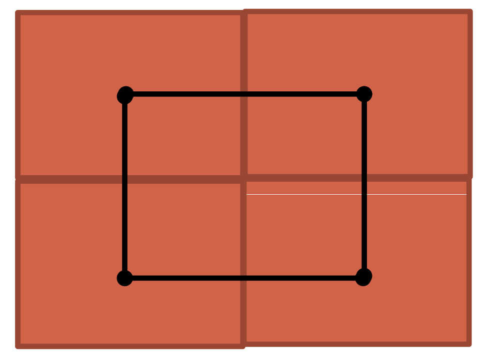


Optimal Schwarz Method

- Optimal Schwarz methods exist when the decomposition has no cycles (Nier 1995)



No Cycle



Cycle

Optimal Schwarz Method

- Conjecture : no optimal method when cycles are present, but
- Does there exist an iteration by subdomains that converges in a finite number of iterations if we are allowed to communicate more than just boundary data?

Schur complement

- For any subdomain Ω_j, we can rewrite the linear system (after permutation) as

$$\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \begin{pmatrix} u_j \\ u^0 \end{pmatrix} = \begin{pmatrix} f_j \\ f_j^0 \end{pmatrix}$$

Inside Ω_j

Outside Ω_j

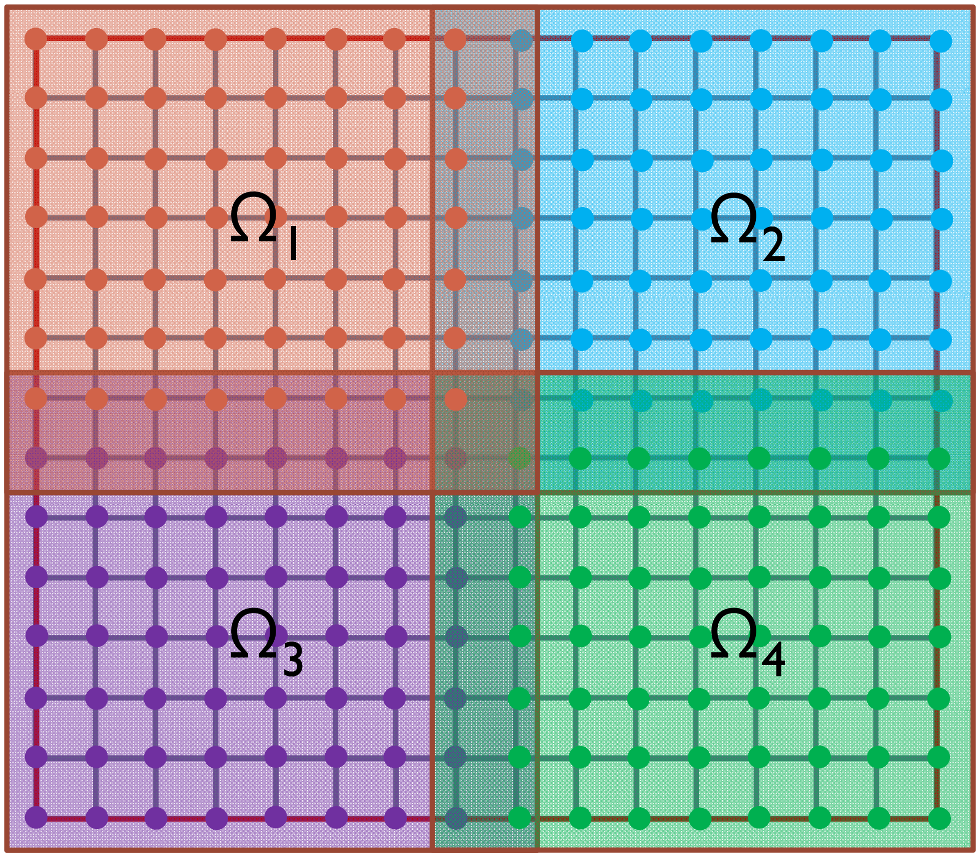
which is equivalent to

$$\left(A_j - B_j D_j^{-1} C_j\right) u_j = f_j - B_j D_j^{-1} f_j^0$$

which can be solved in parallel for each *j*.

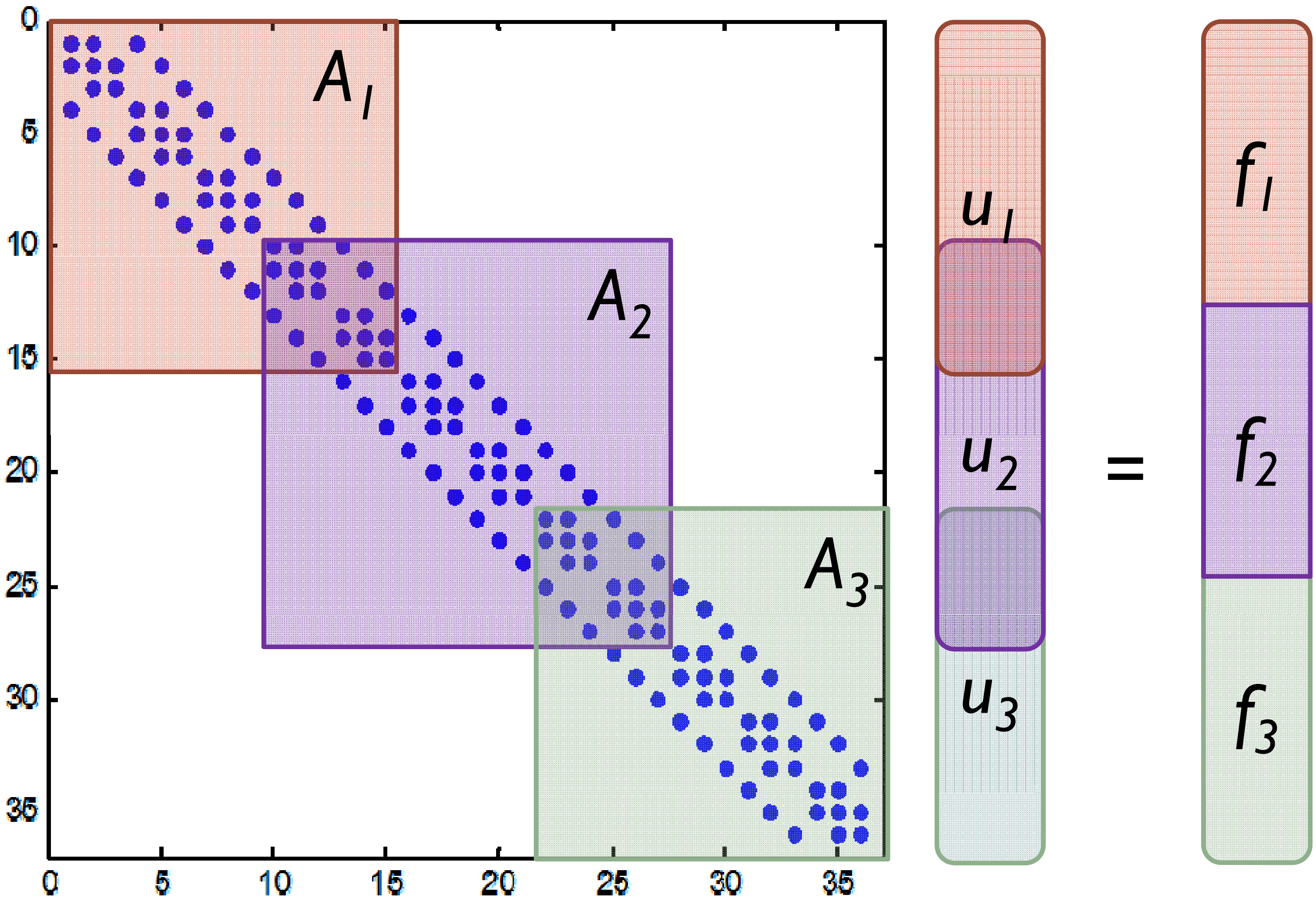
- *How to reconstruct f_j^o (RHS outside Ω_j) using solutions from other subdomains?*

