A COUPLED PREDICTION SCHEME FOR SOLVING THE NAVIER–STOKES AND CONVECTION-DIFFUSION EQUATIONS *

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Abstract.
This paper presents a new algorithm for the numerical solution of the Navier-Stokes equations coupled with the convection-diffusion equation. After establishing convergence of the semi-discrete formulation at each time step, we introduce a new iterative scheme based on a projection method called *Coupled Prediction Scheme* (CPS). We show that even though the predicted temperature is advected by a velocity prediction which is not necessarily divergence free, the theoretical time accuracy of the global scheme is conserved. From a numerical point of view, this new approach gives a faster and more efficient algorithm compared to the usual fixed-point approaches.

Key words.
Boussinesq, Navier-Stokes equations, free convection, fractional time stepping, iterative scheme.

AMS subject classifications. 76R10 65M12 65M60 35Q35

1. Introduction. Heat transfer is an important factor in many fluid dynamics applications. Whenever there is a temperature difference between the fluid and the confining area, heat will be transferred and the flow will be affected in non trivial ways. Natural convection is such an example in which the driving forces are density variations and gravity (see Jiji [28] for instance). Natural convection flows are observed in different situations such as geophysics, weather, ocean movement and are also exploited in numerous applications: double-glazed windows, cooling in electronic devices, building insulation, etc.

The model is generally described using the Boussinesq approximation. In this approximation, the density of the fluid is assumed to be constant and the gravitational source force (the buoyancy term in the momentum equation) depends on the temperature (Martynenko and Khramtsov [34]).

Typically, in the Boussinesq approximation, the coupling between the fluid and the temperature appears through two terms: a source term depending linearly on the temperature, and a convective term based on the velocity of the fluid (see system (2.1)). In this paper we propose a reinforcement of this coupling by adding an explicit dependency to the temperature for the viscosity and the diffusion coefficients. Moreover, since the assumptions on the source term for the momentum equation are not essentials (Remark 2.1), we will consider a more general source term. Owing to this departure from the usual Boussinesq equations, the proposed model can be viewed more generally as a thermally coupled Navier-Stokes problem.

Thermally coupled incompressible flow problems present two major difficulties requiring special attention: solving the incompressible Navier-Stokes equations on very fine three-dimensional meshes in a reasonable computational time is a difficult task; the strong coupling between the Navier-Stokes and convection-diffusion equations often leads to very complex time dependent dynamics requiring efficient solvers.

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From a theoretical point of view, the time dependent case was studied in numerous works (see for example Joseph [29] and Sattinger [41]). For the stationary case, a proof of existence and uniqueness requiring important restrictions on the physical parameters was presented in Gaulthier and Lezaun [15]. In the case of a bounded domain of $\mathbb{R}^2$ or $\mathbb{R}^3$, Bernardi *et al.* [6] obtained a local existence and uniqueness result without any condition on the physical parameters. Theoretical results for a monolithic finite element approximation were also presented.

From a numerical point of view, the Navier-Stokes equations are central in the effectiveness of any method aiming at solving this problem. Since we are interested in three dimensional problems on very fine meshes, direct methods are essentially excluded. One way to circumvent their intrinsic saddle point structure, due to the incompressibility constraint, is to follow the pioneering works of Chorin [13, 14] and Temam [47] who introduced projection methods. The idea is to decouple the incompressibility constraint from the diffusion operator. Numerous variants have been proposed over the years ([3, 40, 43, 21, 24]). For an interesting overview, we also refer to Guermond and coauthors [23, 24].

Several approaches have been proposed for solving natural convection problems, finite element, finite volume, variational multiscale methods (VMS), pseudo spectral approaches, etc (see for examples [20, 7, 31, 33, 35, 30, 51, 45]). The monolithic approach (the resolution of a single system at each time step, as in [6, 30]) leads to large discrete non linear systems, and, for reasonable numerical performances, elaborate algebraic solvers are needed (see [30]). Obviously the arguments that makes the projection method appealing for the Navier-Stokes resolution still applies for the thermally coupled problem. It should come as no surprise that with the exception of monolithic methods (as [30]) and the VMS approach (as [51]), most of the methods, from "pseudo monolithic" finite element methods (see [20, 33, 35]) to the most recent finite volume approaches (for example [36, 44, 32, 45]), are based on a fractional time step or a predictor-corrector approach.

Concerning numerical efficiency, a final note should be made concerning the use of explicit terms allowing a "numerical" decoupling of both equations (thus eliminating the need of a fixed-point) (see for example [44, 35]) or the explicit treatment of the non linear terms yielding a numerically less expansive coupled system. Even with today's technology, the general considerations found in Gresho *et al.* [20, p. 207], are still relevant : explicit methods imposes restrictions on the time step (or the mesh size) which can be severely detrimental to the *overall* performance; even if the implicit/semi-implicit approach is sub-optimal in certain cases, it is generally more robust and gives good performance in most cases. In view of this and the fact that we reinforced the coupling via the physical parameters (and possibly a more complex source term), we will make no effort to study the explicit approaches. Nevertheless, most of the results presented here would still hold, provided a general Courant-Friedrichs-Lewy (CFL) conditions is established (Remark 5.2 and §8).

Two schemes will be presented in this paper, the *Basic Projection Scheme* (BPS, see §5.1) and the *Coupled Prediction Scheme* (CPS, see §5.2). Both schemes are first order in time and rely on an *incremental projection* scheme as proposed in [19, 49]. This incremental projection can be seen as an improvement over projection scheme of Chorin and Temam. Rannacher [40] and Shen [42] showed that the original scheme is not fully first-order even when using higher-order time stepping. Shen [43], Guermond [21, 22], Guermond *et al.* [25], proved that the incremental scheme can be of first-order.

The *Basic Projection Scheme* is introduced uniquely to establish the validity and time accuracy (precision) of the new scheme (CPS) since we know the theoretical global rate of convergence in time for this relatively simple scheme. Although, we could not find precise reference presenting the basic projection scheme, it can be recognize in the literature. The algorithm proposed in [36] is in fact a BPS based on the finite volume method (since the calculation of the pressure in the fixed point loop, this version of BPS is less efficient than §5.1). The algorithm proposed in [44], also based on the finite volume method, is a BPS, although, in that case the approach is totally explicit, making the fixed point loop inactive. As for algorithms based on the finite element method, approaches such as in [20] (a scheme based on a predictor-corrector method), and the works of Nithiarasu *et al.* based on the projection method (for example see [35]), can be regarded as variations of the basic projection scheme. Even with today's tools, it is almost impossible to collect all the communications related to this problem. Although the BPS can be regarded as known scheme, to the best of our knowledge, the CPS is a new approach, based on the projection method, for thermally coupled Navier–Stokes problems.

The *Coupled Prediction Scheme* proposed here, can be described summarily as a modified BPS in which the fixed point loop is reduced to a bare minimum. This scheme rely on the fact that the velocity prediction of a projection method is rich enough to produce a good estimate of the temperature. This idea stems from the arguments raised in [24] concerning the quality of the velocity prediction that can put forward for the temperature. Therefore, in the CPS, *the convection-diffusion equation is coupled to the predicted velocity instead of the corrected velocity* (divergence free). Theorem 5.3 is the theoretical foundation of the CPS as it justify this original coupling. Accordingly, in the CPS, the coupled problem is simpler and the loop contains less calculations, compared to its BPS equivalent. Thus any algorithm based on CPS (different algorithm can be constructed depending on the treatment of the non linear terms) will obviously perform better (i.e. computing time and efficiency, see Table 6.1) than its BPS counterpart. More generally, for any first order implicit or semi-implicit finite element procedure, the amount of calculations involved in the iterative loop is predominant (consider the monolithic first order implicit algorithm for example). Consequently, it is reasonable to presume that the CPS compare advantageously to any equivalent method. Finally, although the theoretical results rely on the variational form (i.e. the finite element formulation), it should be possible to expand those results to a finite volume version of the CPS.

The outline of the paper is as follows:

1. Section 2 presents the problem setting.

2. In § 3, we present the time discretization of the model and we prove in Theorem 3.2 the existence of the solution of the semi-discrete variational formulation in suitable Sobolev spaces.

3. In § 4, we introduce the iterative algorithm based on the fixed point method at each time step in order to solve the coupled problem efficiently. Theorem 4.3 establishes the convergence of the scheme under suitable assumptions while Corollary 4.6 proves the uniqueness of the weak solution.

4. In § 5, we propose the new scheme CPS and a reference scheme BPS and we prove that they both have the same accuracy in time and we establish the quality of the predicted temperature (see Theorem 5.3).

5. Finally, we present some numerical tests in § 6. These preliminary tests were realized with FreeFEM++ [27] and are in agreement with our theoretical results.

2. Problem setting. Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 2$ or 3 which is either convex or of class $C^{1,1}$. Let $\partial\Omega = \Gamma_D \cup \Gamma_N$ the boundary of Ω and Ω_t the open set $\Omega \times (0, T_f)$, where $T_f > 0$ is the final time. We are aiming at solving the Boussinesq equations for the fluid velocity $\mathbf{u}$, the pressure p and the temperature T, which lead to the following system

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nabla\cdot(\nu(T)\nabla\mathbf{u}) + \nabla p = \mathbf{F}(T) & \text{in } \Omega_t, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega_t, \\ \dfrac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)\,T - \nabla\cdot(\lambda(T)\nabla T) = H & \text{in } \Omega_t, \end{cases} \tag{2.1}$$

where the function H represents an external heat source and depends only on the position vector $\mathbf{x} \in \mathbb{R}^d$. $\mathbf{F}$, which represents external volumic forces (such as gravity), depends on temperature T.

Remark 2.1. In the Boussinesq approximation, all physical parameters are assumed to be constant (see [28, 34]) and $\mathbf{F}$ is proportional to the variation of the density and therefore to the variation of temperature ($\mathbf{F} \propto (T - T_0)$). Nevertheless, in this work, those assumptions are not essential and we will allow for a temperature dependance of the viscosity ν and consider more general hypothesis on $\mathbf{F}$.

Hypothesis 2.2. $\mathbf{F} : \mathbb{R} \to \mathbb{R}^d$ is a $C^1(\mathbb{R})$ function. There exists a real T_0 such that $\mathbf{F}(T_0) = 0$, and a non-negative real $\alpha > 0$ such that

$$\|\mathbf{F}'\|_\infty \leq \alpha. \tag{2.2}$$

For the rest of our presentation, the system in the form (2.1) is not well suited. Let us introduce $\theta = T - T_0$ and the function $\mathbf{f}(\theta) = \dfrac{1}{\alpha}\mathbf{F}(T)$. Then, from (2.2), we can write

$$\mathbf{f}(0) = 0, \qquad \|\mathbf{f}'\|_\infty \leq 1 \quad \text{and} \quad \forall s \in \mathbb{R},\ |\mathbf{f}(s)| \leq s. \tag{2.3}$$

Now we can rewrite (2.1) with θ and $\mathbf{f}$

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nabla\cdot(\nu(\theta)\nabla\mathbf{u}) + \nabla p = \alpha\,\mathbf{f}(\theta) & \text{in } \Omega_t, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega_t, \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\,\theta - \nabla\cdot(\lambda(\theta)\nabla\theta) = H & \text{in } \Omega_t. \end{cases} \tag{2.4}$$

System (2.4) is completed with the following initial data:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \in L^2(\Omega)^d \text{ with } \nabla\cdot\mathbf{u}_0 = 0 \qquad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}) \in L^2(\Omega), \tag{2.5}$$

and boundary conditions

$$\mathbf{u} = 0 \ \text{ on } \partial\Omega, \quad \theta = \theta_D \ \text{ on } \Gamma_D, \quad \lambda(\theta)\frac{\partial\theta}{\partial\mathbf{n}} = \theta_N \quad \text{on } \Gamma_N. \tag{2.6}$$

For the sake of simplicity, we consider homogeneous Dirichlet boundary conditions for $\mathbf{u}$ and both Γ_D and Γ_N of positive measure but the general case follows the same lines.

