Optimized Schwarz Method for Problems with Cross Points

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Objectives

The goal of this work is to analyze the behaviour of optimized Schwarz Methods with Robin transmission conditions when cross points are present. In particular, by studying a simple 4-subdomain model problem, we will:

- give necessary and sufficient conditions on the Robin parameters for convergence,
- calculate the optimal scaling for the edge and corner parameters.

1. Optimized Schwarz method

We would like to solve the elliptic PDE

$$\mathcal{L}u = f \quad \text{on } \Omega, \qquad u = g \quad \text{on } \partial\Omega \qquad (*)$$

in parallel by dividing the computational domain Ω into several subdomains $\Omega_1, \ldots, \Omega_N$ (see Fig. 1 for $N = 2$).

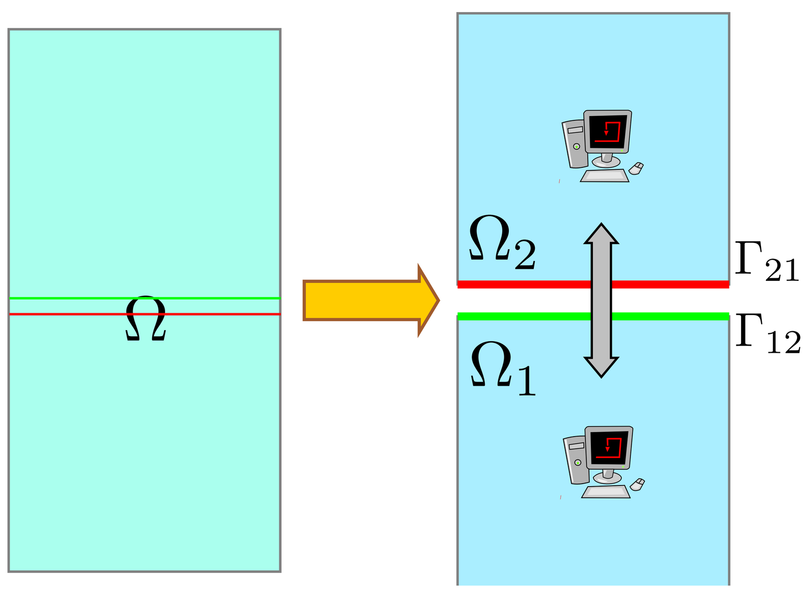


Figure 1: *Domain decomposition.*

The artificial interfaces (shown in red and green) need to obtain boundary data from the other subdomains. The subdomain problems are solved in parallel, and each subdomain transmits the updated boundary data to its neighbours. This is repeated until convergence.

Algorithm 1 ($N = 2$). For $k = 0, 1, 2, \ldots$, solve

$$\begin{cases} \mathcal{L}u_1^{k+1} = f & \text{on } \Omega_1, \\ u_1^{k+1} = g & \text{on } \partial\Omega \cap \partial\Omega_1, \\ \mathcal{B}_{12}u_1^{k+1} = \mathcal{B}_{12}u_2^{k} & \text{on } \Gamma_{12}; \end{cases} \qquad \begin{cases} \mathcal{L}u_2^{k+1} = f & \text{on } \Omega_1, \\ u_2^{k+1} = g & \text{on } \partial\Omega \cap \partial\Omega_2, \\ \mathcal{B}_{21}u_2^{k+1} = \mathcal{B}_{21}u_1^{k} & \text{on } \Gamma_{21}. \end{cases}$$

$\mathcal{B}_{ij}$ can be

- Id (Dirichlet conditions),
- ∂_n (Neumann conditions),
- $\partial_n + p$, $p > 0$ (Robin conditions),
- Higher order conditions.