Sufficient Overlap

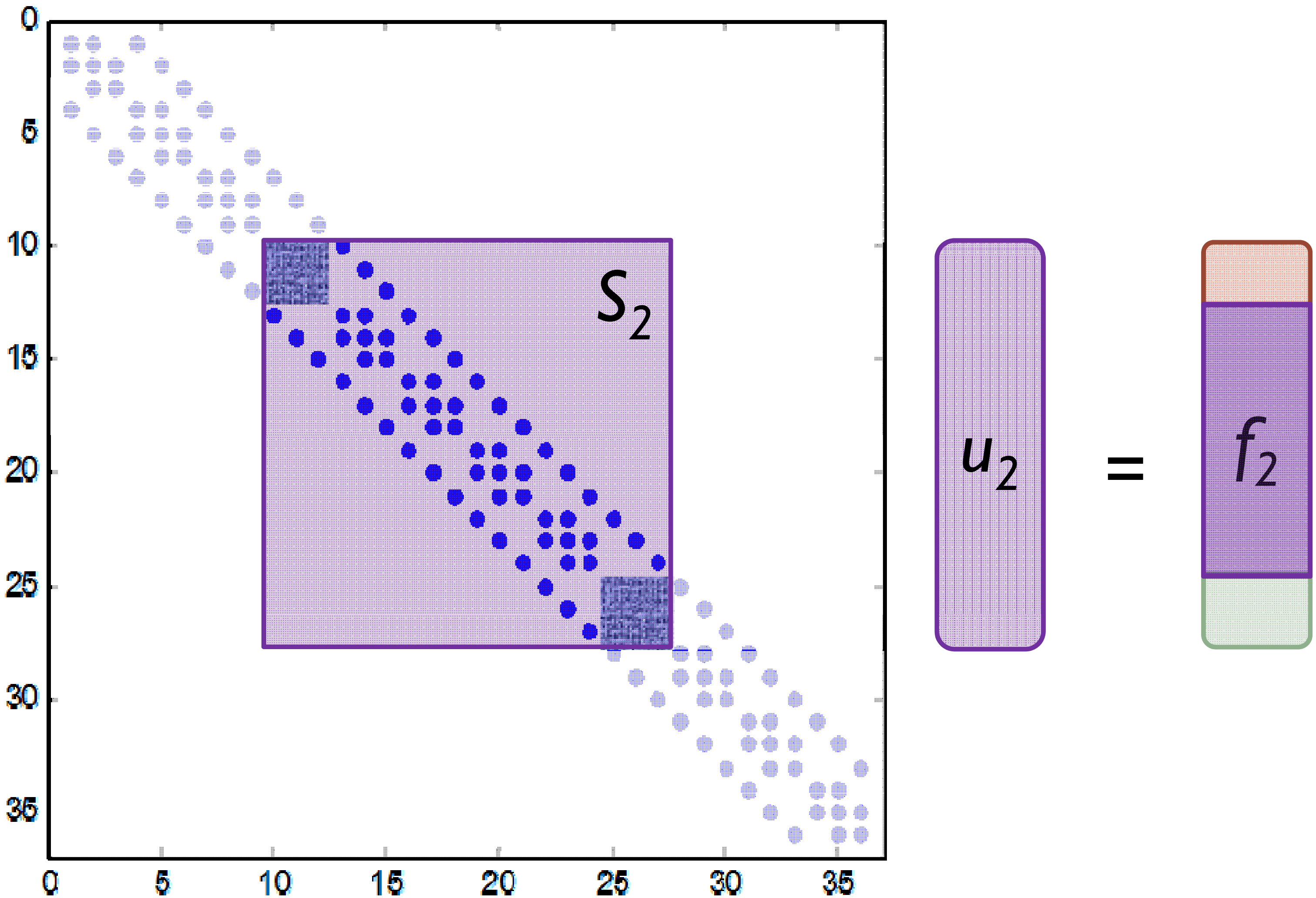


- Assume each grid point lies in the *interior* of at least one subdomain (away from interface)

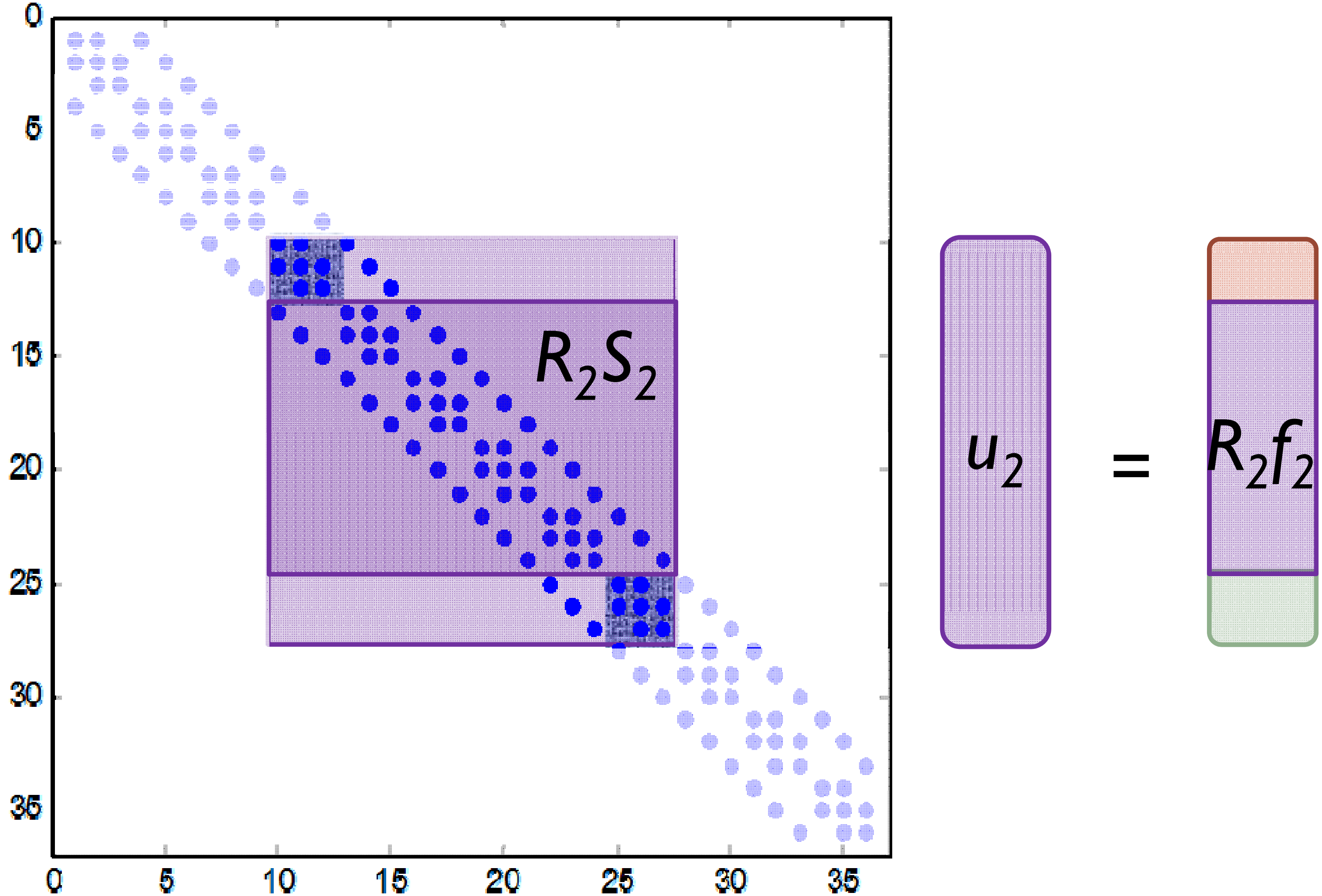
Extracting f_j^o



Extracting f_j^o



Extracting f_j^o



Parallel Algorithm – Version I

$$\left(A_j - B_j D_j^{-1} C_j\right) u_j = f_j - B_j D_j^{-1} f_j^0$$

$$\Downarrow$$

$$\left(A_j - B_j D_j^{-1} C_j\right) u_j^{k+1} = f_j - \sum_{i \neq j} B_j D_j^{-1} \begin{bmatrix} 0 \\ R_i A_i \\ 0 \end{bmatrix} u_i^k$$

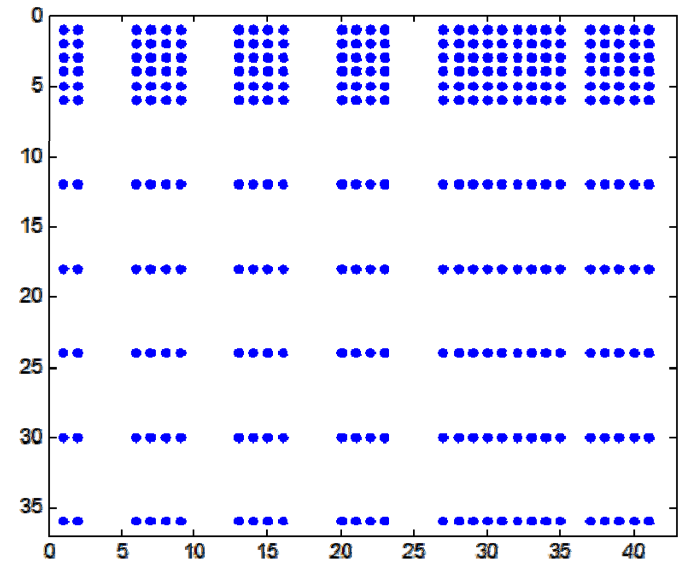
- u_j^{k+1} will yield the exact solution as long as each u_i^k satisfies $R_i A_i u_i = R_i f_i$ $(i \neq j)$
- *Algorithm converges in two steps!*