Hypothesis 2.3. We assume that ν and λ are bounded functions of $W^{1,\infty}(\Omega)$, with

$$\begin{cases} 0 < \nu_0 \leq \nu(r) \leq \nu_1, \quad \|\nu'\|_\infty = \nu_2 \\ 0 < \lambda_0 \leq \lambda(r) \leq \lambda_1, \quad \|\lambda'\|_\infty = \lambda_2. \end{cases} \qquad \forall r \in \mathbb{R} \tag{2.7}$$

3. Analysis of the weak formulation. This section is devoted to the variational formulation of the time discretization associated to system (2.4). We introduce some spaces definitions, supplementary assumptions and apply a time discretization to the initial problem. Finally following the works of Bernardi *et al.* [6], we prove in Theorem 3.2 the existence of a solution of the variational formulation in suitable spaces.

3.1. Time discretization. An implicit time discretization with time step Δt of the coupled system (2.4) result in a sequence of (generalized) Oseen problems of the form

$$\left\{\begin{array}{rcll} \gamma_u \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nabla\cdot(\nu(\theta)\nabla\mathbf{u}) + \nabla p &=& \alpha\,\mathbf{f}(\theta) + \gamma_u \mathbf{r}_u & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &=& 0 & \text{in } \Omega, \\ \gamma_\theta\theta + (\mathbf{u}\cdot\nabla)\,\theta - \nabla\cdot(\lambda(\theta)\nabla\theta) &=& h = H + \gamma_\theta r_\theta & \text{in } \Omega, \end{array}\right. \tag{3.1}$$

completed with the same initial and boundary conditions (2.5)-(2.6). In this system γ_u, γ_θ, $\mathbf{r}_u$ and r_θ are related to the approximation of the time derivatives. Several strategies are available to solve (3.1). The choice of a scheme for both time derivatives as well as the treatment of the two non linear terms $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and $\nabla\cdot(\lambda(\theta)\nabla\theta)$ will lead to different resolution schemes.

For the time derivatives, we will use the same discretization for both the Navier-Stokes and convection-diffusion equations. This gives $\gamma_u = \gamma_\theta = \gamma$ and $\mathbf{r}_u, r_\theta$ are two *known* quantities depending on the solution $(\mathbf{u}, p, \theta)$ at previous times. By applying the same derivation rule for both equations, the usual setting for the resolution of unsteady differential problem is relevant, therefore the usual convergence and stability results are valid (see [37] for instance). Since the proposed scheme for (3.1) is based on a projection method, the theoretical results for this method for the Navier-Stokes problem ([40, 42, 26]) should be mentioned. In particular a backward Euler method will lead to a first order scheme (in L^2-norm for $\mathbf{u}$) and a second order scheme (BDF2) to a second order scheme (in L^2-norm for $\mathbf{u}$). We refer the reader to [23] for a detailed review of the various form of the projection method and its error estimates. It should be noted that all projection schemes (as splitting schemes) have an inherent splitting error of order two in H^1-norm (see [40, 42, 26]). Therefore the proposed algorithm, relying on a projection scheme, is at best of second order in H^1-norm.

For the non linear terms, they can be treated the usual way: implicitly (leading to a fixed-point algorithm), semi-explicitly (for example by using a Richardson extrapolation for both terms) or totally explicitly (leading to a coupled linear system). Even if the explicit treatment of the non linear terms can be seen as an easy technique to reduce numerical costs, we chose to postpone the study of those approaches as they lead inevitably to conditional stability and possibly severe conditions on the time step (for example see [30, 44]). We must emphasizes that for most of these strategies (implicit, semi-explicit and explicit), the system (3.1) will still be a coupled system. The velocity of the fluid $\mathbf{u}$ depends on the temperature θ through the viscosity and the right member, for the temperature, we have a convective term depending on the fluid velocity. Therefore an iterative scheme must be introduced to solve (3.1) at each time step.

The totally implicit approach has been retained since in all cases considered, a fixed-point will be needed (to deal with the coupling of the unknowns). Furthermore, the theoretical results presented in both sections are easily expanded to semi-explicit (and even explicit) approaches (Corollaries 3.3 and 4.5).

3.2. Continuous variational formulation. In what follows, we consider the usual Sobolev spaces: $H^m(\Omega)$, with norm $\|\cdot\|_{m,\Omega}$; $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ its dual space. The duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ is denoted by $<\cdot,\cdot>_\Omega$. The scalar product in $L^2(\Omega)$ is denoted by $(\,\cdot\,)$ and its norm by $\|\cdot\|$. We define $\mathbf{V}$, $L_0^2(\Omega)$ and $\mathcal{T}$ as

$$
\begin{array}{rcl}
\mathbf{V} & = & \left\{\mathbf{v} \in H_0^1(\Omega)^d;\ \nabla\cdot\mathbf{v} = 0 \text{ on } \Omega\right\} \\
L_0^2(\Omega) & = & \left\{q \in L^2(\Omega);\ \displaystyle\int_\Omega q(\mathbf{x})\,d\mathbf{x} = 0\right\} \\
\mathcal{T} & = & \left\{\phi \in H^1(\Omega);\ \phi = 0 \text{ on } \Gamma_D\right\}.
\end{array}
\tag{3.2}
$$

The space $\mathcal{T}$ can be provided with the $H_0^1(\Omega)$-norm $|\phi|_{1,\Omega} = \|\nabla\phi\|$ (based on the Poincaré-Friedrichs inequality). Its dual will be noted by $\mathcal{T}^*$ and the duality product still denoted by $<\cdot,\cdot>_\Omega$. Moreover, we denote by $<\cdot,\cdot>_{\Gamma_N}$ the duality product between the space of function traces $H^{\frac{1}{2}}(\Gamma_N)$ and its dual $H^{-\frac{1}{2}}(\Gamma_N)$.

Since we consider h, θ_D and θ_N belonging to $\mathcal{T}^*$, $H^{\frac{1}{2}}(\Gamma_D)$ and $H^{-\frac{1}{2}}(\Gamma_N)$ respectively, we can introduce c_0 the following constant

$$
c_0 = \sup_{\phi\in\mathcal{T}} \frac{< H, \phi >_\Omega + < \theta_N, \phi >_{\Gamma_N}}{\|\phi\|_{1,\Omega}},
\tag{3.3}
$$

and from Babuška [2] and Brezzi [9], there exists a positive constant β such that

$$
\inf_{q\in L_0^2(\Omega)} \sup_{\mathbf{v}\in H_0^1(\Omega)^d} \frac{-\displaystyle\int_\Omega (\nabla\cdot\mathbf{v})\,q}{\|q\|\,|\mathbf{v}|_{1,\Omega}} \geq \beta.
\tag{3.4}
$$

The variational formulation of the system (3.1) can be written as:
Find $(\mathbf{u}, p, \theta) \in H_0^1(\Omega)^d \times L_0^2(\Omega) \times \mathcal{T}$ such that for all $(\mathbf{v}, p, \theta) \in H_0^1(\Omega)^d \times L_0^2(\Omega) \times \mathcal{T}$

$$
\left\{
\begin{array}{l}
\gamma \displaystyle\int_\Omega \mathbf{u}\cdot\mathbf{v} + \int_\Omega \nu(\theta)\nabla\mathbf{u} : \nabla\mathbf{v} + \int_\Omega (\mathbf{u}\cdot\nabla)\,\mathbf{u}\cdot\mathbf{v} - \int_\Omega (\nabla\cdot\mathbf{v})\,p \\
\hfill - \displaystyle\int_\Omega (\alpha\mathbf{f}(\theta) + \gamma\,\mathbf{r}_u)\cdot\mathbf{v} = 0 \\
\displaystyle\int_\Omega (\nabla\cdot\mathbf{u})\,q \hfill = 0 \\
\gamma \displaystyle\int_\Omega \theta\phi + \int_\Omega \lambda(\theta)\,\nabla\theta\cdot\nabla\phi + \int_\Omega (\mathbf{u}\cdot\nabla)\,\theta\,\phi - < h, \phi >_\Omega - < \theta_N, \phi >_{\Gamma_N} = 0
\end{array}
\right.
\tag{3.5}
$$

and based on (3.4) and the definition of $\mathbf{V}$ (see [18] for instance), system (3.5) is equivalent to the problem: Find $(\mathbf{u}, \theta) \in \mathbf{V}\times\mathcal{T}$ such that $\forall\,(\mathbf{v}, \phi) \in \mathbf{V}\times\mathcal{T}$:

$$
\left\{
\begin{array}{l}
\gamma \displaystyle\int_\Omega \mathbf{u}\cdot\mathbf{v} + \int_\Omega \nu(\theta)\nabla\mathbf{u} : \nabla\mathbf{v} + \int_\Omega (\mathbf{u}\cdot\nabla)\,\mathbf{u}\cdot\mathbf{v} - \int_\Omega (\alpha\mathbf{f}(\theta) + \gamma\,\mathbf{r}_u)\cdot\mathbf{v} = 0 \\
\gamma \displaystyle\int_\Omega \theta\phi + \int_\Omega \lambda(\theta)\,\nabla\theta\cdot\nabla\phi + \int_\Omega (\mathbf{u}\cdot\nabla)\,\theta\,\phi - < h, \phi >_\Omega - < \theta_N, \phi >_{\Gamma_N} = 0
\end{array}
\right.
\tag{3.6}
$$

3.3. Existence of a solution. In this section we establish the existence of a solution (Theorem 3.2) using Brouwer's fixed-point theorem.

For all $\mathbf{u} \in \mathbf{V}$, $\mathbf{v} \in H^1(\Omega)^d$ and $\eta, \psi \in H^1(\Omega)$, we have (see [18])

$$\int_\Omega (\mathbf{u} \cdot \nabla)\, \mathbf{v} \cdot \mathbf{v} = 0, \qquad \int_\Omega (\mathbf{u} \cdot \nabla)\, \psi\, \psi = 0, \qquad \int_\Omega (\mathbf{u} \cdot \nabla)\, \eta\, \psi = -\int_\Omega (\mathbf{u} \cdot \nabla)\, \psi\, \eta \tag{3.7}$$

Let us denote

$$\gamma_\nu = \min(\gamma, \nu_0) \qquad \gamma_\lambda = \min(\gamma, \lambda_0) \tag{3.8}$$

Lemma 3.1. *Assuming Hypothesis 2.2 and 2.3 hold and* $(\mathbf{u}, \theta)$ *is a solution of* (3.6) *. There is two constant* c_θ *and* c_u*, depending on the datum only, such that*

$$\|\theta\|_{1,\Omega} \le c_\theta/\gamma_\lambda, \qquad \|\mathbf{u}\|_{1,\Omega} \le \frac{\alpha\, c_u}{\gamma_\nu\, \gamma_\lambda} \tag{3.9}$$

Proof. The technique used for the proof can be found in [17] where it is used for the backward Euler scheme applied to the Navier-Stokes equation. The proof is made of two similar and relatively basic steps and is essentially based on the algebraic identity

$$(a-b)a = \frac{1}{2}(a^2 - b^2 + (a-b)^2) \qquad \forall a, b \in \mathbb{R}$$

and that in the approximation of the derivatives, written as $\gamma(\mathbf{u} - \mathbf{r}_u)$ and $\gamma(\theta - r_\theta)$, $\mathbf{r}_u$ and r_θ are linear combination of the velocity field (respectively temperature) at multiple preceding times.