The Robin parameter p can be optimized to obtain asymptotically faster convergence than Dirichlet conditions when the grid parmeter h tends to zero:

- Dirichlet ($O(h)$ overlap): $\rho = 1 - Ch$
- Robin (no overlap, $p = O(1/\sqrt{h})$): $\rho = 1 - C\sqrt{h}$

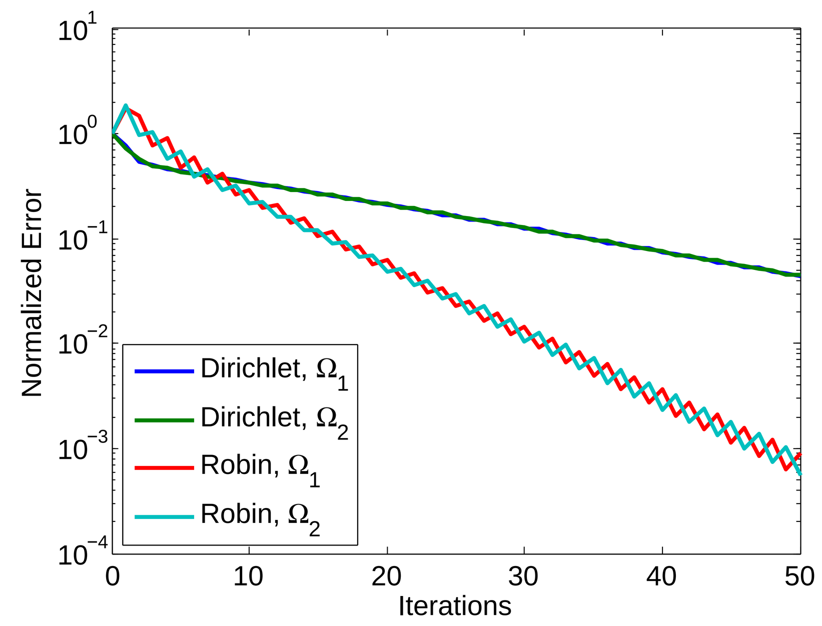


Figure 2: *Convergence of Dirichlet vs. Robin transmission conditions for two subdomains.*

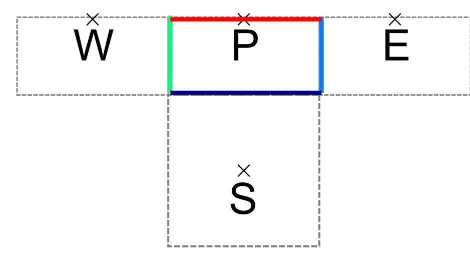
If $\mathcal{A}\mathbf{u} = \mathbf{f}$ is the discretization of $(*)$, then the discrete version of the iteration Algorithm 1 can be written as

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{pmatrix} \mathbf{u}_1^{k+1} \\ \mathbf{u}_2^{k+1} \end{pmatrix} = -\begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix}\begin{pmatrix} \mathbf{u}_1^{k} \\ \mathbf{u}_2^{k} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix},$$

Since each interface variable appears twice (once in each of $\mathbf{u}_1$ and $\mathbf{u}_2$), we need to enforce the consistency condition

$$\mathcal{A}\mathbf{u} = \mathbf{f} \iff \begin{bmatrix} A_1 & B_{12} \\ B_{21} & A_2 \end{bmatrix}\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}.$$

To calculate the matrices A_1 and A_2 corresponding to the subproblem with Robin conditions $\partial_n u + pu = g$, we derive the stencil using a finite-volume formulation:



$$\text{Flux} = \frac{h}{2}\cdot\frac{u_E - u_P}{h}$$
$$\text{Flux} = \frac{h}{2}\cdot\frac{u_W - u_P}{h}$$
$$\text{Flux} = h\cdot\frac{u_S - u_P}{h}$$
$$\text{Flux} = h\cdot(g - p\cdot u)$$

The stencil for the transmission matrices B_{ij} are chosen to be consistent with $\mathcal{A}\mathbf{u} = \mathbf{f}$. For the discrete Laplacian, we have the following stencil:

