A Domain Decomposition Method that Converges in Two Iterations for any Subdomain Decomposition and PDE

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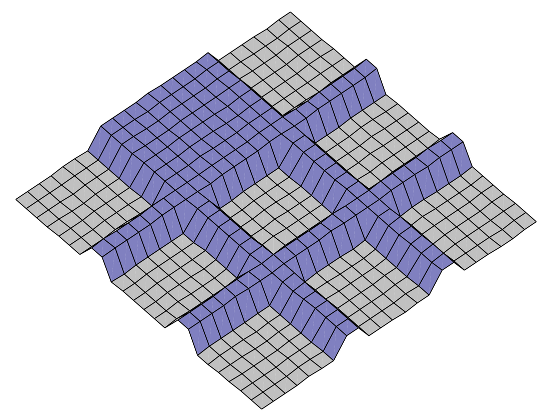


Abstract

All domain decomposition methods are based on a decomposition of the physical domain into many subdomains and an iteration, which uses subdomain solutions only (and maybe a coarse grid), in order to compute an approximate solution of the problem on the entire domain.

We show in this poster that it is possible to formulate such an iteration, only based on subdomain solutions, which converges in two steps to the solution of the underlying problem, independently of the number of subdomains and the PDE solved.

This method is mainly of theoretical interest, since it contains sophisticated non-local operators (and a natural coarse grid component), which need to be approximated in order to obtain a practical method.

Discretized Equations

We describe the method for a linear system

$$A\mathbf{u} = \mathbf{f}$$

which comes from the discretization of a partial differential equation (PDE). While the method is formulated purely at the algebraic level, we consider Laplace's equation to have a concrete example,

$$-\Delta u = f \quad \text{on } \Omega \subset \mathbb{R}^2,$$
$$u = g \quad \text{on } \partial\Omega.$$

Using a five point finite difference discretization, we obtain the discretization stencil

$$\frac{4u_P - u_N - u_E - u_S - u_W}{h^2} = f_P.$$

Domain Decomposition

In order to have a concrete example, we consider the subdomain decomposition shown in Fig. 1.

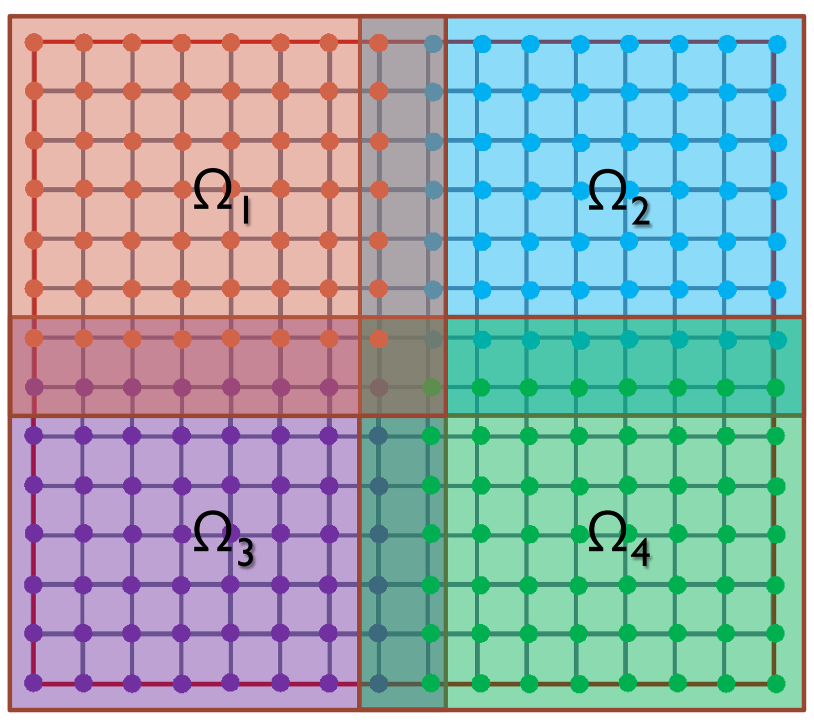


Figure 1: *Domain decomposition.*

We assume that each grid point lies in the *interior* of at least one subdomain (away from the interface, see the illustration in Fig. 1)

Classical Schwarz Method

A classical Schwarz method for this example is (Lions 1988)

for $k = 0, 1, 2, \dots$
for $j = 1, 2, 3, 4$ solve

$$-\Delta u_j^{k+1} = f \quad \text{on } \Omega_j$$
$$u_j^{k+1} = g \quad \text{on } \partial\Omega \cap \overline{\Omega}_j$$
$$u_j^{k+1} = u_i^k \quad \text{on } \Gamma_{ji}$$

This method has a contraction factor of $1 - O(h)$ and is thus very slow on fine grids!

