# Convergence of Substructuring Methods for Elliptic Optimal Control Problems

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## 1 Introduction

We are interested in an Optimal Control Problem (OCP) where the constraint is given by an elliptic partial differential equation (PDE):

$$\begin{aligned} -\nabla \cdot (\kappa(\boldsymbol{x}) \nabla y(\boldsymbol{x})) &= u(\boldsymbol{x}) \quad && \boldsymbol{x} \in \Omega, \\ y(\boldsymbol{x}) &= 0 && \boldsymbol{x} \in \partial\Omega. \end{aligned} \tag{1}$$

The goal is to choose a control variable $u$ from an admissible set $U_{\text{ad}}$ to minimize the discrepancy between the solution and the desired state $\hat{y}(\boldsymbol{x})$, i.e. to minimize the objective functional

$$J(y,u) = \frac{1}{2}\int_{\Omega} |y(\boldsymbol{x}) - \hat{y}(\boldsymbol{x})|^2 d\boldsymbol{x} + \frac{\lambda}{2}\int_{\Omega} |u(\boldsymbol{x})|^2 d\boldsymbol{x}. \tag{2}$$

We formulate and analyze substructuring algorithms for the model elliptic OCP (1)–(2), which originates from the optimal stationary heating example with controlled heat source, on a bounded domain $\Omega \subset \mathbb{R}^d$. In our setting, $y$ denotes the temperature at a particular point, $\kappa(\boldsymbol{x})$ is the thermal conductivity of $\Omega$, and $\lambda > 0$ is a regularization parameter. We assume $u, \hat{y} \in L^2(\Omega)$ to ensure a solution of the problem. For simplicity, we consider $U_{\text{ad}} = L^2(\Omega)$ as the set of all feasible controls. Then from the first-order optimality conditions (cf. [8]), we obtain the adjoint equation corresponding to the problem (1)–(2)

$$\begin{aligned} -\nabla \cdot (\kappa(\boldsymbol{x}) \nabla p(\boldsymbol{x})) &= y(\boldsymbol{x}) - \hat{y}(\boldsymbol{x}) \quad && \boldsymbol{x} \in \Omega, \\ p(\boldsymbol{x}) &= 0 && \boldsymbol{x} \in \partial\Omega, \end{aligned} \tag{3}$$

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together with the optimality condition

$$p(\boldsymbol{x}) + \lambda u(\boldsymbol{x}) = 0. \tag{4}$$

We apply Domain Decomposition (DD) methods, more specifically substructuring methods to solve the state and corresponding adjoint equations. For similar applications of substructuring methods to solve linear-quadratic elliptic OCPs, see [6]. Although our techniques can be extended to multiple subdomains, we only consider a decomposition into two non-overlapping subdomains for the sake of simplicity and compact presentation. For further details on DD methods applied to OCPs, see [1, 2]. We analyze the convergence of Dirichlet-Neumann (DN) [3] and Neumann-Neumann (NN) [4] DD methods for the underlying elliptic PDEs (1)–(3). For more details on DN and NN methods, see [7]. By linearity it suffices to consider the homogeneous problems, $\hat{y}(\boldsymbol{x}) = 0$, and to analyze convergence to zero, since the corresponding error equations coincide with these homogeneous equations.

## 2 Dirichlet-Neumann algorithm

We first apply the Dirichlet-Neumann algorithm to solve the PDEs (1) and (3), coupled through the condition (4). Suppose the domain $\Omega$ is decomposed into two non-overlapping subdomains, $\Omega_1$ and $\Omega_2$. We denote by $y_i, u_i, p_i$ the restriction of $y, u, p$ to $\Omega_i$, and by $n_i$ the unit outward normal for $\Omega_i$ on the interface $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$. Then given two initial guesses $h_y^0(\boldsymbol{x})$ and $h_p^0(\boldsymbol{x})$ along the interface $\Gamma$, we write the DN algorithm for both state and adjoint equations (we do not write explicitly the homogeneous boundary conditions on the outer boundaries satisfied by the iterates): for $k = 1, 2, \ldots$ compute