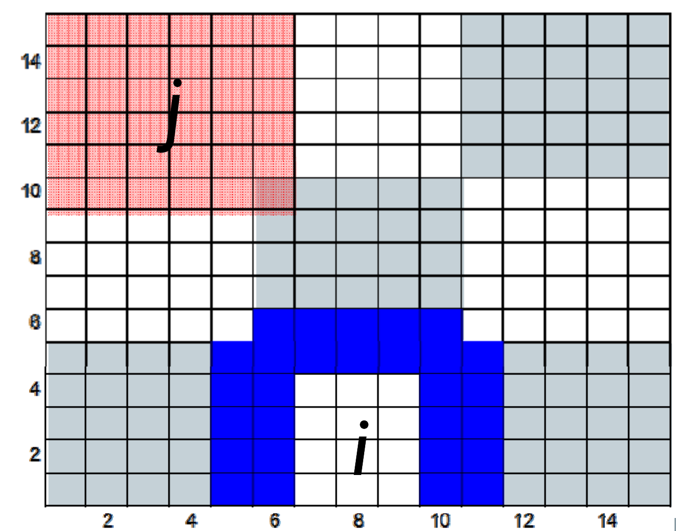
Reducing Communication

- Observation:

$$B_j D_j^{-1} \begin{bmatrix} 0 \\ R_i A_i \\ 0 \end{bmatrix}$$

  has a very specific sparsity pattern
- Column is nonzero only at interfaces between subdomains
- *Values of interior nodes not needed!*





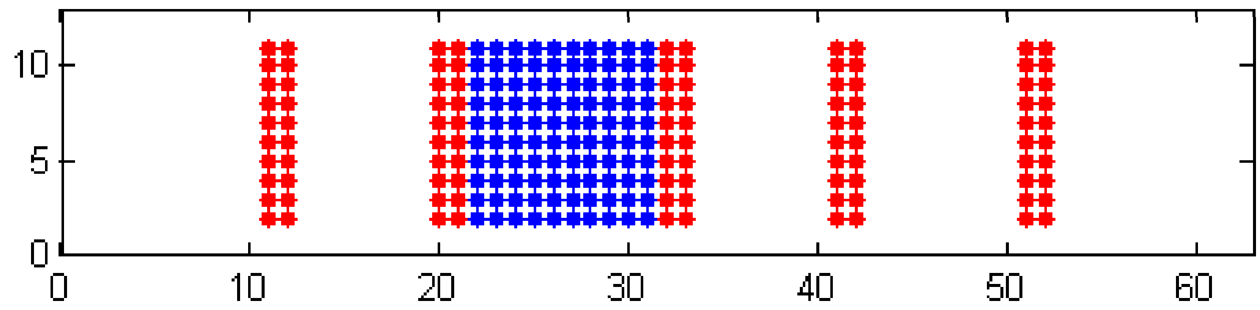
Parallel Algorithm – Version II

$$\left(A_j - B_j D_j^{-1} C_j\right) u_j^{k+1} = f_j - \sum_{i \neq j} B_j D_j^{-1} \begin{bmatrix} 0 \\ R_i A_i P_{ji}^T \\ 0 \end{bmatrix} P_{ji} u_i^k$$

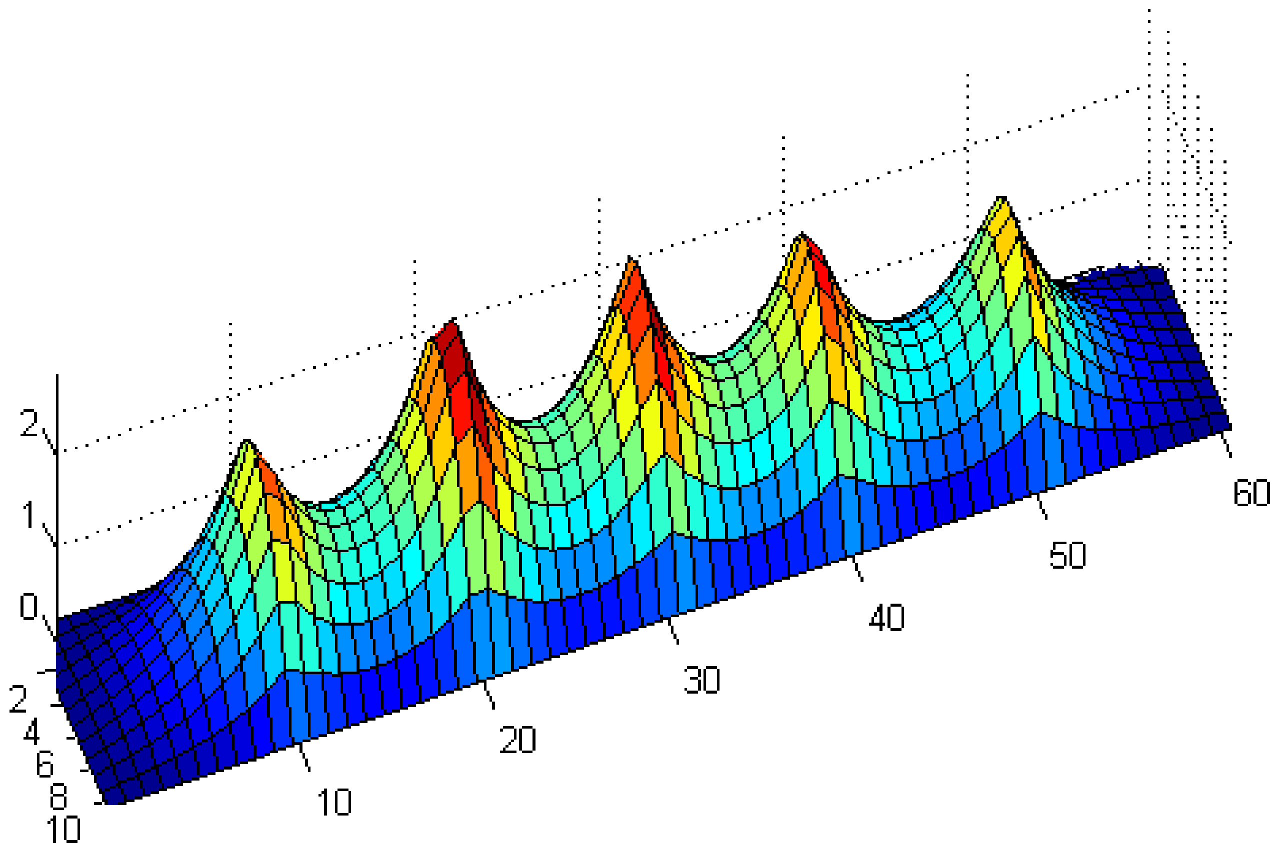
(P_{ji} restricts u_i^k to the "boundary")

- Identical iterates for the two versions
- Convergence in two steps, *even though f_j^0 is no longer reconstructed faithfully*
- Communication reduced by a factor of *H/h* !