Using $\phi = \theta$ in the second equation of (3.6) and the fact that the velocity is divergence free, we get a first bound, in L^2, for θ; from which we get the bound in H^1. As for the bound on $\mathbf{u}$, the result is obtained using $\mathbf{v} = \mathbf{u}$ in the first equation of (3.6), the hypothesis on $\mathbf{f}$ and the bound on θ. Once again, we get a first bound, in L^2, from which we get the bound in H^1. $\square$

Theorem 3.2 (Existence). *Assuming Hypothesis 2.2 and 2.3 holds, for any data* $h \in \mathcal{T}^*$ *and* $(\theta_D, \theta_N) \in H^{\frac{1}{2}}(\Gamma_D) \times H^{-\frac{1}{2}}(\Gamma_N)$*, problem* (3.6) *admits at least a solution* $(\mathbf{u}, \theta) \in H_0^1(\Omega)^d \times H^1(\Omega)$*. Moreover, this solution satisfies the following estimate:*

$$\|\mathbf{u}\|_{1,\Omega} + \|\theta\|_{1,\Omega} \le c\left(\|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)} + \kappa\right), \tag{3.10}$$

where constants $c > 0$ *and* $\kappa > 0$ *depend only on the datum.*

Proof. The existence of a solution is established using a fixed-point theorem and a *topological degree of mapping* technique (see for instance Rabinowitz [38] and Rabinowitz *et al.* [39]). A similar idea was used by Bernardi *et al.* in [4].

Lemma 2.8 in [6] implies that for all $\varepsilon > 0$, there exists a lifting $\mathcal{R}_\theta \in H^1(\Omega)$ of the value of θ on Γ_D satisfying:

$$\|\mathcal{R}_\theta\|_{L^4(\Omega)} \le \varepsilon \|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)} \quad \text{and} \quad \|\mathcal{R}_\theta\|_{H^1(\Omega)} \le c_{\mathcal{R}_\theta} \|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)}, \tag{3.11}$$

where the constant $c_{\mathcal{R}_\theta}$ depends only on the lifting $\mathcal{R}_\theta$.

Since $\mathbf{V}$ is separable (it is a closed subspace of $H^1(\Omega)^d$ which is separable), there exists increasing sequence of finite-dimensional Hilbert subspaces $\mathbf{V}_m$ of $\mathbf{V}$. Also, there exists increasing sequence of finite-dimensional subspaces $\mathbb{T}_m$ of $H_0^1(\Omega)$, such that $\mathbf{V} \times H_0^1(\Omega) = \cup_{m\geq 0} \mathbf{V}_m \times \mathbb{T}_m$. We define a mapping Φ_m from $\mathbf{V}_m \times \mathbb{T}_m$ into itself by

$$
\begin{aligned}
\langle \Phi_m(\mathbf{u},\theta),(\mathbf{v},\phi)\rangle =& \gamma \int_\Omega \mathbf{u}\cdot\mathbf{v} + \int_\Omega \nu\,(\theta+\mathcal{R}_\theta)\,\nabla\mathbf{u} : \nabla\mathbf{v} + \int_\Omega (\mathbf{u}\cdot\nabla)\,\mathbf{u}\cdot\mathbf{v} \\
& + \gamma \int_\Omega (\theta+\mathcal{R}_\theta)\,\phi + \int_\Omega \lambda\,(\theta+\mathcal{R}_\theta)\;\nabla\,(\theta+\mathcal{R}_\theta)\cdot\nabla\phi \\
& + \int_\Omega (\mathbf{u}\cdot\nabla)\,(\theta+\mathcal{R}_\theta)\;\phi - \alpha\int_\Omega \mathbf{f}\,(\theta+\mathcal{R}_\theta)\cdot\mathbf{v} \\
& - \gamma\int_\Omega \mathbf{r}_u\cdot\mathbf{v} - \,<h,\phi>_\Omega - <\theta_N,\phi>_{\Gamma_N} \\
& \qquad \forall(\mathbf{u},\theta)\in\mathbf{V}_m\times\mathbb{T}_m,\ \forall(\mathbf{v},\phi)\in\mathbf{V}_m\times\mathbb{T}_m
\end{aligned}
\tag{3.12}
$$

The mapping Φ_m is well-defined and continuous on $\mathbf{V}_m \times \mathbb{T}_m$ since the functions ν and λ are bounded; the embedding of $H^1(\Omega)$ into $L^6(\Omega)$ is continuous and the trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\Gamma_N)$ is also continuous.

In order to use Brouwer's fixed point theorem, we replace $(\mathbf{v},\phi)$ by $(\mathbf{u},\theta)$ in (3.12). Combining (2.7) and (3.7)

$$
\begin{aligned}
\langle \Phi_m(\mathbf{u},\theta),(\mathbf{u},\theta)\rangle \geq & \gamma\|\mathbf{u}\|^2 + \nu_0\|\nabla\mathbf{u}\|^2 + \gamma\|\theta\|^2 + \lambda_0\|\nabla\theta\|^2 \\
& + \gamma\int_\Omega \mathcal{R}_\theta\theta + \int_\Omega \lambda\,(\theta+\mathcal{R}_\theta)\,\nabla\mathcal{R}_\theta\cdot\nabla\theta - \int_\Omega (\mathbf{u}\cdot\nabla)\,\theta\,\mathcal{R}_\theta \\
& - \alpha\int_\Omega \mathbf{f}\,(\theta+\mathcal{R}_\theta)\cdot\mathbf{u} - \gamma\int_\Omega \mathbf{r}_u\cdot\mathbf{v} - (<h,\phi>_\Omega + <\theta_N,\phi>_{\Gamma_N})
\end{aligned}
$$

Using the Cauchy-Schwarz and Hölder inequalities, relations (2.7), (3.3) and (2.3)

$$
\begin{aligned}
\langle \Phi_m(\mathbf{u},\theta),(\mathbf{u},\theta)\rangle \geq & \gamma\|\mathbf{u}\|^2 + \nu_0\|\nabla\mathbf{u}\|^2 + \gamma\|\theta\|^2 + \lambda_0\|\nabla\theta\|^2 \\
& - \gamma\|\mathcal{R}_\theta\|\,\|\theta\| + \lambda_1\|\nabla\mathcal{R}_\theta\|\,\|\nabla\theta\| - \|\mathbf{u}\|_{L^4(\Omega)^d}\,\|\nabla\theta\|\,\|\mathcal{R}_\theta\|_{L^4(\Omega)} \\
& - \alpha\|\theta\|\,\|\mathbf{u}\| - \alpha\|\mathcal{R}_\theta\|\,\|\mathbf{u}\| - \gamma\|\mathbf{r}_u\|\,\|\mathbf{u}\| - c_0\,\|\theta\|_{1,\Omega}.
\end{aligned}
$$

Let us denote by $|\Omega|$ the volume of domain Ω, using (3.11)

$$
\begin{aligned}
\langle \Phi_m(\mathbf{u},\theta),(\mathbf{u},\theta)\rangle \geq & \ \gamma\|\mathbf{u}\|^2 + \nu_0\|\nabla\mathbf{u}\|^2 + \gamma\|\theta\|^2 + \lambda_0\|\nabla\theta\|^2 \\
& - \varepsilon\,\gamma\sqrt{|\Omega|}\,\|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)}\left(\|\theta\| + \frac{\alpha}{\gamma}\,\|\mathbf{u}\|\right) - \lambda_1\,c_{\mathcal{R}_\theta}\,\|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)}\,\|\nabla\theta\| \\
& - \varepsilon\,\|\mathbf{u}\|_{L^4(\Omega)^d}\,\|\nabla\theta\|\,\|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)} \\
& - \alpha\|\theta\|\,\|\mathbf{u}\| - \gamma\,\|\mathbf{r}_u\|\,\|\mathbf{u}\| - c_0\,\|\theta\|_{1,\Omega}.
\end{aligned}
$$

To simplify the notation we introduce

$$
m_0 = min\,(\gamma_\nu,\gamma_\lambda)\,, \quad m_1 = c_0 + \gamma\,\|\mathbf{r}_u\| + \frac{c_\theta\,\alpha}{\gamma_\lambda}, \quad m_2 = max\left(1,\frac{\alpha}{\gamma}\right)\gamma\sqrt{|\Omega|},
$$

Using Lemma 3.1 and the embedding from $H^1(\Omega)^d$ into $L^4(\Omega)^d$, there exists a positive constant c', such that

$$\begin{aligned}\langle \Phi_m(\mathbf{u},\theta),(\mathbf{u},\theta)\rangle \geq\, & m_0 \left(\|\mathbf{u}\|^2_{1,\Omega} + \|\theta\|^2_{1,\Omega}\right) \\ & - \left(\varepsilon m_2 + \lambda_1\, c_{\mathcal{R}_\theta}\right) \|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)} \left(\|\mathbf{u}\|_{1,\Omega} + \|\theta\|_{1,\Omega}\right) \\ & - \frac{\varepsilon\, c'}{2} \|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)} \left(\|\mathbf{u}\|^2_{1,\Omega} + \|\theta\|^2_{1,\Omega}\right) \\ & - m_1 \left(\|\mathbf{u}\|_{1,\Omega} + \|\theta\|_{1,\Omega}\right).\end{aligned}$$

Choosing ε such that

$$\varepsilon\, c' \|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)} \leq m_0$$

we get

$$\begin{aligned}\langle \Phi_m(\mathbf{u},\theta),(\mathbf{u},\theta)\rangle \geq\, & \frac{m_0}{2} \left(\|\mathbf{u}\|^2_{1,\Omega} + \|\theta\|^2_{1,\Omega}\right) \\ & - \left(\lambda_1\, c_{\mathcal{R}_\theta} \|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)} + m_1 + \varepsilon m_2\right) \left(\|\mathbf{u}\|_{1,\Omega} + \|\theta\|_{1,\Omega}\right).\end{aligned}$$

Using the inequality $(\beta+\eta) \leq \sqrt{2}\left(\beta^2+\eta^2\right)^{1/2}, \forall \beta,\eta \in \mathbb{R}$, we deduce that

$$\begin{aligned}\langle \Phi_m(\mathbf{u},\theta),(\mathbf{u},\theta)\rangle \geq\, & \frac{m_0}{2} \left(\|\mathbf{u}\|^2_{1,\Omega} + \|\theta\|^2_{1,\Omega}\right) \\ & - \sqrt{2}\left(\lambda_1\, c_{\mathcal{R}_\theta} \|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)} + m_1 + \varepsilon m_2\right) \left(\|\mathbf{u}\|^2_{1,\Omega} + \|\theta\|^2_{1,\Omega}\right)^{\frac{1}{2}}.\end{aligned}$$

So, the right-hand side is non-negative on the sphere of radius r defined by:

$$r = \frac{2\sqrt{2}}{m_0} \left(\lambda_1\, c_{\mathcal{R}_\theta} \|\theta_D\|_{H^{\frac{1}{2}}(\Gamma_D)} + m_1 + \varepsilon m_2\right). \tag{3.13}$$

Consequently, applying Brouwer's fixed point theorem (Girault-Raviart [18]) we get the existence of a solution $(\mathbf{u}^m,\theta^m)$ of

$$\Phi_m(\mathbf{u}^m,\theta^m) = 0, \tag{3.14}$$

which satisfies,

$$\left(\|\mathbf{u}^m\|^2_{1,\Omega} + \|\theta^m\|^2_{1,\Omega}\right)^{\frac{1}{2}} \leq r. \tag{3.15}$$

Since the sequence $(\mathbf{u}^m,\theta^m)_m$ is bounded, there exists a subsequence, still denoted by $(\mathbf{u}^m,\theta^m)_m$ for simplicity, which converges to $(\mathbf{u}^*,\theta^*)$ weakly in $H^1(\Omega)^d \times H^1(\Omega)$. From Rellich-Kondrachov's theorem in dimension $d=3$, the continuous injection from $H^1(\Omega)$ into $L^q(\Omega)$, $q \in [1,6]$ is compact (see for instance Brézis [8]), therefore the mapping $(\mathbf{u}^m,\theta^m)_m$ converges to $(\mathbf{u}^*,\theta^*)$ in $L^q(\Omega)^d \times L^q(\Omega)$ strong.