$$\begin{aligned} -\nabla\cdot\big(\kappa(\boldsymbol{x})\nabla y_1^k\big) &= u_1^k && \text{in } \Omega_1, & -\nabla\cdot\big(\kappa(\boldsymbol{x})\nabla p_1^k\big) &= y_1^k && \text{in } \Omega_1,\\ y_1^k &= h_y^{k-1} && \text{on } \Gamma, & p_1^k &= h_p^{k-1} && \text{on } \Gamma, \end{aligned} \tag{5}$$

$$\begin{aligned} -\nabla\cdot\big(\kappa(\boldsymbol{x})\nabla y_2^k\big) &= u_2^k && \text{in } \Omega_2, & -\nabla\cdot\big(\kappa(\boldsymbol{x})\nabla p_2^k\big) &= y_2^k && \text{in } \Omega_2,\\ \partial_{n_2} y_2^k &= -\partial_{n_1} y_1^k && \text{on } \Gamma, & \partial_{n_2} p_2^k &= -\partial_{n_1} p_1^k && \text{on } \Gamma, \end{aligned} \tag{6}$$

together with the update conditions:

$$h_y^k(\boldsymbol{x}) = \theta_y y_2^k\,|_\Gamma + (1-\theta_y)h_y^{k-1}(\boldsymbol{x}),\ h_p^k(\boldsymbol{x}) = \theta_p p_2^k\,|_\Gamma + (1-\theta_p)h_p^{k-1}(\boldsymbol{x}), \tag{7}$$

where $\theta_y, \theta_p$ are two relaxation parameters, one for the state variable and another for the adjoint variable. Note that the adjoint problem in (5) can be derived from the first order stationarity conditions for the modified objective function

$$J_1(y,u) = \frac{1}{2}\int_{\Omega_1} |y-\hat{y}|^2\,d\boldsymbol{x} + \frac{\lambda}{2}\int_{\Omega_1} |u|^2\,d\boldsymbol{x} - \int_\Gamma \kappa\frac{\partial y}{\partial n}\cdot h_p^{k-1}\,dS(\boldsymbol{x}).$$

The adjoint system for (6) can be interpreted similarly.

We analyze the convergence of the DN algorithm (5)-(6)-(7) for the 1d case with $\Omega_1 = (0,\alpha), \Omega_2 = (\alpha, 1)$ and $\kappa(x) = 1$. By the condition (4), we write $u_i^k = -p_i^k/\lambda$ for $i = 1, 2$. We denote by $D^{(m)} := \frac{d^m}{dx^m}$. Eliminating $p_1^k, p_2^k$ from (5)–(6), we obtain

$$\begin{aligned} D^{(4)}y_1^k + \tfrac{1}{\lambda}y_1^k &= 0, & D^{(4)}y_2^k + \tfrac{1}{\lambda}y_2^k &= 0,\\ y_1^k(\alpha) &= h_y^{k-1}, & D^{(1)}y_2^k(\alpha) &= D^{(1)}y_1^k(\alpha),\\ D^{(2)}y_1^k(\alpha) &= \frac{h_p^{k-1}}{\lambda}, & D^{(3)}y_2^k(\alpha) &= D^{(3)}y_1^k(\alpha), \end{aligned} \tag{8}$$

with the homogenous boundary conditions $y_1^k(0) = 0$, $D^{(2)}y_1^k(0) = 0$, $y_2^k(1) = 0$, and $D^{(2)}y_2^k(1) = 0$ at the outer boundaries. Since $\lambda > 0$, we set $\mu^4 := 1/\lambda$. To simplify notation later, we set

$$\gamma_1 = \cosh\left(\tfrac{\mu\alpha}{\sqrt{2}}\right), \gamma_2 = \cosh\left(\tfrac{\mu(1-\alpha)}{\sqrt{2}}\right), \sigma_1 = \sinh\left(\tfrac{\mu\alpha}{\sqrt{2}}\right), \sigma_2 = \sinh\left(\tfrac{\mu(1-\alpha)}{\sqrt{2}}\right),$$
$$\eta_1 = \cos\left(\tfrac{\mu\alpha}{\sqrt{2}}\right), \quad \eta_2 = \cos\left(\tfrac{\mu(1-\alpha)}{\sqrt{2}}\right), \quad \rho_1 = \sin\left(\tfrac{\mu\alpha}{\sqrt{2}}\right), \quad \rho_2 = \sin\left(\tfrac{\mu(1-\alpha)}{\sqrt{2}}\right).$$