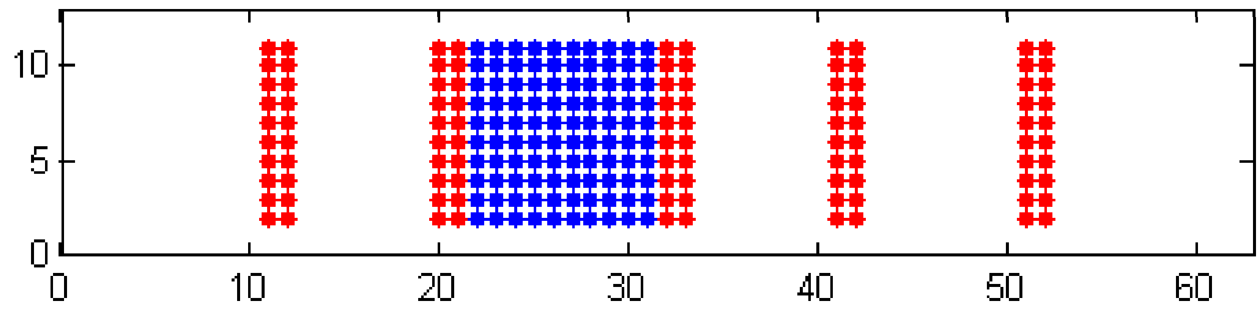
6 × 1 Decomposition



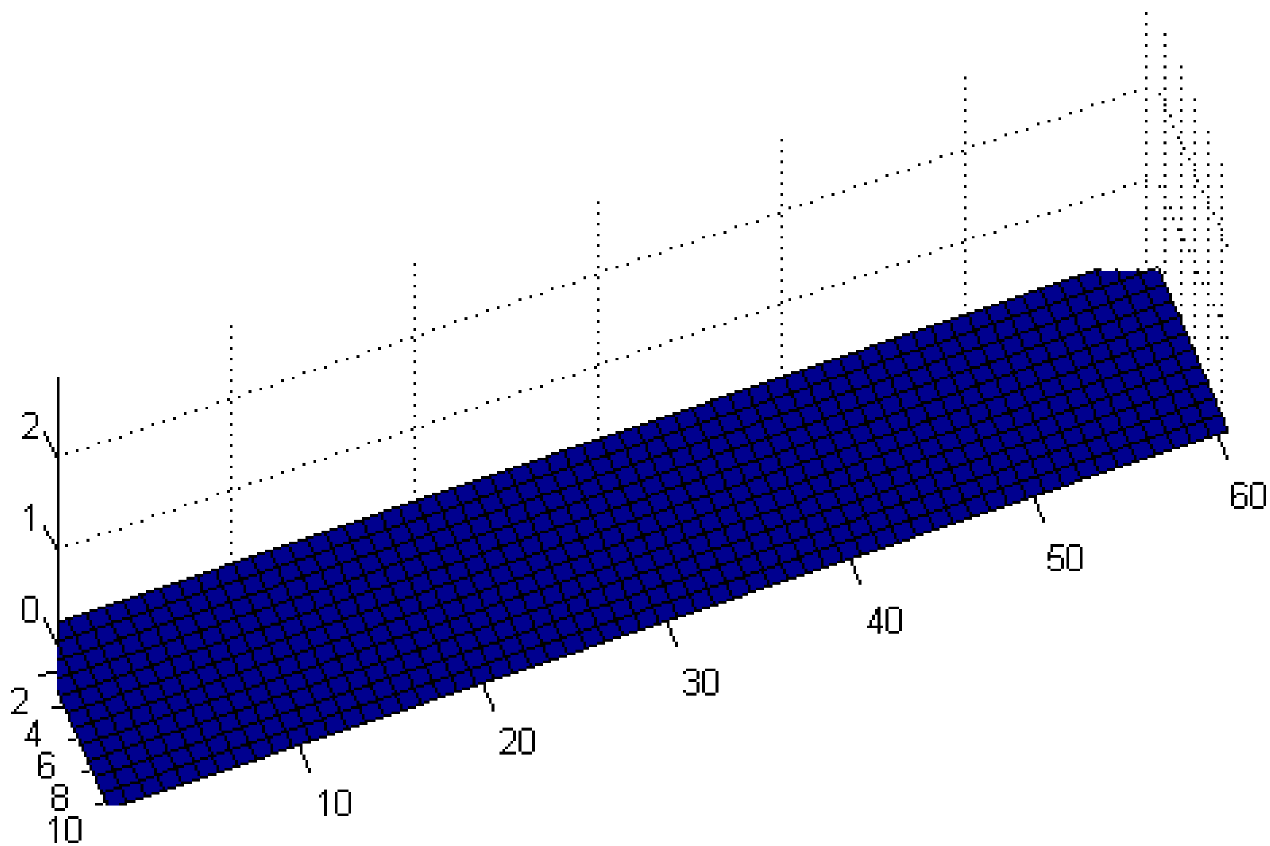
k = 1, Max. Error = 2.86e+000



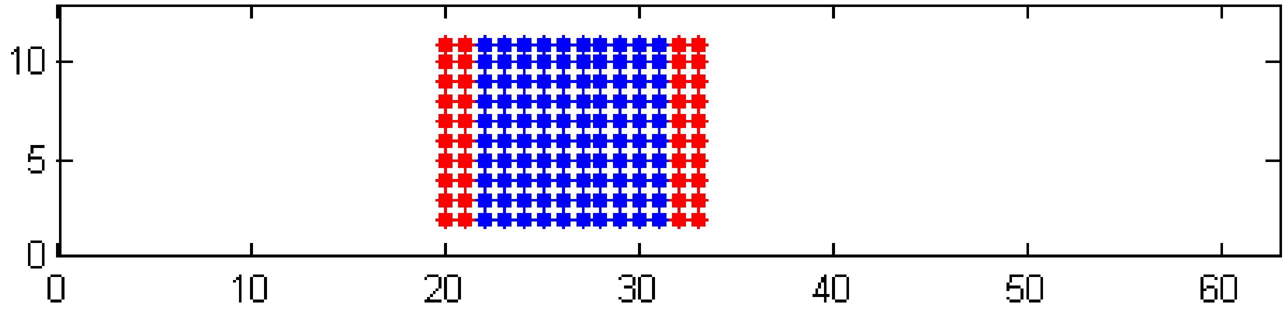
6 × 1 Decomposition

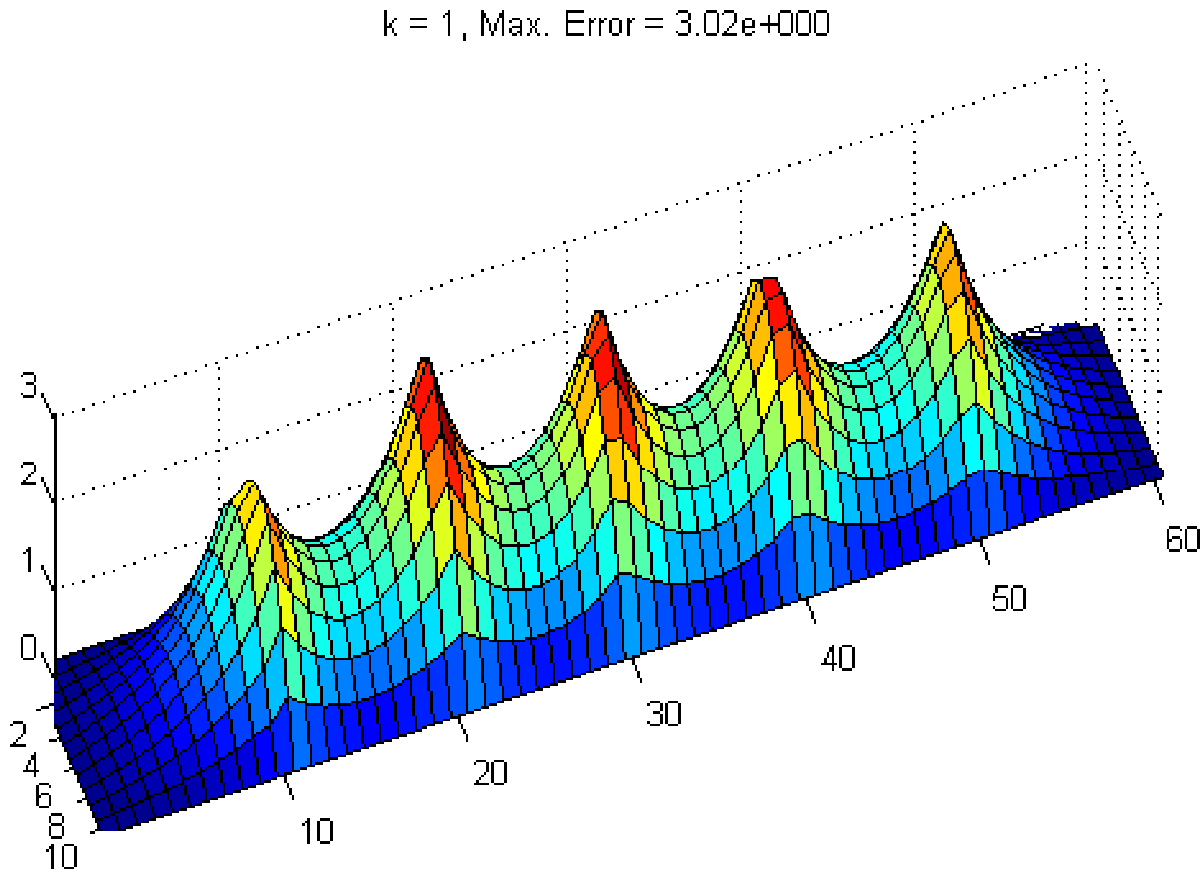


k = 2, Max. Error = 9.77e-015



6 × 1 Decomposition





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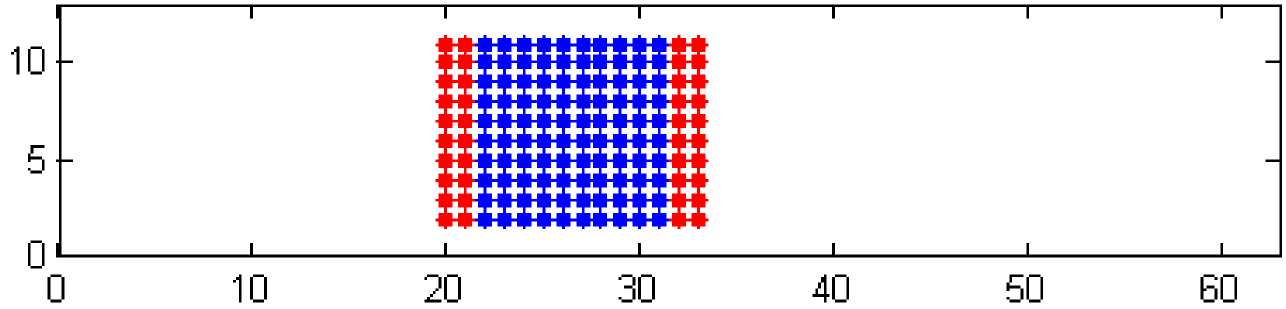