Optimal Schwarz Method

An optimal Schwarz method uses different transmission conditions in the same algorithm

$$\mathcal{B}_{ij} u_j^{k+1} = \mathcal{B}_{ij} u_i^k \quad \text{on } \Gamma_{ij}$$

Here the $\mathcal{B}_{ij}$ are linear operators acting on u along the interfaces Γ_{ij}. $\mathcal{B}_{ij}$ can be:

Local: differential operators (compact stencil), e.g. Dirichlet, Neumann, Robin, etc.

Nonlocal: convolutions (dense matrix blocks), e.g. Steklov-Poincaré, Dirichlet-to-Neumann, etc.

Important historical contributions:

Nataf et al. (1994): For a decomposition into strips, choosing $\mathcal{B}_{ij} := \partial_n - DtN_{ij}$ leads to convergence in a number of iterations equal to the number of subdomains

Nier (1995): Such optimal Schwarz methods exist when the decomposition has no cycles

Local approximations of these optimal operators can lead to methods which are faster than the classical Schwarz method (so called *optimized Schwarz methods*)

- Robin (no overlap): $1 - O(h^{1/2})$
- Robin (with overlap): $1 - O(h^{1/3})$
- 2nd order (no overlap): $1 - O(h^{1/4})$
- 2nd order (with overlap): $1 - O(h^{1/5})$

An Algebraic Optimal Schwarz Method

- Truly optimal: method converges in two iterations
- No restriction on subdomain decomposition (cycles allowed)

For any subdomain Ω_j, we rewrite the linear system after permulation as

$$\begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix} \begin{pmatrix} \mathbf{u}_j \\ \mathbf{u}_j^o \end{pmatrix} = \begin{pmatrix} \mathbf{f}_j \\ \mathbf{f}_j^o \end{pmatrix} \quad \begin{matrix} \text{inside } \Omega_j \\ \text{outside } \Omega_j \end{matrix}$$

Using a Schur complement, we can eliminate the outer variables $\mathbf{u}_j^o$,

$$(A_j - B_j D_j^{-1} C_j)\mathbf{u}_j = \mathbf{f}_j - B_j D_j^{-1} \mathbf{f}_j^o. \qquad (*)$$

These systems can all be solved in parallel, if $\mathbf{f}_j^o$ was known. The purpose of the optimal Schwarz method is therefore to reconstruct the RHS in $(*)$ using solutions from other subdomains, i.e. we must reconstruct $\mathbf{f}_j^o$ from $\mathbf{u}_j^k$ on each Ω_j.

Let v be a node in subdomain Ω_k situated away from its boundary, i.e $\mathbf{e}_v^T B_k = 0$. Then we can recover the corresponding value of $\mathbf{f}$ from the subdomain solution u_k,

$$\mathbf{e}_v^T (A_k - B_k D_k^{-1} C_k)\mathbf{u}_k = \mathbf{e}_v^T (\mathbf{f}_k - B_k D_k^{-1} \mathbf{f}_k^o),$$