Then the general solution of (8) becomes

$$y_1^k(x) = A\sinh\left(\frac{\mu x}{\sqrt{2}}\right)\cos\left(\frac{\mu x}{\sqrt{2}}\right) + B\cosh\left(\frac{\mu x}{\sqrt{2}}\right)\sin\left(\frac{\mu x}{\sqrt{2}}\right), \tag{9}$$

where $A = \frac{h_y^{k-1}\sigma_1\eta_1 - \mu^2 h_p^{k-1}\gamma_1\rho_1}{\sigma_1^2+\rho_1^2}, B = \frac{h_y^{k-1}\gamma_1\rho_1 + \mu^2 h_p^{k-1}\sigma_1\eta_1}{\sigma_1^2+\rho_1^2}$, and

$$y_2^k(x) = C\sinh\left(\tfrac{\mu(1-x)}{\sqrt{2}}\right)\cos\left(\tfrac{\mu(1-x)}{\sqrt{2}}\right) + E\cosh\left(\tfrac{\mu(1-x)}{\sqrt{2}}\right)\sin\left(\tfrac{\mu(1-x)}{\sqrt{2}}\right), \tag{10}$$

with

$$\begin{aligned} C &= -A\frac{\sigma_1\sigma_2\rho_1\rho_2 + \gamma_1\gamma_2\eta_1\eta_2}{\eta_2^2+\sigma_2^2} + B\frac{\gamma_1\eta_1\sigma_2\rho_2 - \sigma_1\rho_1\gamma_2\eta_2}{\eta_2^2+\sigma_2^2},\\ E &= -A\frac{\gamma_1\eta_1\sigma_2\rho_2 - \sigma_1\rho_1\gamma_2\eta_2}{\eta_2^2+\sigma_2^2} - B\frac{\sigma_1\sigma_2\rho_1\rho_2 + \gamma_1\gamma_2\eta_1\eta_2}{\eta_2^2+\sigma_2^2}. \end{aligned}$$

Using (9) and (10), the update conditions (7) are simplified to

$$\begin{aligned} h_y^k &= (1-\theta_y)\, h_y^{k-1} + \theta_y\left(h_y^{k-1}v - \mu^2 h_p^{k-1}w\right),\\ h_p^k &= (1-\theta_p)\, h_p^{k-1} + \theta_p\left(\frac{h_y^{k-1}}{\mu^2}w + h_p^{k-1}v\right), \end{aligned} \tag{11}$$

with the two functions

$$v(\alpha,\mu) = -\frac{\rho_1\rho_2\eta_1\eta_2 + \sigma_1\sigma_2\gamma_1\gamma_2}{(\sigma_1^2+\rho_1^2)\,(\eta_2^2+\sigma_2^2)}, \quad w(\alpha,\mu) = \frac{\gamma_1\sigma_1\rho_2\eta_2 - \rho_1\eta_1\gamma_2\sigma_2}{(\sigma_1^2+\rho_1^2)\,(\eta_2^2+\sigma_2^2)}, \tag{12}$$

and we obtain the following convergence results.