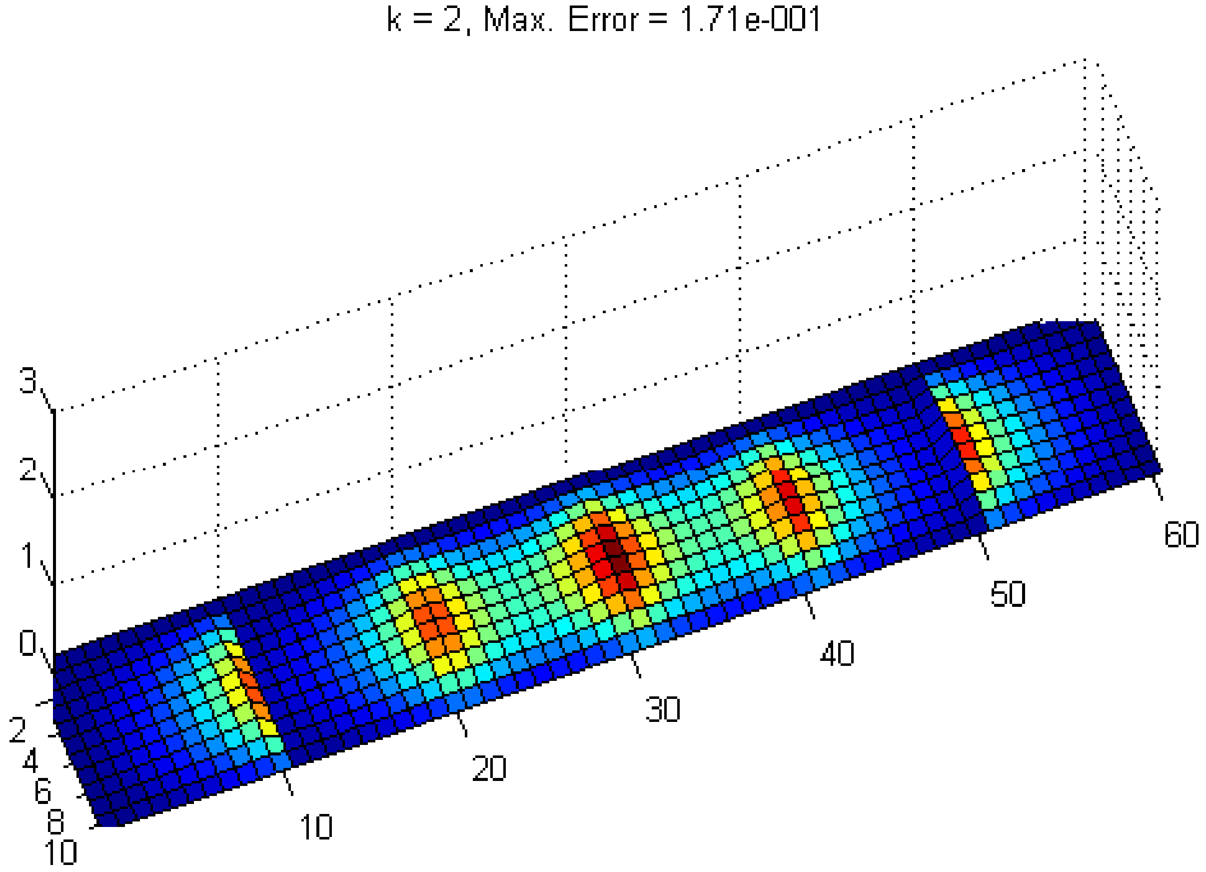

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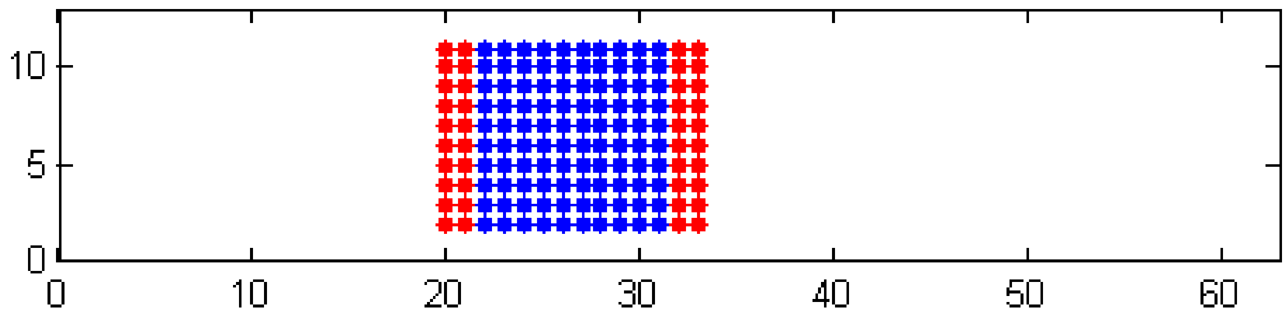