In order to finish the proof of existence, we must verify that $(\mathbf{u}^*,\theta^*)$ satisfies system (3.6). Relation (3.14) implies that for all $(\mathbf{v},\phi) \in \mathbf{V}_m \times \mathbb{T}_m$

$$\begin{aligned}0 = \,& \gamma \int_\Omega \mathbf{u}^m \cdot \mathbf{v} + \int_\Omega (\theta^m + \mathcal{R}_\theta)\,\phi + \int_\Omega \nu\,(\theta^m + \mathcal{R}_\theta)\,\nabla \mathbf{u}^m : \nabla \mathbf{v} \\ & + \int_\Omega \lambda\,(\theta^m + \mathcal{R}_\theta)\,\nabla(\theta^m + \mathcal{R}_\theta)\cdot\nabla\phi + \int_\Omega (\mathbf{u}^m\cdot\nabla)\,\mathbf{u}^m\cdot\mathbf{v} \\ & + \int_\Omega (\mathbf{u}^m\cdot\nabla)\,(\theta^m + \mathcal{R}_\theta)\,\phi - \alpha \int_\Omega \mathbf{f}\,(\theta^m + \mathcal{R}_\theta)\cdot\mathbf{v} \\ & - \gamma \int_\Omega \mathbf{r}_u\cdot\mathbf{v} - <h,\phi>_\Omega - <\theta_N,\phi>_{\Gamma_N}.\end{aligned} \tag{3.16}$$

From the strong convergence of $(\mathbf{u}^m,\theta^m)_m$ to $(\mathbf{u}^*,\theta^*)$ in $L^2(\Omega)^d\times L^2(\Omega)$ and in $L^4(\Omega)^d\times L^4(\Omega)$, we can easily show the convergence of all the terms on the right hand side except for the third and fourth. We will work on the third term of the right hand side. Since both term have the same form, the same argument is used for the fourth one.

The sequence $(\theta^m)_m$ converges to θ^* strongly in $L^2(\Omega)$ and $\nu(\cdot)$ is continuous and bounded. Therefore, for all $\mathbf{v}\in\mathbf{V}_m$

$$\lim_{m\to\infty}\nu\left(\theta^m+\mathcal{R}_\theta\right)\,\nabla\mathbf{v}=\nu\left(\theta^*+\mathcal{R}_\theta\right)\,\nabla\mathbf{v},\qquad\text{a.e in }\Omega.$$

On the other hand

$$\|\nu\left(\theta^m+\mathcal{R}_\theta\right)\,\nabla\mathbf{v}\|\le\nu_1\|\nabla\mathbf{v}\|,$$

Writing the difference for the third term, we get the inequality

$$\begin{aligned}&\int_\Omega\left(\nu\left(\theta^m+\mathcal{R}_\theta\right)\nabla\mathbf{u}^m-\nu\left(\theta^*+\mathcal{R}_\theta\right)\nabla\mathbf{u}^*\right):\nabla\mathbf{v}\\&\quad=\int_\Omega\nu\left(\theta^m+\mathcal{R}_\theta\right)\nabla\left(\mathbf{u}^m-\mathbf{u}^*\right):\nabla\mathbf{v}+\int_\Omega\left(\nu\left(\theta^m+\mathcal{R}_\theta\right)-\nu\left(\theta^*+\mathcal{R}_\theta\right)\right)\nabla\mathbf{u}^*:\nabla\mathbf{v}\\&\quad\le\nu_1\int_\Omega\left|\nabla\left(\mathbf{u}^m-\mathbf{u}^*\right):\nabla\mathbf{v}\right|+\int_\Omega\left(\nu\left(\theta^m+\mathcal{R}_\theta\right)-\,\nu\left(\theta^*+\mathcal{R}_\theta\right)\right)\nabla\mathbf{u}^*:\nabla\mathbf{v}.\end{aligned}$$

From the weak convergence of $\nabla\mathbf{u}^m$ to $\nabla\mathbf{u}^*$ in $L^2(\Omega)^{d\times d}$, the first integral goes to 0 when m goes to ∞. Moreover, using the Lebesgue dominated convergence (see for instance Brézis [8]), we deduce the convergence of the second quantity to 0. Then

$$\lim_{m\to\infty}\int_\Omega\nu\left(\theta^m+\mathcal{R}_\theta\right)\nabla\left(\theta^m+\mathcal{R}_\theta\right)\cdot\nabla\phi\;=\;\int_\Omega\nu\left(\theta^*+\mathcal{R}_\theta\right)\nabla\left(\theta^*+\mathcal{R}_\theta\right)\cdot\nabla\phi.$$

From this, we have shown that $(\mathbf{u}^*,\theta^*+\mathcal{R}_\theta)$ satisfies (3.16), therefore we have $(\mathbf{u}^*,\theta^*)$ solution of (3.6) and consequently solution to (3.5). ∎

Finally, if we consider semi-implicit or explicit treatment of the nonlinear term in (3.1), (3.5) or (3.6), the last proof would see little changes since Lemma 3.1, (2.7) and (3.7) would suffice to control the related terms, and we have

Corollary 3.3. *Assuming the same hypothesis as Theorem 3.2, for any linearization of the advection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and diffusion term $\lambda(\theta)\nabla\theta$, the existence of a solution still holds.*

4. Analysis of an iterative scheme. In order to approximate the solution of problem (3.5), we propose an iterative procedure based on a decoupled computation of the fluid and convection-diffusion equations. Note that, in this section we focus only on the study the convergence of the iterative procedure for **one time step** of the unsteady coupled problem (2.4).

1. Initialization: Given $(\mathbf{u}^0,p^0,\theta^0)\in H^1_0(\Omega)^d\times L^2_0(\Omega)\times\mathcal{T}$.

2. Until convergence, compute:

First step: $\left(\mathbf{u}^{k+1},p^{k+1}\right)\in H^1_0(\Omega)^d\times L^2_0(\Omega)$ solution of

$$(4.1)\quad\left\{\begin{aligned}&\gamma\int_\Omega\mathbf{u}^{k+1}\cdot\mathbf{v}+\int_\Omega\nu(\theta^k)\,\nabla\mathbf{u}^{k+1}:\nabla\mathbf{v}+\int_\Omega\left(\mathbf{u}^k\cdot\nabla\right)\mathbf{u}^{k+1}\cdot\mathbf{v}\\&\qquad\qquad-\int_\Omega\nabla\cdot\mathbf{v}\,p^{k+1}=\alpha\int_\Omega\mathbf{f}(\theta^k)\cdot\mathbf{v}\,+\,\gamma\int_\Omega\mathbf{r}_u\cdot\mathbf{v}\\&\int_\Omega\left(\nabla\cdot\mathbf{u}^{k+1}\right)q\qquad\qquad=0.\end{aligned}\right.$$

Second step: Knowing $\mathbf{u}^{k+1}$, compute $\theta^{k+1} \in \mathcal{T}$ solution of

$$
\begin{aligned}
\gamma \int_\Omega \theta^{k+1}\phi + \int_\Omega \lambda(\theta^k)\,\nabla\theta^{k+1}\cdot\nabla\phi + \int_\Omega \left(\mathbf{u}^{k+1}\cdot\nabla\right)\theta^{k+1}\,\phi \\
=< h,\phi >_\Omega + < \theta_N,\phi >_{\Gamma_N}\,.
\end{aligned}
\tag{4.2}
$$

Some additional regularity is needed to prove the convergence of (4.1)-(4.2):

Hypothesis 4.1. The sequence $\left(\mathbf{u}^k,\theta^k\right)_k$ belong to $W^{1,3}(\Omega)^d \times W^{1,3}(\Omega)$ and is uniformly bounded *i.e* there exists positive constant $M > 0$ such that:

$$
\|\mathbf{u}^k\|_{W^{1,3}(\Omega)^d} + \|\theta^k\|_{W^{1,3}(\Omega)} \le M.
$$

Remark 4.2. This hypothesis is totally related to ν and λ, the diffusion terms, if treated explicitly, Hypothesis 4.1 can be dropped.

Theorem 4.3 (Convergence). *Assuming Hypothesis 2.2, 2.3 and 4.1 hold. If $h \in \mathcal{T}^*$ and $(\theta_D,\theta_N) \in H^{\frac{1}{2}}(\Gamma_D)\times H^{-1/2}(\Gamma_N)$, then, there exists a positive constant C which depends only on Ω, such that $\forall\, m \le k \in \mathbb{N}$, the following estimate holds:*

$$
\begin{aligned}
&\|\theta^{k+1}-\theta^{m+1}\|_{1,\Omega} + \|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|_{1,\Omega} \\
&\qquad\le \max(\kappa_1,\kappa_2)^m\left(\|\theta^{k-m}-\theta^0\|_{1,\Omega} + \|\mathbf{u}^{k-m}-\mathbf{u}^0\|_{1,\Omega}\right),
\end{aligned}
\tag{4.3}
$$

where

$$
\kappa_1 = CM\left(\frac{\lambda_2}{\gamma_\lambda} + \frac{\nu_2+\alpha/M}{\gamma_\nu}\left(1+\frac{C^2 c_\theta}{\gamma_\lambda^2}\right)\right) \qquad \kappa_2 = \frac{C^2\alpha c_u}{\gamma_\nu^2\gamma_\lambda}\left(1+\frac{C^2 c_\theta}{\gamma_\lambda^2}\right)
\tag{4.4}
$$

Moreover, if $\max\limits_{i=1,2}\kappa_i < 1$, then the sequence $\left((\mathbf{u}^k,p^k,\theta^k)\right)_{k\in\mathbb{N}}$ converges strongly in $H^1(\Omega)^d\times L^2(\Omega)\times H^1(\Omega)$, and its limit is a solution of system (3.5).

Remark 4.4. From the definition of γ_ν, γ_λ, κ_1, κ_2 and since C depends only on Ω, assuming that κ_i are less than one is in fact a smallness hypothesis on the physical parameters ν_0, ν_2, λ_0 and λ_2.

Proof. The proof of this Theorem is made in four distinct steps and is inspired from Chacon *et al.* in [12] and Yakoubi in [50].