**Theorem 1 (Convergence in the symmetric case).** *For symmetric subdomains, $\alpha = 1/2$ in* (5)-(6)-(7)*, the DN algorithm for the coupled PDEs converges linearly for $0 < \theta_y, \theta_p < 1$, $\theta_y \neq 1/2, \theta_p \neq 1/2$. For $\theta_y = 1/2 = \theta_p$, it converges in two iterations. Convergence is independent of the value of $\lambda$.*

*Proof.* For $\alpha = 1/2$, $v(\alpha,\mu) = -1, w(\alpha,\mu) = 0$. The expressions (11) become

$$h_y^k = (1-2\theta_y)\, h_y^{k-1} = (1-2\theta_y)^k\, h_y^0,\ h_p^k = (1-2\theta_p)\, h_p^{k-1} = (1-2\theta_p)^k\, h_p^0.$$

Therefore the convergence is linear for $0 < \theta_y, \theta_p < 1$, $\theta_y \neq 1/2, \theta_p \neq 1/2$. If $\theta_y = 1/2 = \theta_p$, we have $h_y^1 = 0 = h_p^1$, and hence the desired converged solution is achieved after one more iteration.

We now focus on the more interesting asymmetric subdomain case ($\alpha \neq 1/2$).

**Theorem 2 (Convergence in the asymmetric case).** *Suppose $\alpha \neq 1/2$. Then the DN algorithm* (5)-(6)-(7) *for the coupled PDEs converges in at most three iterations if and only if $(\theta_y, \theta_p)$ equals either $(\Lambda^+, \Lambda^-)$ or $(\Lambda^-, \Lambda^+)$, where*

$$\Lambda^{\pm} := \frac{1}{(1-v)} \pm \frac{|w|}{(1-v)\sqrt{(1-v)^2 + w^2}}. \tag{13}$$

*Proof.* For $\alpha \neq 1/2$, we set $\bar{h}_p^k := \mu h_p^k, \bar{h}_y^k := \frac{h_y^k}{\mu}$. We rewrite the updating terms (11) in the matrix form

$$\begin{pmatrix} \bar{h}_y^k \\ \bar{h}_p^k \end{pmatrix} = \left[ \begin{pmatrix} 1-\theta_y & 0 \\ 0 & 1-\theta_p \end{pmatrix} + \begin{pmatrix} \theta_y v(\alpha,\mu) & -\theta_y w(\alpha,\mu) \\ \theta_p w(\alpha,\mu) & \theta_p v(\alpha,\mu) \end{pmatrix} \right] \begin{pmatrix} \bar{h}_y^{k-1} \\ \bar{h}_p^{k-1} \end{pmatrix}.$$

Note that the matrix of the system on the right side (which we call $S$) is never zero for any particular set of values $\theta_y, \theta_p$. So we do not get two-step convergence for $\alpha \neq 1/2$, unlike in Theorem 1. We claim that there is some positive integer $n$, for which $S^n = 0$. This results in

$$\begin{pmatrix} \bar{h}_y^n \\ \bar{h}_p^n \end{pmatrix} = S^n \begin{pmatrix} \bar{h}_y^0 \\ \bar{h}_p^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so that the DN algorithm converges in $n+1$ iterations. The spectral radius of $S$ is

$$\Upsilon(\theta_y, \theta_p, \alpha, \mu) := \max\left\{ \left| 1 - \frac{1}{2}(\theta_y + \theta_p)(1-v) \pm \frac{1}{2}\sqrt{(\theta_y - \theta_p)^2 (1-v)^2 - 4\theta_y\theta_p w^2} \right| \right\}.$$

For each $\alpha \in (0,1)$ and $\mu > 0$, we solve the system

$$1 - \frac{1}{2}(\theta_y + \theta_p)(1-v) = 0, \quad (\theta_y - \theta_p)^2 (1-v)^2 - 4\theta_y\theta_p w^2 = 0 \tag{14}$$

simultaneously for $\theta_y, \theta_p$ to obtain $(\Lambda^+, \Lambda^-)$, as in equation (13). $\Upsilon$ being symmetric with respect to $\theta_y, \theta_p$, $(\Lambda^-, \Lambda^+)$ is also a solution of the system

(14). Therefore $\Upsilon(\Lambda^{\pm}, \Lambda^{\mp}, \alpha, \mu) = 0$, resulting in $S^2 = 0$ and hence three step convergence to the exact solution. For any other values of $(\theta_y, \theta_p)$, the spectral radius of $S$ is non-zero, so the algorithm cannot converge to the exact solution in a finite number of iterations.