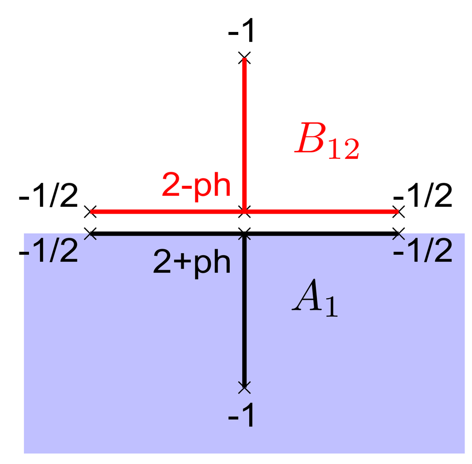
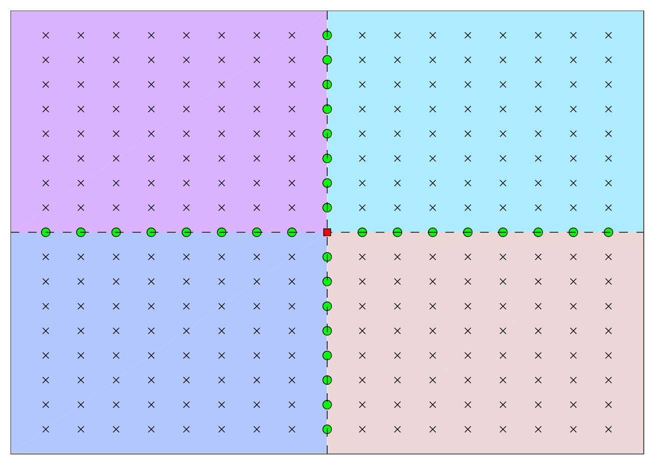


Figure 4: *Stencil for the two-subdomain problem.*

2. A cross point problem

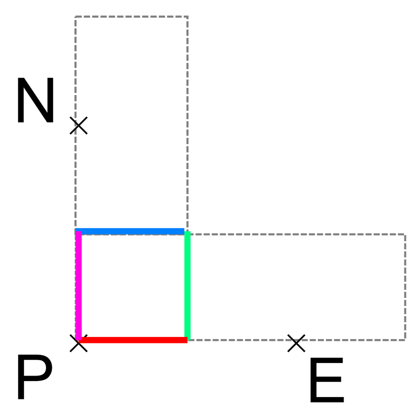
Consider the 4-subdomain decomposition below:



The centre point (in red) is a *cross point* because it touches more than 2 subdomains. The subdomain iteration becomes

$$\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{bmatrix}\begin{pmatrix} \mathbf{u}_1^{k+1} \\ \mathbf{u}_2^{k+1} \\ \mathbf{u}_3^{k+1} \\ \mathbf{u}_4^{k+1} \end{pmatrix} = -\begin{bmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & B_{32} & 0 & B_{34} \\ B_{41} & B_{42} & B_{43} & 0 \end{bmatrix}\begin{pmatrix} \mathbf{u}_1^{k} \\ \mathbf{u}_2^{k} \\ \mathbf{u}_3^{k} \\ \mathbf{u}_4^{k} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \\ \mathbf{f}_4 \end{pmatrix}. \quad (\dagger)$$

The stencils at the corner point corresponding to A_i and B_{ij} with Robin conditions are shown in Figures 5 and 6.



$$\text{Flux} = \frac{h}{2}\cdot\frac{u_N - u_P}{h}$$
$$\text{Flux} = \frac{h}{2}\cdot\frac{u_E - u_P}{h}$$
$$\text{Flux} = \frac{h}{2}\cdot(g_S - p\cdot u)$$
$$\text{Flux} = \frac{h}{2}\cdot(g_W - p\cdot u)$$

Figure 5: *Finite volume stencil for A_i at the centre.*

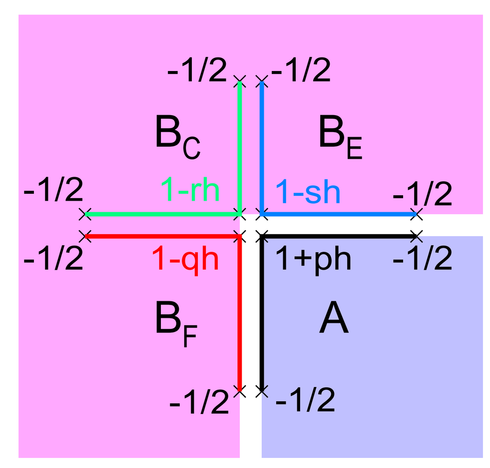


Figure 6: *Stencil for 4-subdomain problem at the centre. Consistency requires $p = q + r + s$.*

$(\dagger)$ is essentially a block-Jacobi iteration, where convergence requires the diagonal dominance of the augmented system. Since A_i has weight $1 + ph$ and the B_{ij} have a combined weight $3 - ph$ at the corner, the augmented matrix would not be diagonally dominant unless p is large enough.