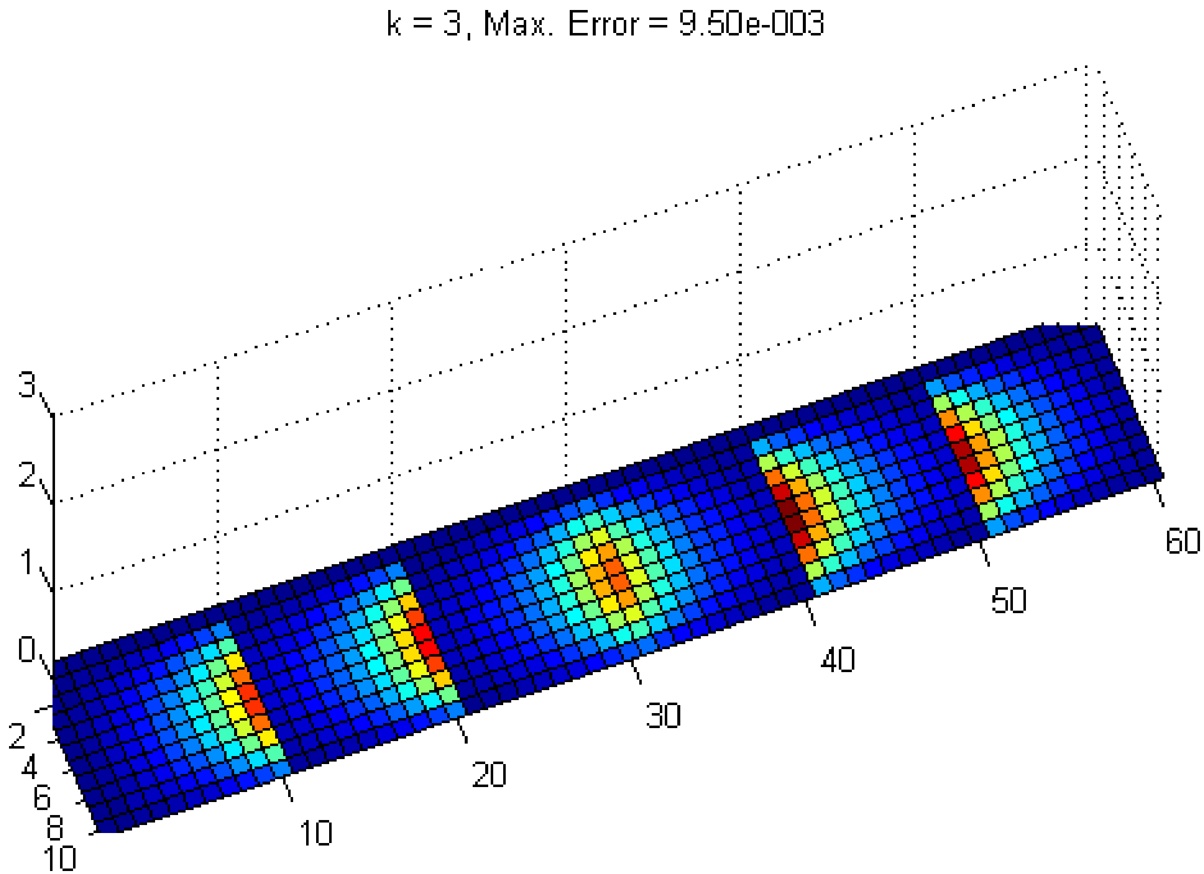

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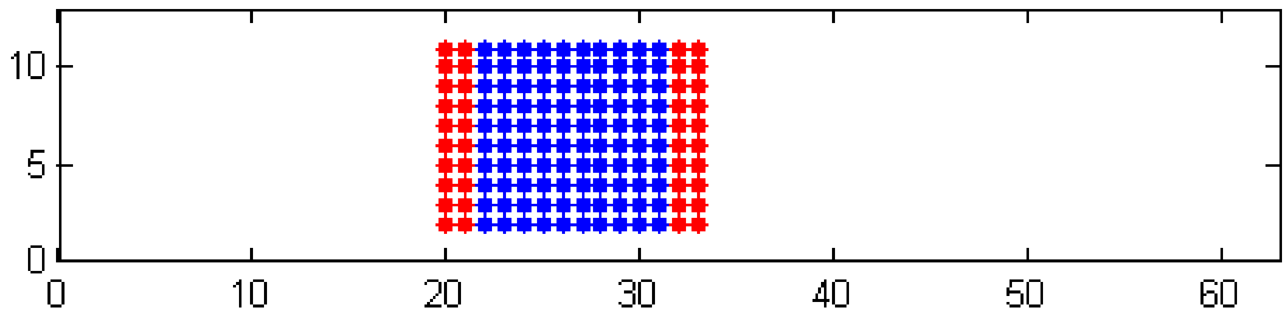

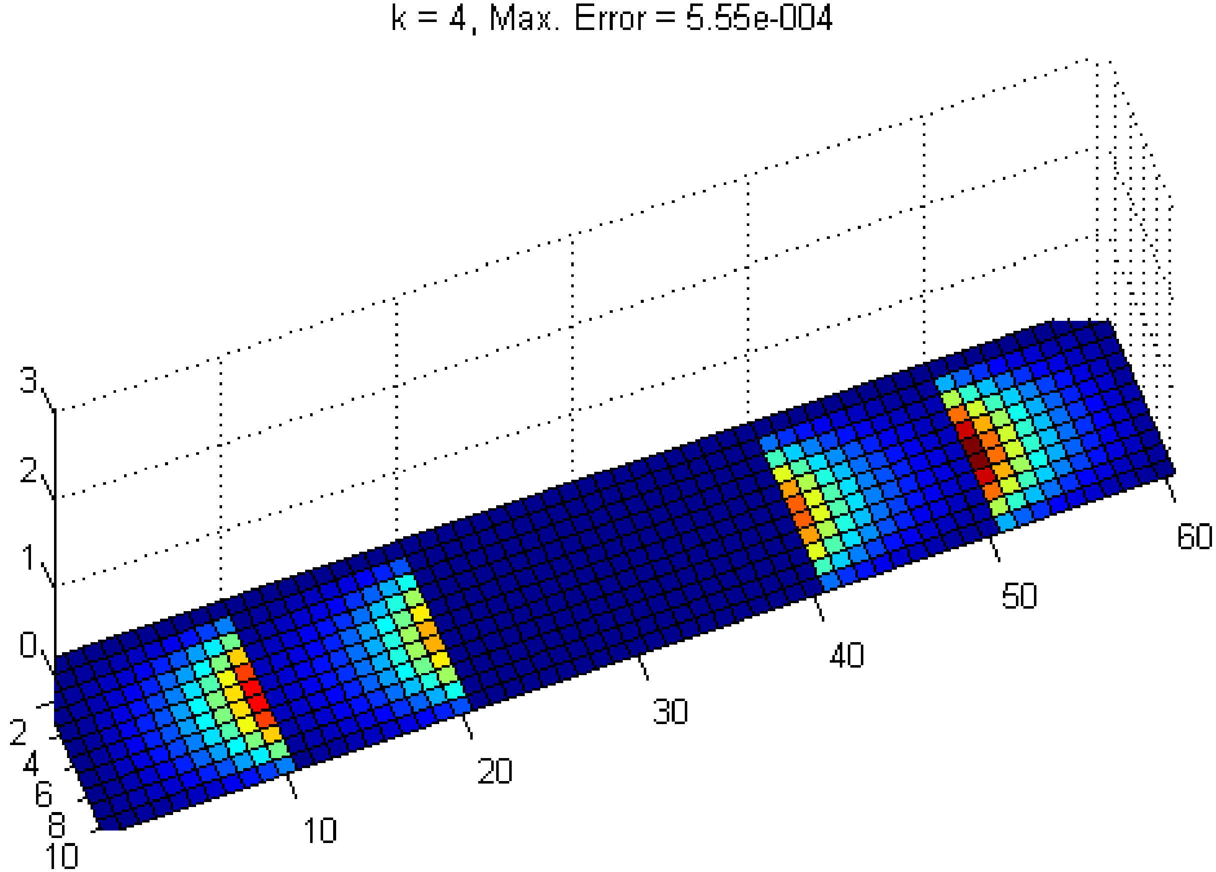

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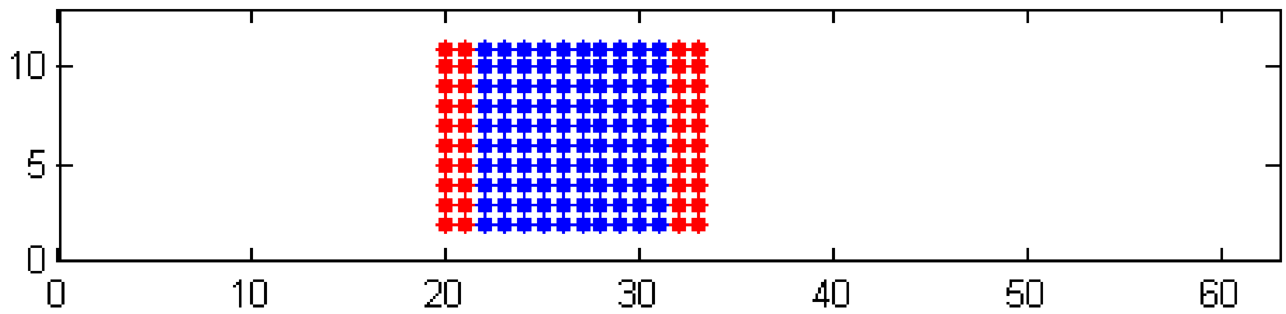