*Remark 1.* Since $v(\alpha, \mu) \leq 0$ (which can be seen from (12) by noting that $\gamma_i \geq |\eta_i|, \sigma_i \geq |\rho_i|$ for all $\alpha, \mu$), equation (13) implies that $\Lambda^- \in (0, 1)$ and $\Lambda^+ \in (0, 2)$. Note that unlike the symmetric case $\alpha = 1/2$, it is possible to have convergence for $\theta_y > 1$; for $\alpha = 0.99$ and $\mu = \sqrt{8}$, convergence in three steps occurs for $(\theta_y, \theta_p) = (1.000685490, 0.9621364448)$.

*Remark 2.* For a symmetric decomposition of a rectangular domain in 2D into two equal subdomains, it can be shown that $\Lambda^{\pm} = 0.5$ still gives two-step convergence in the DN method. For an asymmetric decomposition, however, the optimal values may be different, see the last example in Section 4.

## 3 Neumann-Neumann algorithm

To write the NN algorithm for both state and adjoint equations (1)-(3), we again divide $\Omega$ into two non-overlapping subdomains, $\Omega_1$ and $\Omega_2$. We use the same notations as in Section 2. Given two initial guesses $g_y^0(\boldsymbol{x})$ and $g_p^0(\boldsymbol{x})$ along the interface $\Gamma$, the NN algorithm is (again we do not write explicitly the homogeneous boundary conditions on the outer boundaries satisfied by the iterates): for $k = 1, 2, \ldots$ compute the approximations

$$\begin{aligned} -\nabla \cdot \left(\kappa(\boldsymbol{x})\nabla y_i^k\right) &= u_i^k \qquad \text{in } \Omega_i, \\ y_i^k &= g_y^{k-1} \qquad \text{on } \Gamma, \end{aligned} \tag{15}$$

followed by the correction step,

$$\begin{aligned} -\nabla \cdot \left(\kappa(\boldsymbol{x})\nabla \psi_i^k\right) &= 0 \qquad \text{in } \Omega_i, \\ \partial_{n_i}\psi_i^k &= \partial_{n_1} y_1^k + \partial_{n_2} y_2^k \qquad \text{on } \Gamma, \end{aligned} \tag{16}$$

and similarly for the adjoint equation, we compute

$$\begin{aligned} -\nabla \cdot \left(\kappa(\boldsymbol{x})\nabla p_i^k\right) &= y_i^k \qquad \text{in } \Omega_i, \\ p_i^k &= g_p^{k-1} \qquad \text{on } \Gamma, \end{aligned} \tag{17}$$

followed by the correction step,

$$\begin{aligned} -\nabla \cdot \left(\kappa(\boldsymbol{x})\nabla \varphi_i^k\right) &= 0 \qquad \text{in } \Omega_i, \\ \partial_{n_i}\varphi_i^k &= \partial_{n_1} p_1^k + \partial_{n_2} p_2^k \qquad \text{on } \Gamma. \end{aligned} \tag{18}$$

The update conditions for $g_y^k$ and $g_p^k$ are

$$\begin{aligned} g_y^k(\boldsymbol{x}) &= g_y^{k-1}(\boldsymbol{x}) - \theta_y \left(\psi_1^k\,|_\Gamma + \psi_2^k\,|_\Gamma\right), \\ g_p^k(\boldsymbol{x}) &= g_p^{k-1}(\boldsymbol{x}) - \theta_p \left(\varphi_1^k\,|_\Gamma + \varphi_2^k\,|_\Gamma\right). \end{aligned} \tag{19}$$