Step 1: Analysis of the velocity sequence

We take two non-negative integers k and m such that $k \ge m$, and the test function $\mathbf{v}$ equal to $\mathbf{u}^{k+1}-\mathbf{u}^{m+1}$ in (4.1). Computing the difference between the first equation of (4.1) taken at iteration $m+1$ and $k+1$ we get

$$
\begin{aligned}
\gamma\,\|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|^2 &+ \int_\Omega \left(\nu(\theta^k)\,\nabla\mathbf{u}^{k+1} - \nu(\theta^m)\,\nabla\mathbf{u}^{m+1}\right):\nabla\left(\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\right) \\
&+ \int_\Omega \left(\left(\mathbf{u}^k\cdot\nabla\right)\mathbf{u}^{k+1} - \left(\mathbf{u}^m\cdot\nabla\right)\mathbf{u}^{m+1}\right)\cdot\left(\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\right) \\
&= \alpha\int_\Omega \left(\mathbf{f}(\theta^k)-\mathbf{f}(\theta^m)\right)\cdot\left(\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\right).
\end{aligned}
\tag{4.5}
$$

Let us introduce the following identities in the previous equality

$$
\begin{aligned}
&\int_\Omega \left(\left(\mathbf{w}^p\cdot\nabla\right)\mathbf{w} - \left(\mathbf{z}^p\cdot\nabla\right)\mathbf{z}\right)\cdot(\mathbf{w}-\mathbf{z}) = \\
&\qquad \int_\Omega \left(\mathbf{w}^p\cdot\nabla\right)(\mathbf{w}-\mathbf{z})\cdot(\mathbf{w}-\mathbf{z}) + \int_\Omega \left(\left(\mathbf{w}^p-\mathbf{z}^p\right)\cdot\nabla\right)\mathbf{z}\cdot(\mathbf{w}-\mathbf{z})
\end{aligned}
\tag{4.6}
$$

$$\begin{aligned}
(4.7)\qquad &\int_\Omega \left(\nu(\theta^k)\,\nabla \mathbf{w}\ -\nu(\theta^m)\,\nabla \mathbf{z}\right) : \nabla\left(\mathbf{w}-\mathbf{z}\right) = \\
&\qquad \int_\Omega \nu(\theta^k)\ |\nabla\left(\mathbf{w}-\mathbf{z}\right)|^2 + \int_\Omega \left(\nu(\theta^k)-\nu(\theta^m)\right)\ \nabla \mathbf{z} : \nabla\left(\mathbf{w}-\mathbf{z}\right)
\end{aligned}$$

By the mean value Theorem, (2.3), (2.7), (3.7) , (4.6) and (4.7) we get

$$\begin{aligned}
\gamma\|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|^2 + \nu_0\|\nabla\left(\mathbf{u}^{k+1}\ -\mathbf{u}^{m+1}\right)\|^2 \le& \\
&\int_\Omega \nu_2\left|\theta^k-\theta^m\right|\ \left|\nabla \mathbf{u}^{m+1} : \nabla\left(\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\right)\right| \\
&+\int_\Omega \left|\left(\left(\mathbf{u}^k-\mathbf{u}^m\right)\ \cdot\nabla\right)\mathbf{u}^{m+1}\cdot\left(\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\right)\right| \\
&+\alpha\int_\Omega \left|\left(\theta^k-\theta^m\right)\cdot\left(\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\right)\right|
\end{aligned}$$

Due to Hypothesis 4.1, the Sobolev embeddings of $H^1(\Omega)$ onto $L^6(\Omega)$, $H^1(\Omega)^d$ onto $L^4(\Omega)^d$ and from Hölder inequality,

$$\begin{aligned}
\gamma\|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|^2 &+\nu_0\|\nabla\left(\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\right)\|^2 \\
&\le \nu_2\|\theta^k-\theta^m\|_{L^6(\Omega)}\|\nabla \mathbf{u}^{m+1}\|_{L^3(\Omega)^{d\times d}}\|\nabla\left(\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\right)\| \\
&+ \|\mathbf{u}^k-\mathbf{u}^m\|_{L^4(\Omega)^d}\|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|_{L^4(\Omega)^d}\|\nabla \mathbf{u}^{m+1}\| \\
&+ \alpha\|\theta^k-\theta^m\|\,\|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|,
\end{aligned}$$

and, there exists a constant C based on the $H^1_0(\Omega)$-norm and depending only on Ω such that

$$\gamma_\nu\|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|_{1,\Omega} \le CM(\nu_2+\alpha/M)\,|\theta^k-\theta^m|_{1,\Omega}\ +\ C^2\,|\mathbf{u}^k-\mathbf{u}^m|_{1,\Omega}\,\|\nabla \mathbf{u}^{m+1}\|.$$

Finally using the estimation (3.9) in Lemma 3.1

$$(4.8)\quad \|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|_{1,\Omega} \le \frac{CM(\nu_2+\alpha/M)}{\gamma_\nu}\|\theta^k-\theta^m\|_{1,\Omega} + C^2\,\frac{\alpha c_u}{\gamma_\nu^2\,\gamma_\lambda}\|\mathbf{u}^k-\mathbf{u}^m\|_{1,\Omega}.$$

Step 2: Analysis of the temperature sequence

The same method is used to estimate $\|\theta^{m+1}-\theta^{k+1}\|$. Choosing test function $\phi=\theta^{k+1}-\theta^{m+1}$ in formula (4.2), we obtain

$$\begin{aligned}
0 =& \gamma\,\|\theta^{k+1}-\theta^{m+1}\|^2 \\
(4.9)\qquad &+\int_\Omega \left(\lambda(\theta^k)\,\nabla\theta^{k+1}\ -\ \lambda(\theta^m)\,\nabla\theta^{m+1}\right)\cdot\nabla\left(\theta^{k+1}-\theta^{m+1}\right) \\
&+\int_\Omega \left(\left(\mathbf{u}^{k+1}\cdot\nabla\right)\theta^{k+1}\ -\ \left(\mathbf{u}^{m+1}\cdot\nabla\right)\theta^{m+1}\right)\ \left(\theta^{k+1}-\theta^{m+1}\right).
\end{aligned}$$

Using identities similar to (4.6), (4.7) and relying on the Sobolev embedding, the following estimates hold:

$$\begin{aligned}
\gamma\,\|\theta^{k+1}-\theta^{m+1}\|^2 + \lambda_0\,\|\nabla(\ \theta^{k+1}-\theta^{m+1})\|^2& \\
&\le \lambda_2\|\theta^k-\theta^m\|_{L^6(\Omega)}\,\|\nabla\theta^{m+1}\|_{L^3(\Omega)}\,\|\nabla(\theta^{k+1}-\theta^{m+1})\| \\
&+\|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|_{L^4(\Omega)^d}\,\|\nabla\theta^{m+1}\|\,\|\theta^{k+1}-\theta^{m+1}\|_{L^4(\Omega)},
\end{aligned}$$

and from (3.9)

$$\|\theta^{k+1}-\theta^{m+1}\|_{1,\Omega} \le C\,M\,\frac{\lambda_2}{\gamma_\lambda}\,\|\theta^k-\theta^m\|_{1,\Omega}+C^2\,\frac{c_\theta}{\gamma_\lambda^2}\|\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\|_{1,\Omega}. \tag{4.10}$$

Combining the previous inequality with (4.8)

$$\begin{aligned}\|\theta^{k+1}-\theta^{m+1}\|_{1,\Omega} &\le CM\left(\frac{\lambda_2}{\gamma_\lambda}+\frac{(\nu_2+\alpha/M)C^2c_\theta}{\gamma_\lambda^2\gamma_\nu}\right)\|\theta^k-\theta^m\|_{1,\Omega}\\ &\quad + C^4\,\frac{\alpha\,c_u c_\theta}{\gamma_\nu^2\,\gamma_\lambda^3}\,\|\mathbf{u}^k-\mathbf{u}^m\|_{1,\Omega}\end{aligned} \tag{4.11}$$

which gives us (4.3). Consequently, under the assumption $\kappa_i<1$, $i=1,2$, the sequence $(\mathbf{u}^k,\theta^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)^d\times H^1(\Omega)$ and it converges to $(\mathbf{u},\theta)$ in $H^1(\Omega)^d\times H^1(\Omega)$ strongly.

Step 3: Convergence of the pressure sequence

For all two integers $k\ge m$, and for all test function $\mathbf{v}\in H_0^1(\Omega)^d$, we have

$$\begin{aligned}\int_\Omega \nabla\cdot\mathbf{v}\,\left(p^{k+1}-p^{m+1}\right) &=\\ &\gamma\int_\Omega\left(\mathbf{u}^{k+1}-\mathbf{u}^{m+1}\right)\cdot\mathbf{v}+\int_\Omega\left(\nu(\theta^k)\,\nabla\mathbf{u}^{k+1}-\nu(\theta^m)\nabla\mathbf{u}^{m+1}\right):\nabla\mathbf{v}\\ &+\int_\Omega\left[\left(\mathbf{u}^k\cdot\nabla\right)\mathbf{u}^{k+1}-\left(\mathbf{u}^m\cdot\nabla\right)\mathbf{u}^{m+1}\right]\cdot\mathbf{v}-\alpha\int_\Omega\left(\mathbf{f}(\theta^k)-\mathbf{f}(\theta^m)\right)\cdot\mathbf{v}.\end{aligned}$$

Passing to the limit on m, and using the strong convergence of sequences $(\mathbf{u}^k)_k$ and $(\theta^k)_k$, we deduce that

$$\lim_{k,m\to\infty}\int_\Omega\nabla\cdot\mathbf{v}\,\left(p^{k+1}-p^{m+1}\right)=0.$$

Thanks to the inf-sup condition (3.4), this yields

$$\lim_{k,m\to\infty}\beta\|p^{k+1}-p^{m+1}\|=0,$$

proving the convergence of the pressure sequence.

Step 4: Identification of the limit

The last step is devoted to the proof that the limit is a solution of problem (3.5). Let $(\mathbf{u},p,\theta)$ be the limit of $((\mathbf{u}^k,p^k,\theta^k))_k$ in $H^1(\Omega)^d\times L^2(\Omega)\times H^1(\Omega)$-strong. Since the functions λ and ν are continuous and bounded we have

$$\lim_{k\to\infty}\nu(\theta^k)=\nu(\theta) \text{ and } \lim_{k\to\infty}\lambda(\theta^k)=\lambda(\theta),\, a.e \text{ in } \Omega.$$

Next, we use the inverse Lebesgue Theorem (see for instance [8]), there exists a subsequence, still denoted by $\left((\mathbf{u}^k,p^k,\theta^k)\right)_k$ such that

i) $\lim_{k\to\infty}\nabla(\mathbf{u}^k)=\nabla\mathbf{u}$ and $\lim_{k\to\infty}\nabla(\theta^k)=\nabla\theta$ *a.e.* on Ω

ii) there exists $\zeta\in L^1(\Omega)$ such that $|\nabla\mathbf{u}^k|+|\nabla\theta^k|\le\zeta\quad\forall k$

Thus for all $(\mathbf{v},\phi) \in \mathbf{V} \times \mathcal{T}$, using the uniqueness of the limit of all subsequences of $\big((\mathbf{u}^k, p^k, \theta^k)\big)_k$ we have

$$\begin{aligned}
\lim_{k\to\infty} \gamma \int_\Omega \mathbf{u}^{k+1}\cdot\mathbf{v} + \int_\Omega \nu(\theta^k)\,\nabla\mathbf{u}^{k+1} : \nabla\mathbf{v} &= \gamma\int_\Omega \mathbf{u}\cdot\mathbf{v} + \int_\Omega \nu(\theta)\,\nabla\mathbf{u} : \nabla\mathbf{v} \\
\lim_{k\to\infty} \gamma \int_\Omega \theta^{k+1}\phi + \int_\Omega \lambda(\theta^k)\nabla\theta^{k+1}\cdot\nabla\phi &= \gamma\int_\Omega \theta\phi + \int_\Omega \lambda(\theta)\,\nabla\theta\cdot\nabla\phi.
\end{aligned}$$