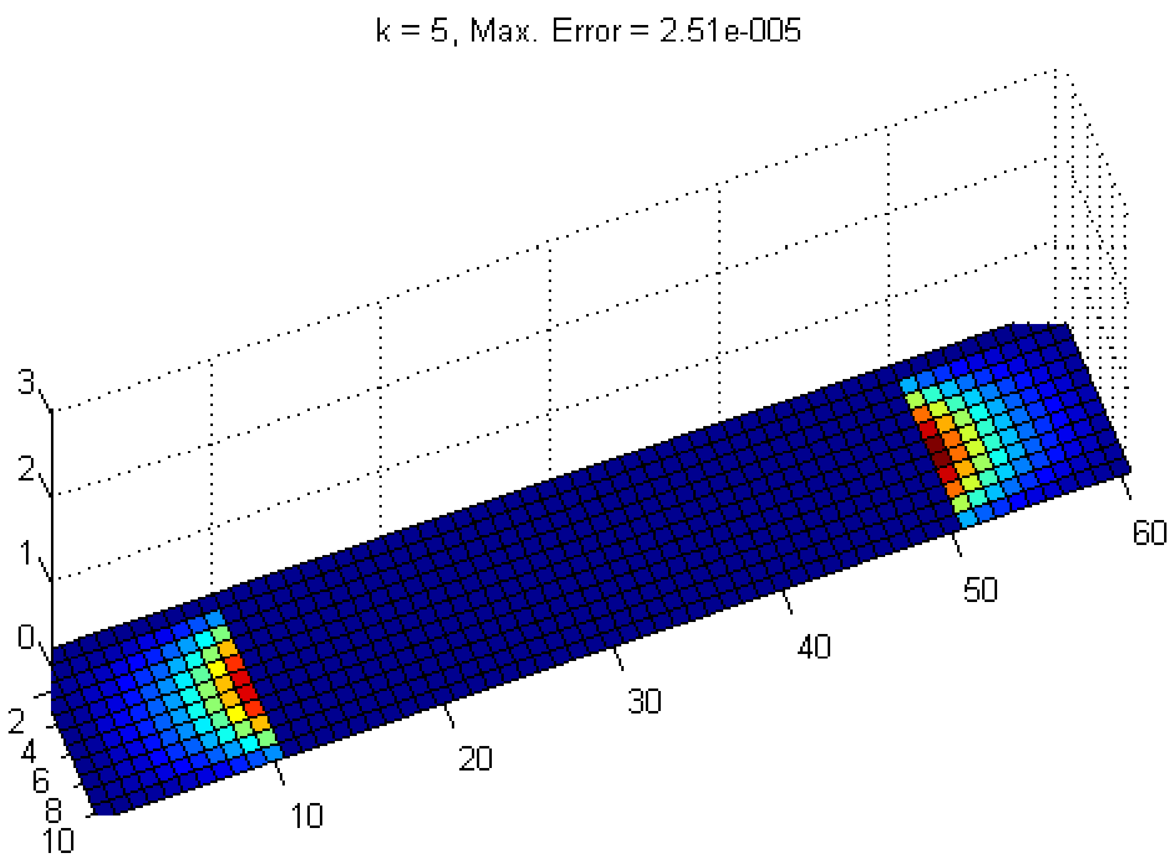

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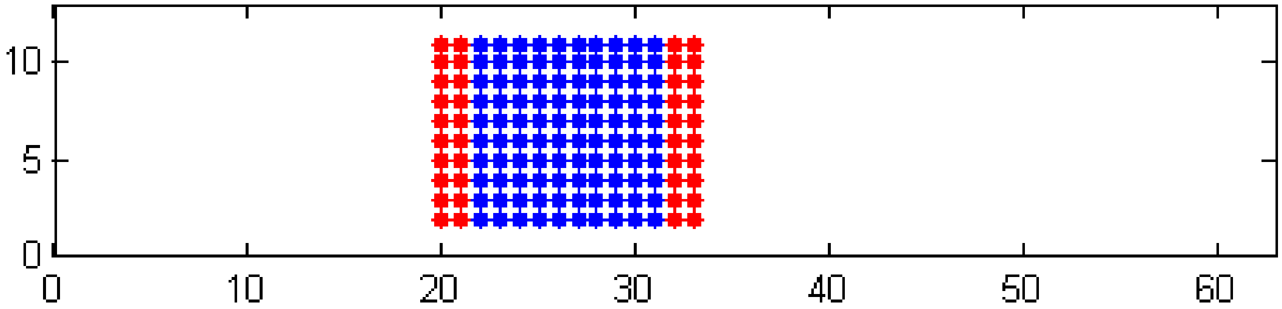
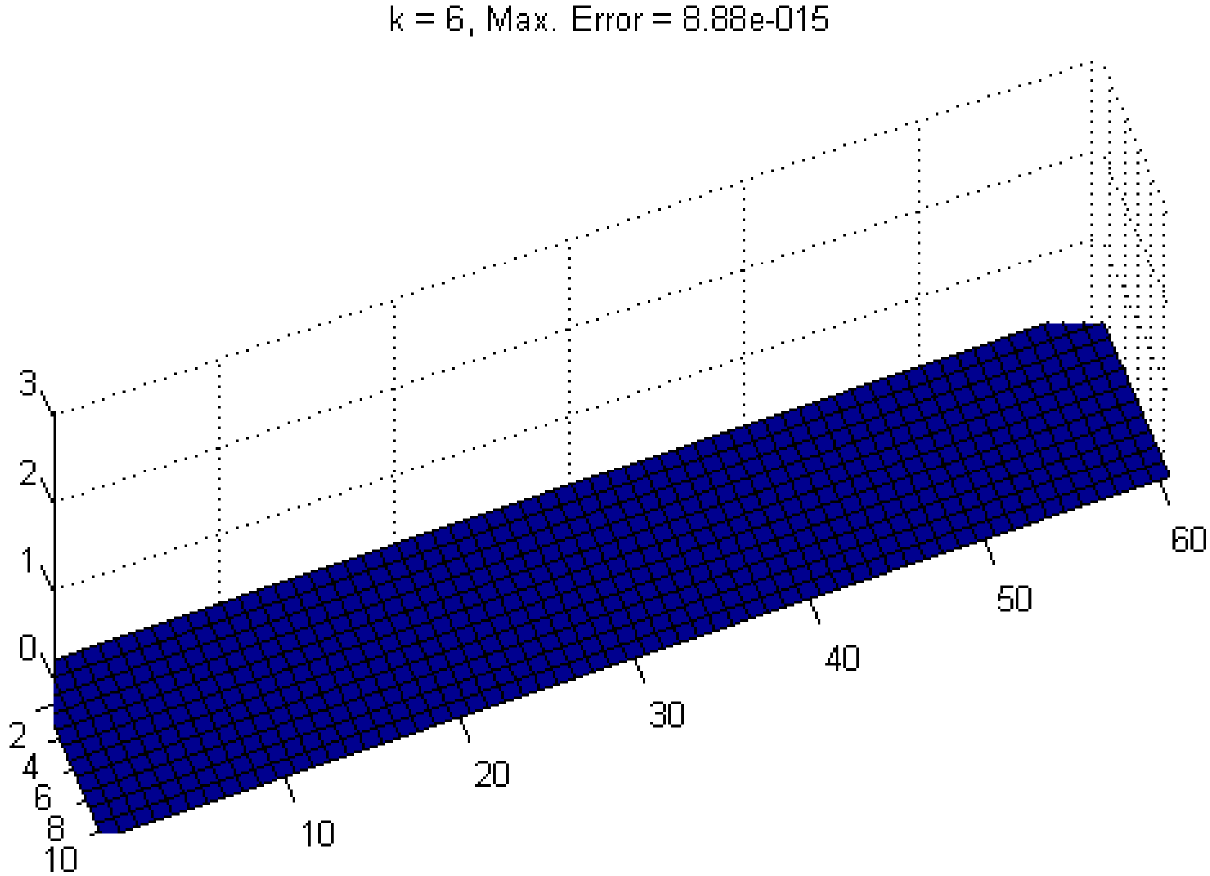
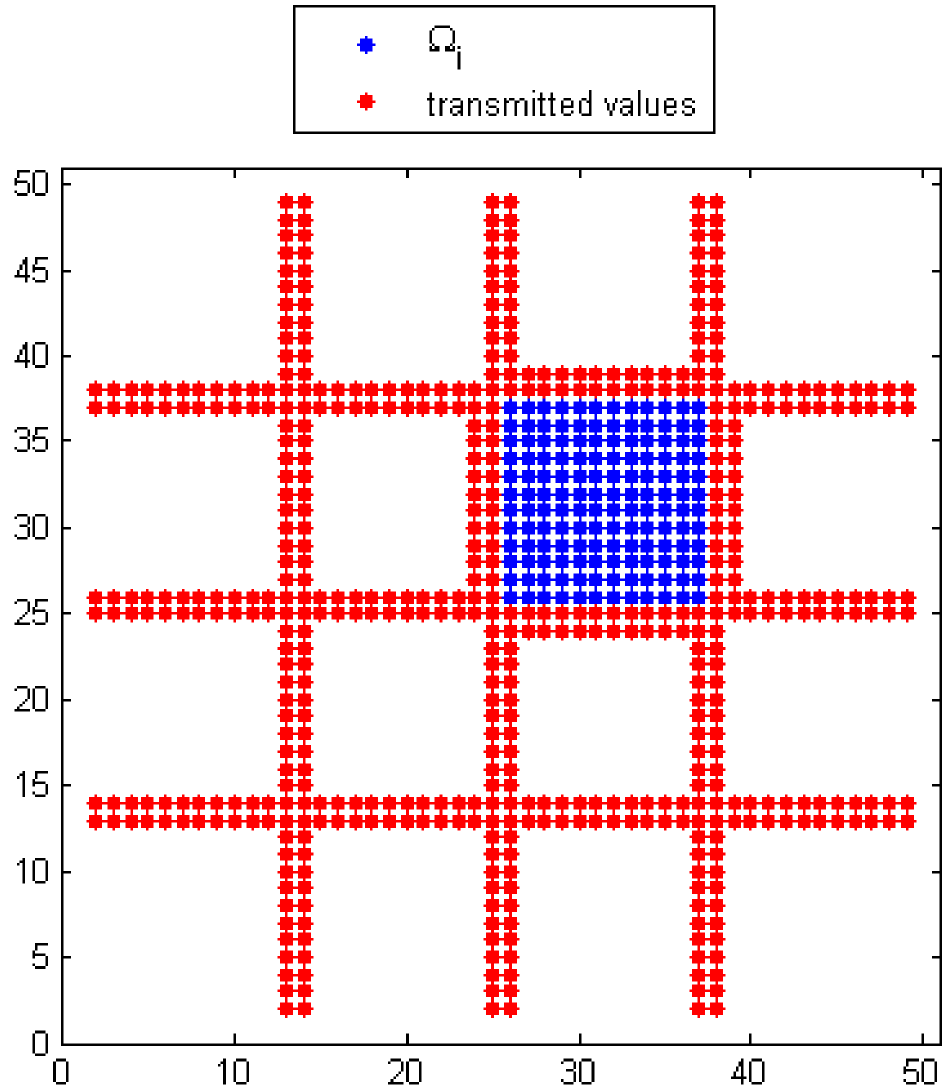