We again analyze the convergence for the NN algorithm (15)–(19) for $\Omega_1 = (0, \alpha), \Omega_2 = (\alpha, 1)$ and $\kappa(x) = 1$. By (4), we have $u_i^k = -p_i^k/\lambda$ for $i = 1, 2$. Eliminating $p_1^k, p_2^k$ from (15)-(17), we obtain

$$\begin{array}{rlrl} D^{(4)} y_1^k + \frac{1}{\lambda} y_1^k = 0, & & D^{(4)} y_2^k + \frac{1}{\lambda} y_2^k = 0, \\ y_1^k(\alpha) = g_y^{k-1}, & & y_2^k(\alpha) = g_y^{k-1}, \\ D^{(2)} y_1^k(\alpha) = \frac{g_p^{k-1}}{\lambda}, & & D^{(2)} y_2^k(\alpha) = \frac{g_p^{k-1}}{\lambda}, \end{array} \tag{20}$$

with the homogenous boundary conditions $y_1^k(0) = 0$, $D^{(2)} y_1^k(0) = 0$, $y_2^k(1) = 0$, and $D^{(2)} y_2^k(1) = 0$ at the outer boundaries. With $\mu^4 := 1/\lambda$, the solutions of (20) become

$$\begin{aligned} y_1^k(x) &= E_1 \sinh\left(\tfrac{\mu x}{\sqrt{2}}\right) \cos\left(\tfrac{\mu x}{\sqrt{2}}\right) + E_2 \cosh\left(\tfrac{\mu x}{\sqrt{2}}\right) \sin\left(\tfrac{\mu x}{\sqrt{2}}\right), \\ y_2^k(x) &= F_1 \sinh\left(\tfrac{\mu(1-x)}{\sqrt{2}}\right) \cos\left(\tfrac{\mu(1-x)}{\sqrt{2}}\right) + F_2 \cosh\left(\tfrac{\mu(1-x)}{\sqrt{2}}\right) \sin\left(\tfrac{\mu(1-x)}{\sqrt{2}}\right), \end{aligned}$$

where

$$E_1 = \frac{g_y^{k-1}\sigma_1\eta_1 - \mu^2 g_p^{k-1}\gamma_1\rho_1}{\sigma_1^2 + \rho_1^2}, \quad E_2 = \frac{g_y^{k-1}\gamma_1\rho_1 + \mu^2 g_p^{k-1}\sigma_1\eta_1}{\sigma_1^2 + \rho_1^2},$$

$$F_1 = \frac{g_y^{k-1}\sigma_2\eta_2 - \mu^2 g_p^{k-1}\gamma_2\rho_2}{\sigma_2^2 + \rho_2^2}, \quad F_2 = \frac{g_y^{k-1}\gamma_2\rho_2 + \mu^2 g_p^{k-1}\sigma_2\eta_2}{\sigma_2^2 + \rho_2^2}.$$

Finally solving $\psi_i^k, \varphi_i^k$ in (16)-(18) and replacing them in (19) we get the updating terms

$$\begin{aligned} g_y^k &= g_y^{k-1} - \theta_y (g_y^{k-1} z_1 + \mu^2 g_p^{k-1} z_2), \\ g_p^k &= g_p^{k-1} - \theta_p (g_p^{k-1} z_1 - \tfrac{1}{\mu^2} g_y^{k-1} z_2), \end{aligned} \tag{21}$$

with the functions $z_1(\alpha, \mu) = \frac{\mu}{\sqrt{2}} \left( \frac{\sigma_1\gamma_1 + \rho_1\eta_1}{\sigma_1^2 + \rho_1^2} + \frac{\sigma_2\gamma_2 + \rho_2\eta_2}{\sigma_2^2 + \rho_2^2} \right)$, and $z_2(\alpha, \mu) = \frac{\mu}{\sqrt{2}} \left( \frac{\sigma_1\gamma_1 - \rho_1\eta_1}{\sigma_1^2 + \rho_1^2} + \frac{\sigma_2\gamma_2 - \rho_2\eta_2}{\sigma_2^2 + \rho_2^2} \right)$.

**Theorem 3 (Convergence of the NN algorithm).** *The NN algorithm for the coupled PDEs* (15)–(19) *converges in at most three iterations if* $(\theta_y, \theta_p)$ *is any of the pairs* $(\Theta^+, \Theta^-), (\Theta^-, \Theta^+)$, *where* $\Theta^\pm := \frac{1}{z_1} \pm \frac{|z_2|}{z_1\sqrt{z_1^2 + z_2^2}}$.