Moreover from the strong convergence of $\mathbf{u}^k$ in $L^4(\Omega)^d$, θ^k in $L^4(\Omega)$, $\nabla\cdot\mathbf{u}^k$ in $L^2(\Omega)$ and p^k in $L^2(\Omega)$ we get

$$\begin{aligned}
\lim_{k\to\infty} \int_\Omega (\mathbf{u}^k\cdot\nabla)\mathbf{u}^{k+1}\cdot\mathbf{v} - \int_\Omega (\nabla\cdot\mathbf{v})\,p^k &= \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - \int_\Omega (\nabla\cdot\mathbf{v})\,p \\
\lim_{k\to\infty} \int_\Omega (\mathbf{u}^{k+1}\cdot\nabla)\theta^{k+1}\,\phi &= \int_\Omega (\mathbf{u}\cdot\nabla)\theta\,\phi. \\
\lim_{k\to\infty} -\int_\Omega (\nabla\cdot\mathbf{u}^k)\,q &= -\int_\Omega (\nabla\cdot\mathbf{u})\,q, \qquad \forall q\in L^2_0(\Omega).
\end{aligned}$$

Finally, from Hypothesis 2.2, we get the convergence for the source term and the solution of (4.1)-(4.2) converges to the solution of problem (3.5). ∎

Apart from the usual embedding results, this last proof uses: Lemma 3.1, (2.3), (2.7), (3.7) , (4.6), (4.7), and the regularity of ν and λ. Therefore, if we consider other types of treatment for the nonlinear term in (4.1) and (4.2), most of the proof would be unchanged (in fact it would be simplified). As before we will sum up with

Corollary 4.5. *Assuming the same hypothesis as Theorem 4.3, for any linearization of the convection term* $(\mathbf{u}\cdot\nabla)\mathbf{u}$ *and diffusion term* $\lambda(\theta)\nabla\theta$*, the convergence Theorem 4.3 is valid.*

This last proof gives us more than the convergence of the iterative scheme. In fact, from Hypothesis 4.1 (regularity of both gradients), and a smallness hypothesis we get

Corollary 4.6 (Uniqueness). *Under the hypothesis of Theorem 4.3. For* κ_1 *and* κ_2 *as defined by* (4.4)*. Assuming that*

$$max\{\kappa_1,\kappa_2\} < 1 \tag{4.12}$$

if (3.5) *admits a solution with* $\mathbf{u}\in W^{1,3}(\Omega)^d$ *and* $\theta \in \times W^{1,3}(\Omega)$*, then this solution is unique.*

5. Iterative schemes to solve the unsteady coupled problem. The main topic of this section revolves around the study of an efficient way to solve (4.1)-(4.2).

In the new *Coupled Prediction Scheme*, at each time step, we solve a coupled system between the velocity prediction and the convection-diffusion equation. It will become clear (Theorem 5.3), that the temperature does not need to be updated to satisfy the global convergence rate (the velocity and pressure are updated).

As a comparative tool, we will construct the *Basic Projection Scheme* for which we know the theoretical global rate of convergence in time (from [37]).

Remark 5.1. For the sake of simplicity and clarity, we opted for a backward Euler time differentiation. This gives $\gamma = 1/\Delta t$ and $\mathbf{r}_u = \mathbf{u}_n$, $r_\theta = \theta_n$ (the solutions at the previous time step). Shen [43], Guermond [21, 22], Guermond *et al.* [25], proved in various situations, that by using BDF2, the error on the velocity in the L^2-norm is $O(\Delta t^2)$, and on the pressure in the L^2-norm is $O(\Delta t)$. Therefore, the use of a second order time differentiation should be beneficiary for the CPS. We leave for future work, the use of higher order time differentiation and enrichment of the projection methods as proposed in Timmermans *et al.* [48] and Guermond *et al.* [23].

5.1. The Basic Projection Scheme. Denoting $t_n = n\Delta t$ and $f_n = f(\cdot, t_n)$. For given initial conditions $\mathbf{u}_0, p_0$ and θ_0, at each time step, we compute $\mathbf{u}_{n+1}, p_{n+1}$ and θ_{n+1}, with the following steps,

1. *Initialization:* $\mathbf{u}^0_{n+1} = \mathbf{u}_n$, $p^0_{n+1} = p_n$, $\theta^0_{n+1} = \theta_n$.
2. *Until convergence, compute:*
   (a) *Velocity prediction:* $\tilde{\mathbf{u}}^{j+1}_{n+1}$ solution of:

$$\frac{\tilde{\mathbf{u}}^{j+1}_{n+1} - \mathbf{u}_n}{\Delta t} + \left(\tilde{\mathbf{u}}^{j}_{n+1} \cdot \nabla\right) \tilde{\mathbf{u}}^{j+1}_{n+1} - \nabla \cdot \left(\nu(\theta^j_{n+1}) \nabla \tilde{\mathbf{u}}^{j+1}_{n+1}\right) + \nabla p_n = \alpha \mathbf{f}(\theta^j_{n+1})$$

   (b) *Projection step:* ψ^{j+1}_{n+1} solution of Poisson problem with suitable boundary conditions

$$-\Delta \psi^{j+1}_{n+1} = -\frac{1}{\Delta t} \nabla \cdot \tilde{\mathbf{u}}^{j+1}_{n+1},$$

   (c) *Correction step:* $\mathbf{u}^{j+1}_{n+1}$ such that

$$\mathbf{u}^{j+1}_{n+1} = \tilde{\mathbf{u}}^{j+1}_{n+1} - \Delta t \nabla \psi^{j+1}_{n+1} \tag{5.1}$$

   (d) *Convection-diffusion equation :* θ^{j+1}_{n+1} solution of

$$\frac{\theta^{j+1}_{n+1} - \theta_n}{\Delta t} + \left(\mathbf{u}^{j+1}_{n+1} \cdot \nabla\right) \theta^{j+1}_{n+1} - \nabla \cdot \left(\lambda(\theta^j_{n+1}) \nabla \theta^{j+1}_{n+1}\right) = h. \tag{5.2}$$

3. *Pressure correction:* denote ψ_{n+1} the converged solution of step 2 (b),

$$p_{n+1} = p_n + \psi_{n+1}$$

5.2. The Coupled Prediction Scheme. For given initial conditions $\mathbf{u}_0, p_0$ and θ_0. At each time step we compute $\mathbf{u}_{n+1}, p_{n+1}$ and θ_{n+1} with the following steps,

1. *Initialization:* $\tilde{\mathbf{u}}^0_{n+1} = \mathbf{u}_n$, $\tilde{\theta}^0_{n+1} = \tilde{\theta}_n$.
2. *Until convergence, compute:*
   (a) *Velocity prediction:* $\tilde{\mathbf{u}}^{j+1}_{n+1}$ solution of:

$$\frac{\tilde{\mathbf{u}}^{j+1}_{n+1} - \mathbf{u}_n}{\Delta t} + \left(\tilde{\mathbf{u}}^{j}_{n+1} \cdot \nabla\right) \tilde{\mathbf{u}}^{j+1}_{n+1} - \nabla \cdot \left(\nu(\tilde{\theta}^j_{n+1}) \nabla \tilde{\mathbf{u}}^{j+1}_{n+1}\right) + \nabla p_n = \alpha \mathbf{f}(\tilde{\theta}^j_{n+1})$$

   (b) *Temperature prediction:* $\tilde{\theta}^{j+1}_{n+1}$ solution of

$$\frac{\tilde{\theta}^{j+1}_{n+1} - \tilde{\theta}_n}{\Delta t} + \left(\tilde{\mathbf{u}}^{j+1}_{n+1} \cdot \nabla\right) \tilde{\theta}^{j+1}_{n+1} - \nabla \cdot \left(\lambda(\tilde{\theta}^j_{n+1}) \nabla \tilde{\theta}^{j+1}_{n+1}\right) = h. \tag{5.3}$$

3. *Projection step :* denote $\left(\tilde{\mathbf{u}}_{n+1}, \tilde{\theta}_{n+1}\right)$ the solution of step 2. Compute ψ_{n+1} solution of Poisson problem with the suitable boundary conditions:

$$-\Delta \psi_{n+1} = -\frac{1}{\Delta t} \nabla \cdot \tilde{\mathbf{u}}_{n+1} \tag{5.4}$$

4. *Velocity and pressure correction :*

$$\mathbf{u}_{n+1} = \tilde{\mathbf{u}}_{n+1} - \Delta t \nabla \psi_{n+1} \qquad p_{n+1} = p_n + \psi_{n+1}.$$

Remark 5.2.

1. For the BPS, applying Theorem 4.3 we get the convergence. For the CPS, adapting the proof of Theorem 4.3 we get the convergence of step 2.
2. For the CPS, to correct the predicted temperature, a fixed point is necessary. In that case, Theorem 4.3 can by applied directly; however, the BPS would clearly be more efficient.
3. If $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and $\nabla\cdot(\lambda(\theta)\nabla\theta)$ are treated implicitly or semi implicitly, following [37], the scheme is unconditionally stable and of order 1. However, if one of those terms is treated explicitly, the stability becomes conditional (again see [37] for the Courant-Friedrichs-Lewy (CFL) condition related to each terms).
4. Observe that for the BPS, the pressure correction (step 3) is made out of the loop since it has no effect on the rest of the variables.

We will now show that the error estimate between CPS and BPS is at least first order in L^2-norm. Therefore, the CPS is a first order scheme.

THEOREM 5.3. *Assuming Hypothesis 2.2, 2.3, 4.1 and* (4.12) *hold, that* $h\in\mathcal{T}^*$ *and* $(\theta_D,\theta_N)\in H^{\frac{1}{2}}(\Gamma_D)\times H^{-\frac{1}{2}}(\Gamma_N)$. *Let* $(\hat{\mathbf{u}}_n,\hat{\theta}_n)$ *and* $(\tilde{\mathbf{u}}_n,\tilde{\theta}_n)$ *be the results of the BPS and CPS respectively, then*

$$\|\hat{\mathbf{u}}_{n+1}-\tilde{\mathbf{u}}_{n+1}\|=\mathcal{O}(\Delta t^{\frac{3}{2}}),\qquad \|\hat{\mathbf{u}}_{n+1}-\tilde{\mathbf{u}}_{n+1}\|_{1,\Omega}=\mathcal{O}(\Delta t)\quad \forall n\geq 0 \tag{5.5}$$

$$\|\hat{\theta}_{n+1}-\tilde{\theta}_{n+1}\|=\mathcal{O}(\Delta t),\qquad \|\hat{\theta}_{n+1}-\tilde{\theta}_{n+1}\|_{1,\Omega}=\mathcal{O}(\sqrt{\Delta t})\quad \forall n\geq 0 \tag{5.6}$$

Proof. For the first step will use the limit point of both fixed point, $(\hat{\mathbf{u}}_{n+1},\hat{\theta}_{n+1})$ and $(\mathbf{u}^\star_{n+1},\tilde{\theta}_{n+1})$ the solution of step 2 in the CPS. We write

$$\delta\theta_{n+1}=\hat{\theta}_{n+1}-\tilde{\theta}_{n+1},\qquad \forall n\geq 0,$$

from (5.2) and (5.3), using $\phi=\delta\theta_{n+1}$ and (3.7) we get an equation similar to (4.9), using the same argument we have

$$\begin{aligned}\frac{1}{\Delta t}\|\delta\theta_{n+1}\|^2+\lambda_0\|\nabla\delta\theta_{n+1}\|^2\leq C\,M\,\lambda_2\|\nabla\delta\theta_{n+1}\|^2+\frac{1}{\Delta t}\|\delta\theta_n\|\|\delta\theta_{n+1}\|\\ +CM\|\hat{\mathbf{u}}_{n+1}-\mathbf{u}^\star_{n+1}\|_{1,\Omega}\|\delta\theta_{n+1}\|.\end{aligned}$$