Theorem 1 *For the discrete Laplace equation, $(\dagger)$ converges if and only if $p > \frac{1}{h}$ at the corner.*

Theorem 1 can be generalized to N subdomains for $N \geq 2$. Figure 7 shows the error for the 4-subdomain problems with Dirichlet and Robin conditions when p is too small at the corner.

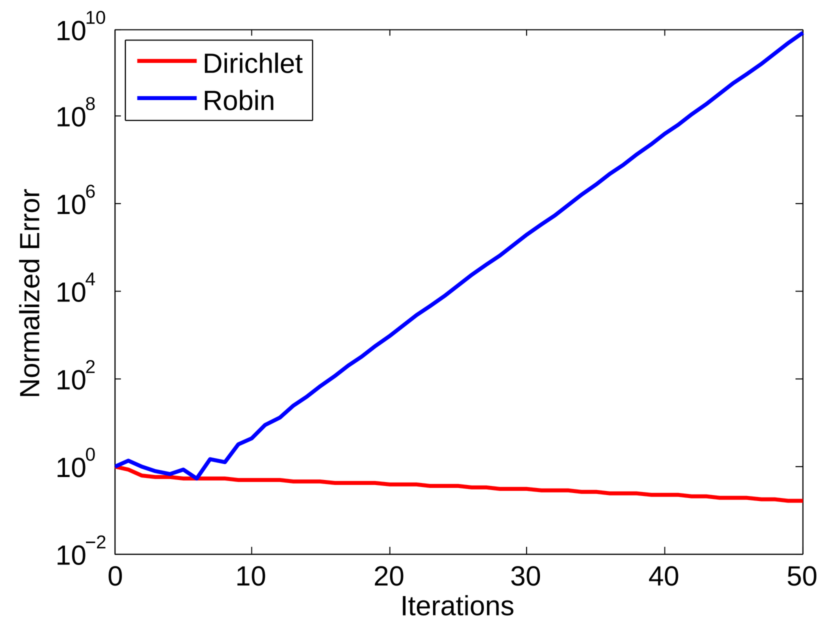


Figure 7: *Convergence of Dirichlet vs. Robin transmission conditions ($p = O(h^{-1/2})$).*

3. Optimal scaling

Let p_E and p_C be fixed Robin parameters for the edge and the corner. The rate of convergence of $(\dagger)$ is given by the spectral radius of the iteration matrix $R(p_E, p_C)$, where

$$R = -\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & A_3 & \\ & & & A_4 \end{bmatrix}^{-1}\begin{bmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & B_{32} & 0 & B_{34} \\ B_{41} & B_{42} & B_{43} & 0 \end{bmatrix}.$$

It is possible to write $R = (I - X)(I + X)^{-1}$, where X has positive eigenvalues $\mu_i > 0$ (otherwise $\rho(R) \geq 1$ and the iteration diverges). To find the best Robin parameters, we must solve the min-max problem

$$\min_{p_E, p_C}\max_i \frac{1 - \mu_i}{1 + \mu_i}.$$

The optimality conditions are

1. $\mu_{\min}\mu_{\max} = 1$,
2. $\mu_{\max}/\mu_{\min} \to \min$.

By estimating $\mu_{\min}$ and μ_{max} in terms of p_E and p_C, we obtain the following asymptotic behaviour.

Theorem 2 *The optimal Robin parameters p_E and p_C satisfy*

$$p_E = O(h^{-1/2}), \quad \frac{c_1}{h} \leq p_C \leq \frac{c_2}{h^{3/2}|\log h|},$$

leading to a contraction factor of $1 - C\sqrt{h}$.