*Proof.* Setting $\bar{g}_p^k := \mu g_p^k, \bar{g}_y^k := \frac{g_y^k}{\mu}$, we rewrite the updating terms (21) as:

$$\begin{pmatrix} \bar{g}_y^k \\ \bar{g}_p^k \end{pmatrix} = \begin{pmatrix} 1 - \theta_y z_1 & -\theta_y z_2 \\ \theta_p z_2 & 1 - \theta_p z_1 \end{pmatrix} \begin{pmatrix} \bar{g}_y^{k-1} \\ \bar{g}_p^{k-1} \end{pmatrix}.$$

The matrix on the right side (we call $P$) is never zero for any set of values $\theta_y, \theta_p$. But like in the DN method, if we have $P^n = 0$, for some $n$, then we get

$$\begin{pmatrix} \bar{g}_y^n \\ \bar{g}_p^n \end{pmatrix} = P^n \begin{pmatrix} \bar{g}_y^0 \\ \bar{g}_p^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

resulting in convergence in $n+1$ iterations. The spectral radius of $P$ is: $\Phi\left(\theta_y, \theta_p, \alpha, \mu\right) := \max\left\{\left|1 - \frac{1}{2}\left(\theta_y + \theta_p\right) z_1 \pm \frac{1}{2}\sqrt{\left(\theta_y - \theta_p\right)^2 z_1^2 - 4\theta_y\theta_p z_2^2}\right|\right\}$. We solve the system $1 - \frac{1}{2}\left(\theta_y + \theta_p\right) z_1 = 0, \left(\theta_y - \theta_p\right)^2 z_1^2 - 4\theta_y\theta_p z_2^2 = 0$ simultaneously for each $\alpha \in (0,1)$ and $\mu > 0$ to obtain a solution $(\Theta^+, \Theta^-)$ as in the Theorem. Due to the symmetric nature of $\Phi$ with respect to $\theta_y, \theta_p$, $(\Theta^-, \Theta^+)$ is another solution pair of the system of equations. Thus $\Phi\left(\Theta^\pm, \Theta^\mp, \alpha, \mu\right) = 0$, resulting in $P^2 = 0$ and therefore three step convergence to the exact solution.

## 4 Numerical Examples

We perform numerical experiments to verify the convergence rate of the DN and NN algorithms for the model problem (1)–(2) with $\lambda = 1/2$, $\hat{y}(x) = 0$. In the top two plots of Figure 1, we observe two-step convergence of the DN method for $\alpha = 1/2$ on the left, and three-step convergence for $\alpha = 0.6$ for the optimal choice of $(\Lambda^+, \Lambda^-) = (0.62, 0.57)$ on the right. The two bottom plots of Figure 1 show the convergence behavior of the DN algorithm for different choices of $\theta_y$ and $\theta_p$. On the left panel, we get $\theta_y = \theta_p = 1/2$ to be the best parameters for the symmetric case, whereas on the right $(\Lambda^+, \Lambda^-)$ yields the fastest convergence for $\alpha = 0.6$. For the NN experiment, we plot on the left panel of Figure 2 the first three iterates of the state variable for the optimal choice of $(\Theta^+, \Theta^-) = (0.30, 0.16)$, and on the right the convergence curves for various values of the parameters $\theta_y, \theta_p$. In Figure 3, we show convergence of the DN and NN methods for the 2D problem:

$$\begin{aligned} -\Delta y(\boldsymbol{x}) &= u(\boldsymbol{x}) \quad && \boldsymbol{x} \in \Omega = (0,1)^2, \\ y(\boldsymbol{x}) &= 0 && \boldsymbol{x} \in \partial\Omega, \end{aligned}$$

with an interface $\Gamma = \{0.6\} \times (0,1)$ and $\lambda = 1/2$. Note that the optimal parameters are different from the 1d case when the decomposition is non-symmetric, as the choice of $(0.5, 0.5)$ appears to perform better than $(\Lambda^+, \Lambda^-)$ in the DN example. A full analysis of the 2D case will be the subject of a future paper. We are also working on the analysis of the case of multiple subdomains, where it is not clear if one can choose relaxation parameters to obtain finite termination of the algorithm; see [5] for the uncontrolled case.