From Theorem 3.2, for all n, $\hat{\mathbf{u}}_{n+1}$ and $\tilde{\mathbf{u}}_{n+1}$ are bounded in $H^1(\Omega)^d$ by a constant depending only on the data. Therefore $\mathbf{u}^\star_{n+1}$ is also bounded, and we have a constant C_U such that

$$\|\hat{\mathbf{u}}_{n+1}-\mathbf{u}^\star_{n+1}\|^2_{1,\Omega}\leq C_U\qquad \forall n.$$

From (4.12) $\eta_\lambda=\lambda_0-\lambda_2 CM$ is a strictly positive constant, (recall that C is based on the semi-norm). Using two appropriate form of Young's inequality we have

$$\frac{1}{4\Delta t}\|\delta\theta_{n+1}\|^2+\eta_\lambda\|\nabla\delta\theta_{n+1}\|^2\leq\frac{1}{2\Delta t}\|\delta\theta_n\|^2+C_U(CM)^2\Delta t. \tag{5.7}$$

For $n=0$ in (5.7), since BPS and CPS have the same initial condition, $\delta\theta_0=0$ and

$$\frac{1}{4\Delta t}\|\delta\theta_1\|^2+\eta_\lambda\|\nabla\delta\theta_1\|^2\leq C_U(CM)^2\Delta t,$$

then (5.6) is verified for $n=0$ and by induction, for all n.

Since $\hat{\mathbf{u}}_{n+1}$ and $\tilde{\mathbf{u}}_{n+1}$ satisfy (3.6) for $\hat{\theta}_{n+1}$ and $\tilde{\theta}_{n+1}$ (resp.), we follow the same path as in Step 1 of Theorem 4.3. Introducing

$$\delta u_{n+1} = \hat{\mathbf{u}}_{n+1} - \bar{\mathbf{u}}_{n+1} \qquad \forall n \geq 0$$

similar to (4.5), we get

$$\begin{aligned} \frac{1}{\Delta t}\|\delta u_{n+1}\|^2 + \nu_0\|\nabla\delta u_{n+1}\|^2 \leq{}& CM(\nu_2 + \alpha/M)\|\delta\theta_{n+1}\|_{1,\Omega}\|\nabla\delta u_{n+1}\| \\ &+ \frac{\alpha C^2 c_u}{\gamma_\nu\gamma_\lambda}\|\nabla\delta u_{n+1}\|^2 + \frac{1}{\Delta t}\|\delta u_n\|\|\delta u_{n+1}\|. \end{aligned}$$

From (4.12) $\eta_\nu = \nu_0 - \dfrac{\alpha C^2 c_u}{\gamma_\nu\gamma_\lambda} > 0$. Using two Young's inequalities, we get

$$\frac{1}{\Delta t}\|\delta u_{n+1}\|^2 + \eta_\nu\|\nabla\delta u_{n+1}\|^2 \leq \frac{(CM(\nu_2 + \alpha/M))^2}{\eta_\nu}\|\delta\theta_{n+1}\|^2 + \frac{2}{\Delta t}\|\delta u_n\|^2. \tag{5.8}$$

For $n = 0$ in (5.8), since BPS and CPS have the same initial condition, $\delta u_0 = 0$ and using (5.6)

$$\frac{1}{\Delta t}\|\delta u_1\|^2 + \eta_\nu\|\nabla\delta u_1\|^2 \leq \frac{(CM(\nu_2 + \alpha/M))^2}{\eta_\nu} 4C_U(CM)^2\Delta t^2 \tag{5.9}$$

then (5.5) is verified for $n = 0$ and, by induction, the result follows for all n. ◻

Remark 5.4. The estimates obtained in the last Theorem are not sharp. Since $\tilde{\theta}_n$ depends on the velocity prediction instead of $\tilde{\mathbf{u}}_n$ we cannot use (5.5) (i.e. bootstrap (5.5)-(5.6)) to sharpen (5.6). A result concerning the predicted velocity is needed to get better results. However, we must keep in mind that those estimates ((5.5) and (5.6)) are not error estimates for the CPS but they are indicative of the minimal order of the time accuracy of the scheme.

Remark 5.5. The semi-explicit or explicit treatment of the non linear terms yields some obvious simplifications in the last proof. In particular, if the diffusive term $\lambda(\theta)\nabla\theta$ is treated semi-explicitly or explicitly, the proof of (5.6) does not rely on the smallness hypothesis 4.12. The same can be said for the convective term and (5.5). In any cases, assuming that we apply the same linearization in both schemes (BPS and CPS), Theorem 5.3 is valid.

6. Numerical experiments. Two groups of tests will be presented. The first series of tests, based on an analytic solution (in 3D), will be used to validate the accuracy of the new scheme. By comparison with the BPS, we will illustrate the numerical efficiency and the sensitivity of the CPS for the two major parameters. The second series is based on a classical (and almost mandatory) 2D Rayleigh-Bénard problem (RBC). The RBC problem is the object of numerous researches regarding natural convection. Therefore, the behavior of this model is quite predicable and can be viewed as some sort of benchmark (see [1, 46, 36] for some literature review on the RBC and bidimensional benchmarks for this problem).

In all cases, the totally implicit BPS and CPS were used and the numerical results performed with FreeFEM++ [27]. The solver uses a stabilized Taylor-Hood finite element (see Brezzi-Fortin [10]) for the space discretization of Navier-Stokes systems and P2 (FEM) for the convection-diffusion equation.

6.1. Accuracy test. In order to evaluate the convergence rates and compare the performance of both scheme, we construct a 3D problem where the exact solution is given in $\Omega = [0,1]^3$ by

$$\left\{\begin{array}{rcl} u_1 &=& \left(x^2+xy-z^2+yz\right)\,\sin(t) \\ u_2 &=& -\left(2xy+0.5y^2+2yz-2xz\right)\,\sin(t) \\ u_3 &=& \left(z^2+y^2-x^2+3xy\right)\,\sin(t) \\ p &=& \left(x-y+3z-\dfrac{3}{2}\right)\,\sin(t) \\ \theta &=& 2+\left(x^2+y^2+z^2+1\right)\,\sin(t). \end{array}\right. \tag{6.1}$$

Since this solution is in the spatial discretization space, the approximation error is only related to the time discretization. The suitable forcing functions are given by

$$\left\{\begin{array}{rcl} \mathbf{f} &=& \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nabla\cdot(\nu(\theta)\,\nabla\,\mathbf{u}) + \nabla p - \alpha\sqrt{\theta}, \\ \\ h &=& \dfrac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\,\theta - \nabla\cdot(\lambda(\theta)\,\nabla\,\theta)\,, \end{array}\right. \tag{6.2}$$

where $\alpha = 10^2$, $\nu(\theta) = \lambda(\theta) = 10^{-4}\sqrt{\theta}$.

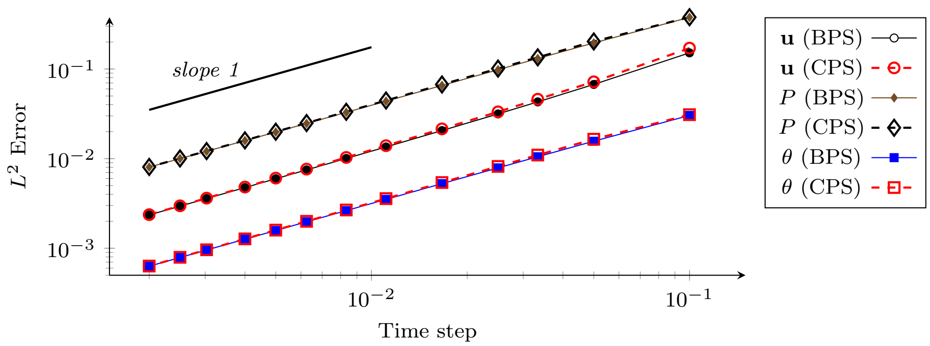

Fig. 6.1. *L^2 Error: Basic Projection Scheme vs Coupled Prediction Scheme*

Table 6.1
Overall computation times at $t = 1$ with $\Delta t = 10^{-2}$ for structured meshes (tetrahedral elements based on a regular grid).

grid (mesh size)	BPS: CPU time(s)	CPS: CPU time(s)	Difference (%)
3x3x3 (0.0370)	954	700	-26.6
7x7x7 (0.0029)	1518	1015	-33.1
10x10x10 (0.001)	3790	2287	-39.7

In Figure 6.1, we plotted the L^2 errors of the velocity, pressure, and temperature between the numerical solution and the exact solution at $t = 1$ for different time step. Observe that the order of accuracy in time for all variables is conserved by the CPS scheme. Recall that for CPS, the temperature is transported by the velocity prediction which is not divergence free. We remark that all BPS approximations are slightly more accurate than those obtained by CPS, but the CPU time of the BPS scheme is significantly higher than for the CPS, (see Table 6.1).

6.2. Sensitivities. Based on (6.1) and (6.2), we propose two series of tests exploring the sensitivities of solutions with respect to the viscosity ν and the thermal conductivity λ. A tetrahedral mesh based on a 7x7x7 grid and a time step of 10^{-2} s were used. To illustrate the effects of the viscosity, the following set of data was used,

$$\alpha = 0, \quad \lambda = \sqrt{\theta}, \quad \nu_k = \frac{10^{-2}}{k} \qquad k = 1, ..., 100 \tag{6.3}$$

and for the second case (effects of the conductivity),

$$\alpha = 100, \quad \nu = 10^{-3}\sqrt{\theta}, \quad \lambda_k = \frac{10^{-3}}{k} \qquad k = 1, ..., 100. \tag{6.4}$$

To compare the CPS and the BPS with respect to variations of either variables, we introduced a measure to the relative differences of the error of each schemes.

$$\Delta = 100\frac{|E_{BPS} - E_{CPS}|}{E_{BPS}} \tag{6.5}$$

where

$$E_{BPS} = (\|\mathbf{u} - \mathbf{u}_{BPS}\|^2 + \|p - p_{BPS}\|^2 + \|\theta - \theta_{BPS}\|^2)^{1/2}$$

$$E_{CPS} = (\|\mathbf{u} - \mathbf{u}_{CPS}\|^2 + \|p - p_{CPS}\|^2 + \|\theta - \theta_{CPS}\|^2)^{1/2}.$$

As predicted by the error estimates (for instance [5]), Figure 6.2 shows that the error is proportional to the inverse of the parameters. Therefore we have an increase of the errors (for the velocity, pressure and temperature) for a decrease of the thermal conductivity or viscosity (leading to more turbulent flow).