which is equivalent to

$$f_v := \mathbf{e}_v^T \mathbf{f}_k = (\mathbf{e}_v^T A_k)\mathbf{u}_k.$$

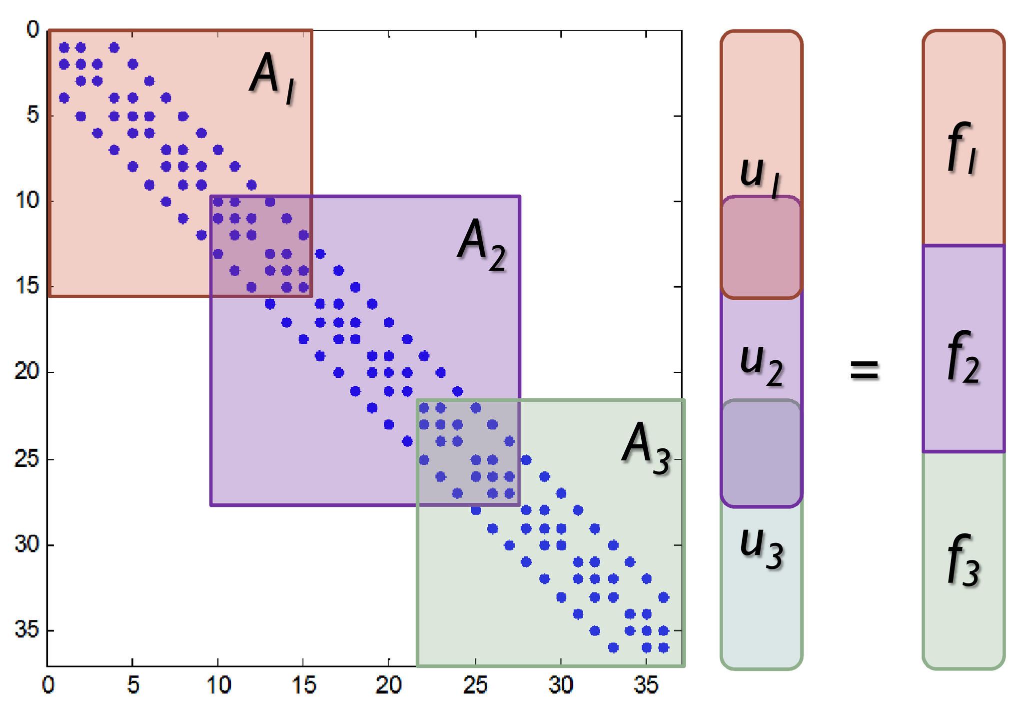


Figure 2: *Extraction of* $\mathbf{f}$ *components from* $\mathbf{u}_k$

Optimal algebraic Schwarz algorithm, Version I:

$$(A_j - B_j D_j^{-1} C_j)\mathbf{u}_j^{k+1} = \mathbf{f}_j - \sum_{i \neq j} B_j D_j^{-1} \begin{bmatrix} 0 \\ R_i A_i \\ 0 \end{bmatrix} \mathbf{u}_i^k$$

$\mathbf{u}_j^{k+1}$ will yield the exact solution as long as each $\mathbf{u}_i^k$ satisfies $R_i A_i \mathbf{u}_i^k = R_i \mathbf{f}_i$, $i \neq j$.

Algorithm converges in two iterations !

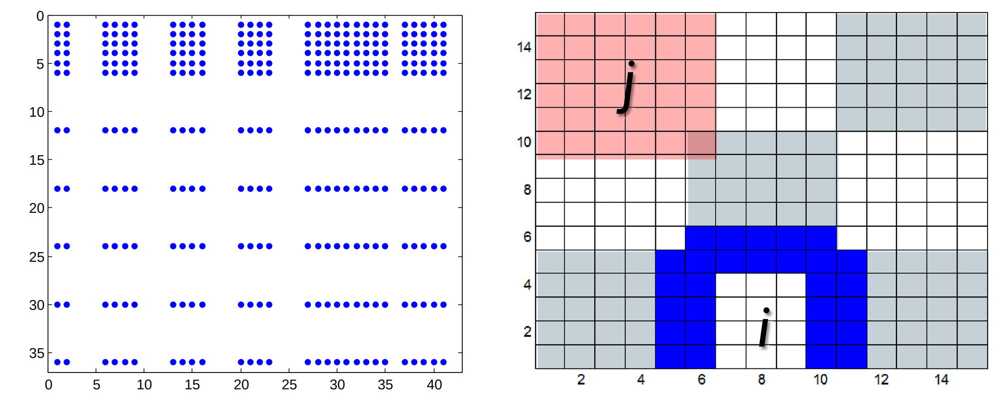
Reducing Communication

The optimal algebraic Schwarz algorithm Version I communicates all subdomain solution values to each subdomain. In contrast, a classical Schwarz method only communicates interface values of neighboring subdomains.

Observation: the term

$$B_j D_j^{-1} \begin{bmatrix} 0 \\ R_i A_i \\ 0 \end{bmatrix}$$

has a very special sparsity pattern: the column is nonzero only at interfaces between subdomains. **The values of interior nodes are not needed !**



Optimal algebraic Schwarz algorithm, Version II:

$$(A_j - B_j D_j^{-1} C_j)\mathbf{u}_j^{k+1} = \mathbf{f}_j - \sum_{i \neq j} B_j D_j^{-1} \begin{bmatrix} 0 \\ R_i A_i \\ 0 \end{bmatrix} P_{ji}^T P_{ji} \mathbf{u}_i^k$$

(P_{ji} restricts $\mathbf{u}_{ik}$ to the "boundary")

Version I and Version II produce identical iterates, and thus Version II also converges in two iterations, even though $\mathbf{f}_j^o$ are no longer reconstructed faithfully!