4 × 4 Decomposition



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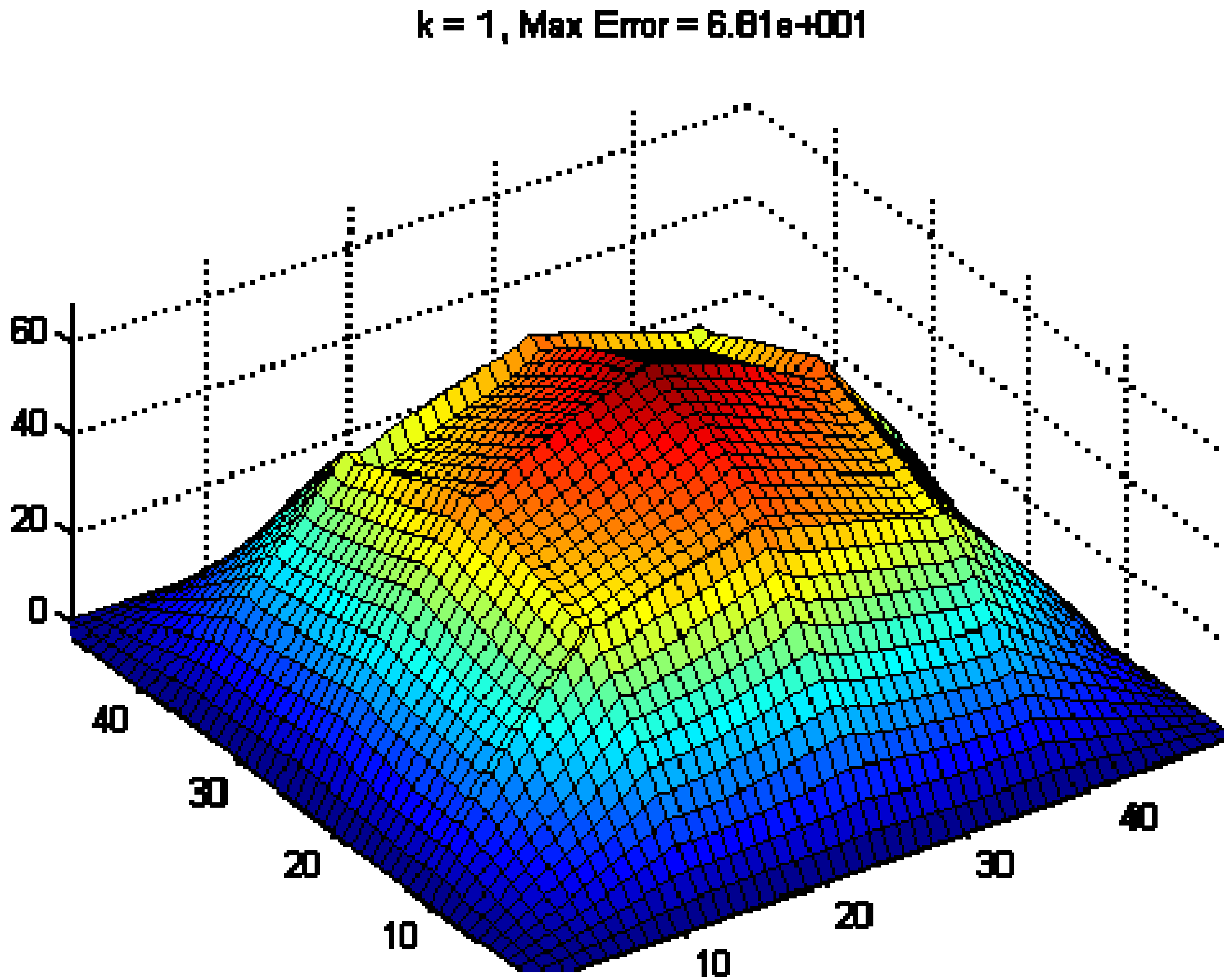
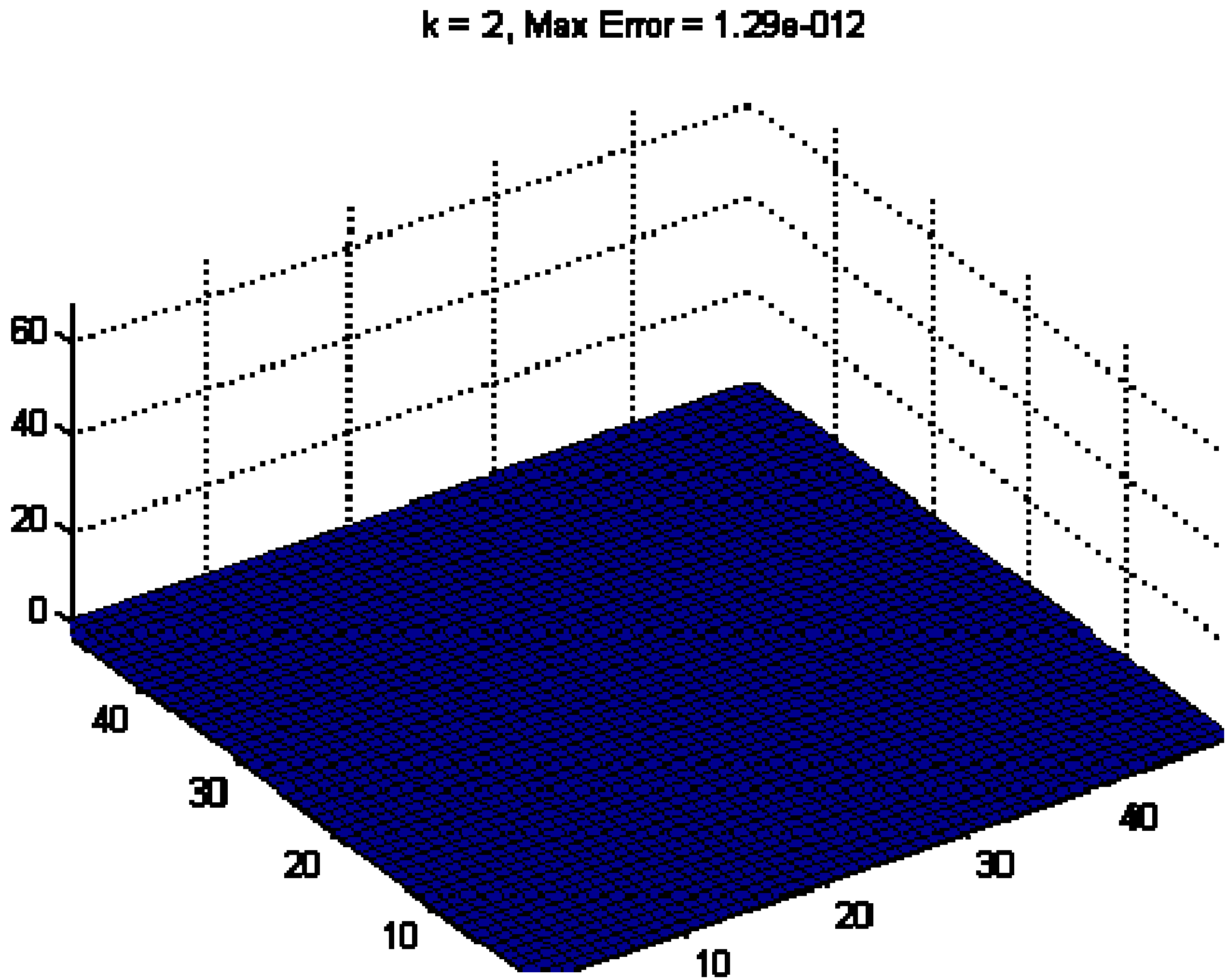


4 × 4 Decomposition



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- LHS is the Schur complement (same as tree case), but
- RHS is a special linear combination of data gathered from each of the other subdomains

Conclusions

- New algebraic method based on Schur complements
- Convergence in two iterations possible if one also uses boundary data from non-neighbours
- Works for arbitrary decompositions into subdomains
- Ongoing work:
  - Derive associated *optimized* methods with local approximations of the Schur complements