Figures 8 and 9 show the convergence behaviour for $p_E = 1.65/\sqrt{h}$, $p_C = 1.7/h$.

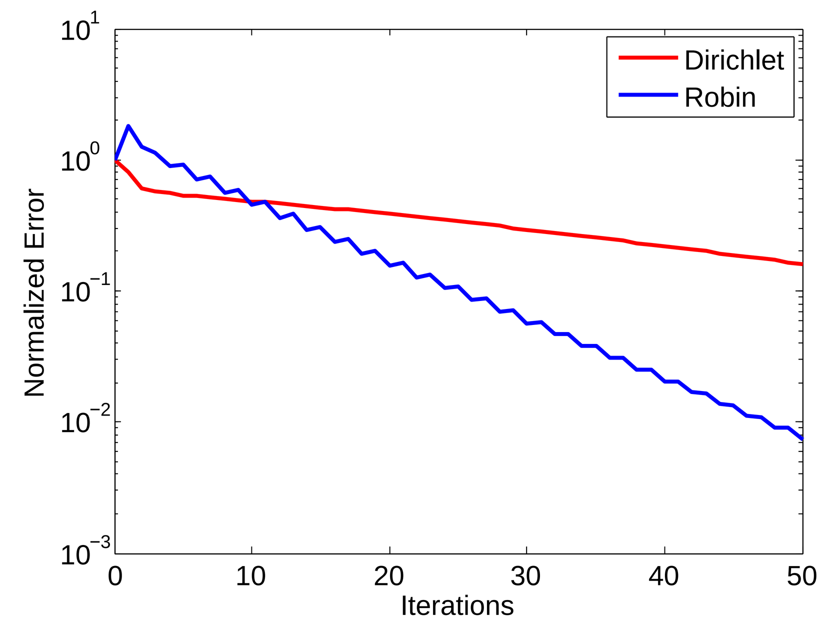


Figure 8: *Convergence of Dirichlet vs. Robin transmission conditions ($p = O(h^{-1})$ at corner).*

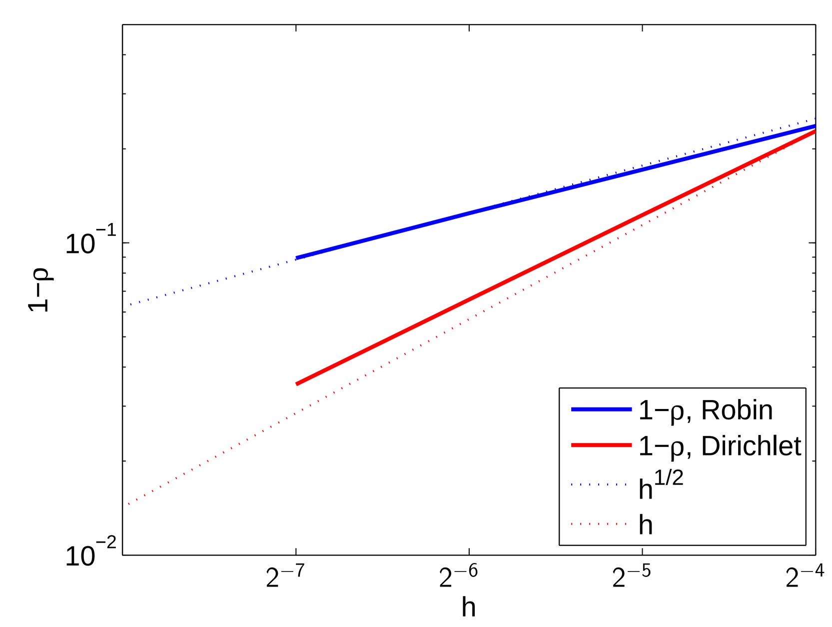


Figure 9: *Contraction factor $(1 - \rho)$ vs. grid parameter (h).*

References

[1] M. J. Gander. *Optimized Schwarz methods*, SIAM J. Numer. Anal. 44, pp. 699–731, 2006.

[2] M. J. Gander and F. Kwok. *Best Robin parameters for Optimized Schwarz methods at cross points*, in preparation, 2010.