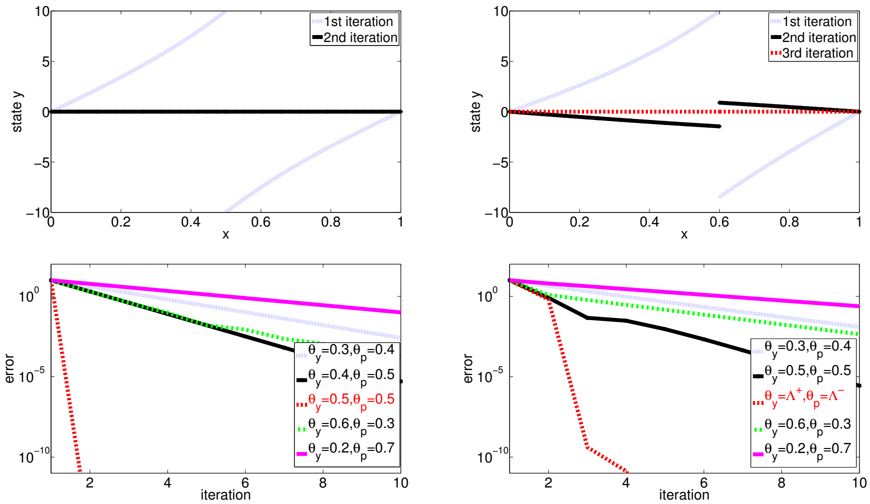


**Fig. 1** Convergence of the iterative solution of the DN method: in two iterations for the symmetric case on the top left, and in three iterations for $\alpha = 0.6$ on the top right; error curves for various values of $\theta_y, \theta_p$ for $\alpha = 1/2$ on the bottom left, and for $\alpha = 0.6$ on the bottom right

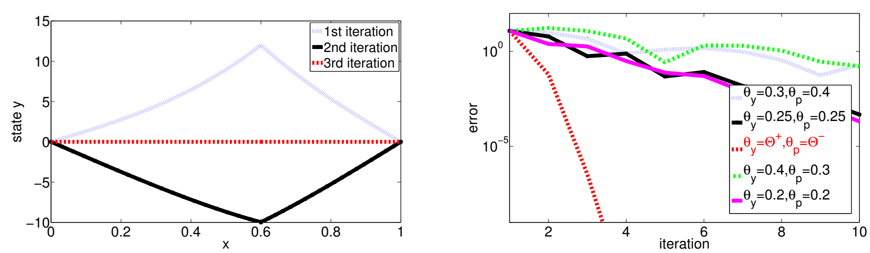


**Fig. 2** Convergence of NN: convergence of the iterative solutions with optimal parameters in three iterations on the left, and convergence for various values of $\theta_y, \theta_p$ on the right

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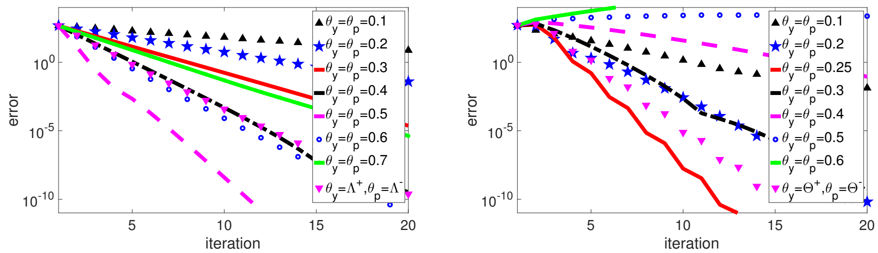


**Fig. 3** Convergence in 2d: convergence of DN method in 2d on the left, and NN method in 2d on the right. $\Lambda^{\pm}, \Theta^{\pm}$ correspond to 1d optimal parameters


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