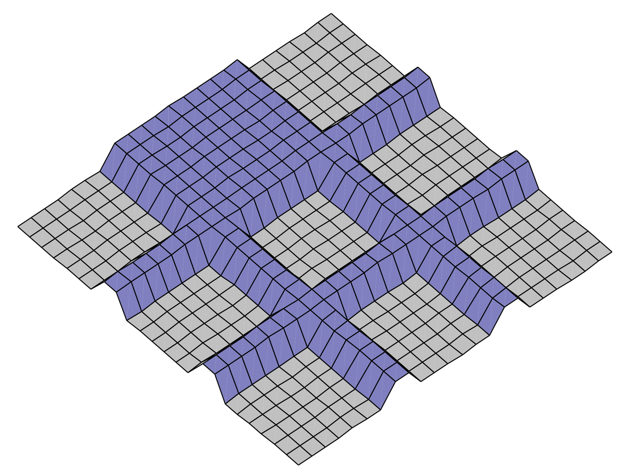


Figure 4: *Optimal interface condition information.*

Numerical Experiments

We show two sets of experiments, as illustrated in Fig. 5. Black squares indicate nodal values required by Ω_j, which is enclosed by thick solid lines. (a) decomposition into vertical strips. The top figure shows the values required by Algorithm II, and the bottom those required by classical Parallel Schwarz with optimal transmission conditions. (b) a 4×4 decomposition, shown with the communication pattern for Algorithm II.

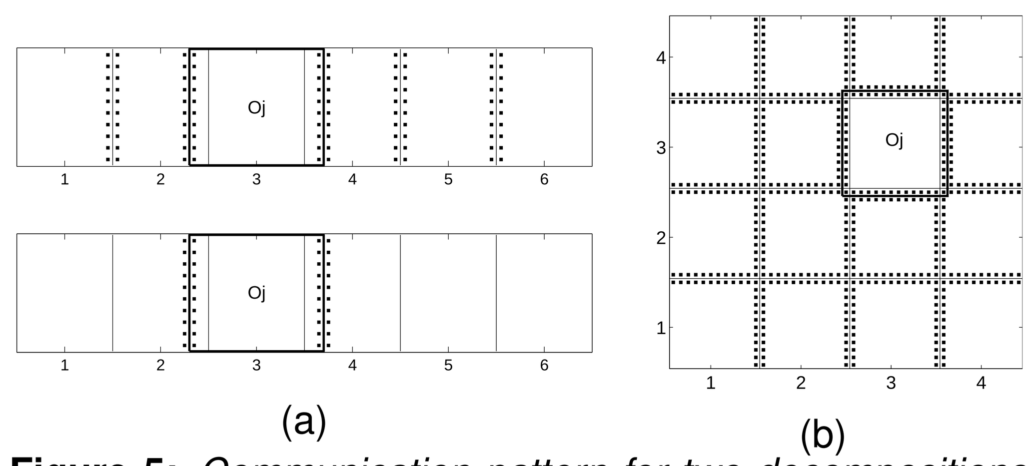


Figure 5: *Communication pattern for two decompositions into subdomains*

In Table 1 we report the maximum L^∞ errors over all subdomains.

Table 1: *Parallel Schwarz with optimal transmission conditions versus Algorithm II.*

	Example 1 (6×1)		Example 2 (4×4)	
k	Schwarz	Algorithm II	Schwarz	Algorithm II
1	3.605×10^0	3.681×10^0	6.987×10^1	6.965×10^1
2	2.176×10^{-1}	1.066×10^{-14}	1.191×10^2	8.527×10^{-13}
3	1.252×10^{-2}		5.438×10^1	
4	7.328×10^{-4}		4.652×10^2	
5	3.278×10^{-5}		1.118×10^3	
6	1.066×10^{-14}		3.894×10^3	

References

[1] M. J. Gander and F. Kwok. *Optimal Interface Conditions for an Arbitrary Decomposition into Subdomains*, Proceedings of the 19th international conference on domain decomposition methods, Springer Verlag, 2011.

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