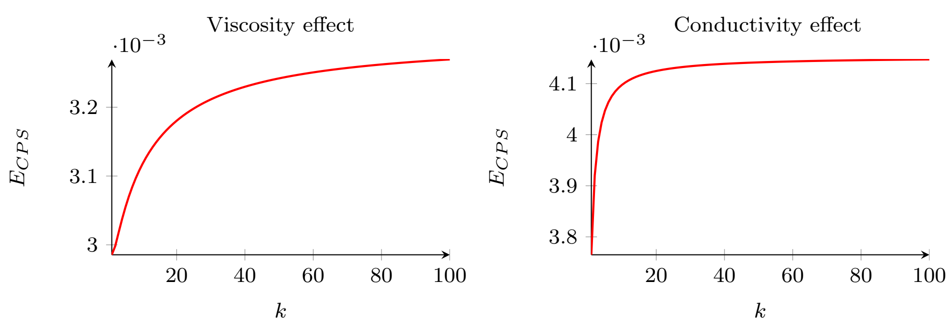

FIG. 6.2. *Sum of the errors for* $\mathbf{u}, p, \theta$ *for* ν_k *(left) and* λ_k *(right) as defined in* (6.3) *and* (6.4)

The Δ function measure the relative variation for the CPS error in relation with the BPS scheme. Figure 6.3 shows a variation of less than 2% between the error of approximations, for a variation of 2 order of magnitude for the physical parameters (less than 1% in case of the viscosity). From those graphics, it seems that both the BPS and the CPS have a similar behavior with respect to ν and λ. We conclude that the new scheme is robust with respect to ν and λ and keeps it advantages compared to the more classical BPS scheme.

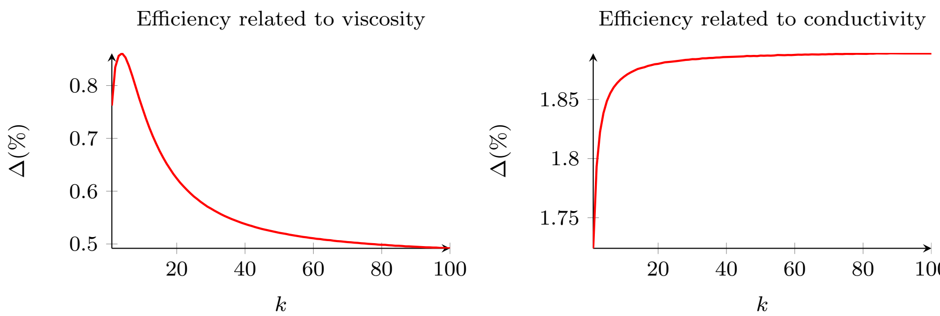

FIG. 6.3. *Relative difference between the approximation errors for CPS and BPS for ν_k (left) and λ_k (right) as defined in* (6.3) *and* (6.4)

6.3. Rayleigh-Bénard convection.

6.3.1. Problem description. Natural convection is frequently associated with the RBC which can be considered as the preferred example of convection from an academic point of view. Even if based on a simple geometry and constant physical parameters, the RBC problem "shares number of important properties with many other pattern-formation mechanisms" (Getling, [16]). It offers a first approach to complex flows as well as the transition from conductive to convective heat transfer modes.

The RBC is a model in which heat transfer occurs via a fluid between two horizontal flat plates at different temperatures. The bidimensional RBC problem is developed with the Boussinesq approximation and the governing equations are:

$$(6.6) \qquad \left\{ \begin{array}{rcl} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu\Delta\mathbf{u} &=& \beta\,\mathbf{g}\,(\theta-\theta_1) \\ \nabla\cdot\mathbf{u} &=& 0 \\ \dfrac{\partial \theta}{\partial t} + (\mathbf{u}\cdot\nabla)\,\theta - \lambda\Delta\theta &=& 0. \end{array} \right.$$

Where $\mathbf{g} = (0, g)$ is the gravitational acceleration and β is the thermal expansion coefficient of the fluid. The domain is heated from below so the temperature at the top θ_1 is less than the temperature θ_2 at the bottom. A dimensional analysis shows that there are two dimensionless groups: the Rayleigh number Ra and the Prandtl number Pr

$$Ra = \frac{\beta g(\theta_2-\theta_1)\,\ell^3}{\nu\,\lambda L^3}, \qquad Pr = \frac{\nu}{\lambda}$$

both defined using the viscosity coefficient ν and the conductivity coefficient λ. The Rayleigh number is associated with buoyancy driven flow. When Ra is below the critical value Ra_c, heat transfer is primarily in the form of conduction. For Rayleigh number over the critical value, heat transfer is primarily in the form of unstable convection. For Rayleigh number moderately over the critical value we can observe the formation of a horizontal arrangement of Bénard (or Rayleigh-Bénard) cells rotating alternatively from clock-wise to counter-clockwise. In this context, Busse *et al.* [11] proved the stability of straight parallel convection rolls and used them to explain many experimental observations. Finally, for very large Rayleigh number, the flow becomes turbulent and chaotic behavior is observed.

The first two cases can easily be illustrated using the temperature difference of the two plates. If $\theta_2 - \theta_1$ is sufficiently small, the associate Ra is under the the critical value. The fluid is quiescent and the temperature increases linearly in the vertical direction. This is a pure conduction state. If $\theta_2 - \theta_1$ increases beyond the critical Rayleigh number, the pure conduction state becomes unstable and convection starts. Since the number of Bénard cells can be establish theoretically, it can be used as a validation for moderately high Rayleigh number.

In the first two tests presented (loosely based on the benchmark in [46]), the domain is rectangular with an aspect ratio of 2. The Rayleigh number is either under or slightly over the critical value Ra_c. These conditions produce stable solutions for which a comparison to a Direct Numerical Simulation (DNS) on a fine structured mesh was carried out.

The last tests uses moderate Rayleigh numbers (a dimensionless formulation was used on a square domain). These cases allowed us to demonstrate the robustness of the CPS and the efficiency of the method at high Rayleigh numbers. To this end, we used the benchmark solutions of [36].

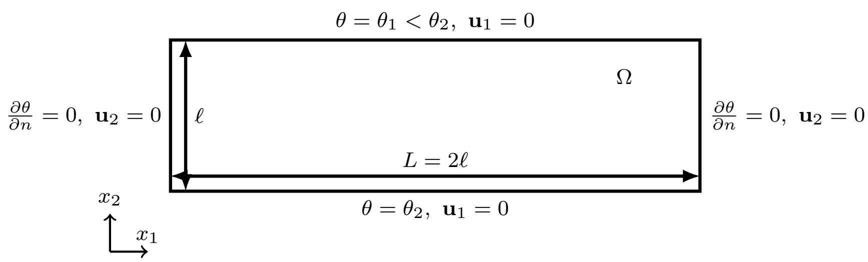

FIG. 6.4. *Problem definition and boundary conditions.*

6.3.2. Numerical results for low Rayleigh numbers. We assume that Ω is rectangular with boundary conditions as illustrated in Figure 6.4. Computations were carried out for:

- computational domain: $\Omega = [0, 2\ell] \times [0, \ell]$, where $\ell = 0.0100028m$
- thermal expansion coefficient: $\beta = 3 \cdot 10^{-3}\, K^{-1}$
- thermal conductivity: $\lambda = 2.2 \cdot 10^{-5}\, m^2 \cdot s^{-1}$
- dynamic viscosity: $\nu = 1.54 \cdot 10^{-5}\, m^2 \cdot s^{-1}$
- gravity acceleration: $g = 9.8\, m \cdot s^{-1}$
- regular triangular mesh (grid size : 30x60 for CPS and 128x256 for DNS)
- a constant time step of 10^{-2} was used for all computations
- the convergence of the fixed point used a criteria of 10^{-9} on $\mathbf{u}$ and θ.
- the steady state determined with a criteria of 10^{-6} on $\mathbf{u}$ and θ.

For the first test $\theta_1 = 313K$, $\theta_2 = 323K$ ($\Delta\theta = 10K$), so that $Ra = 868.5$ and for the second test we doubled the value of the temperature difference ($\theta_2 = 333K$ and $\Delta\theta = 20K$) leading to $Ra = 1737$.

Table 6.2 presents the differences between DNS and CPS (in L^2-norm) for the temperature and velocity fields. The relative difference between both method is minute (both solution are at steady state) confirming the good behavior of the CPS.

For $Ra = 868.5$, there is no convective motion, in fact, at $\Delta\theta = 10K$, there is no flow, and the heat is transmitted by conduction through the fluid. For $Ra = 1737$, as was predicted by the theory ($Ra_c \approx 1708$) , two Bénard cells are formed and their

Table 6.2
Comparison DNS versus CPS at low Rayleigh number. In parenthesis, the relative error in percentage.

	$R_a = 868.5$	$R_a = 1737$
$\Vert\theta_{DNS} - \theta_{CPS}\Vert$	3.82686e-06 (0.85e-06%)	6.98331e-05 (1.5e-03%)
$\Vert\mathbf{u}_{DNS} - \mathbf{u}_{CPS}\Vert$	2.74762e-08 (0.02%)	5.72609e-07 (0.17%)

rotations alternate from clockwise to counter-clockwise with hot fluid rising and cold fluid falling, see Figure 6.5.

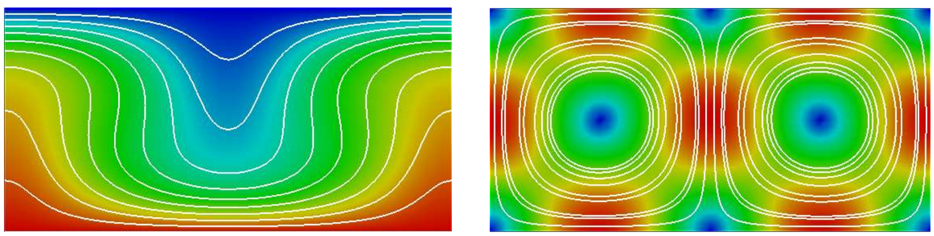

Fig. 6.5. *Rayleigh $Ra = 1737 > Ra_c$. On the left temperature isolines, on the right the velocity field streamlines, exposing the Bénard cells.*

6.3.3. Benchmark for different Rayleigh numbers. To illustrate the capability of CPS to deal with various Rayleigh number, we used the benchmark tests proposed in [36] (we refer the interested readers to this paper for a detailed review of those tests). It consists in three tests on a unit square with increasing values of Rayleigh number (10^4, 10^5 and 10^6). To have accurate solutions in all cases, a triangular mesh based on a sufficiently fine grid (256x256) is used. The comparison will be made with 3 values calculated at steady states: the maximal values of each component of the velocity field (u_{max}, v_{max}) and the average Nusselt number Nu. Recall that for a unit square, the average Nusselt number (for the bottom plate $y = 0$) is defined as

$$Nu = -\int_0^1 \frac{\partial\theta}{\partial y}(x, 0)\, dx.$$

It must be noted that in [36], the authors have established these values (noted here as u_{max}^{ref}, v_{max}^{ref} and Nu^{ref}) using a finite volume approach that can be regarded as a BPS method with implicit treatment of the non linear terms. Table 6.3 presents the values obtained using CPS. Once again, the difference with the predicted values is negligible. The largest difference, found for $Ra = 10^6$, is less than 0.6% of deviation. This gap can easily be explained by the use of different convergence criteria (10^{-9} here and 10^{-7} in [36]) and criteria for the steady state detection (10^{-6} and 10^{-5} resp.). Figure 6.6 depict the behavior of the temperature and velocity at permanent regime. All three cases are relatively stable and shows presence of smaller cells in corners. Observe that $Ra = 10^5$ is the only case developing smaller cells in opposite corners (lower left, upper right) compared to the others cases (upper left, lower right). This is due to the clockwise motion of the fluid (counter clockwise for the other cases) and is explained by the chaotic nature of the phenomenon.

Table 6.3
Comparison CPS and [36] for high Rayleigh number.

Ra	u_{max}	u_{max}^{ref}	v_{max}	v_{max}^{ref}	Nu	Nu^{ref}
10^4	0.25230	0.25228	0.26370	0.26369	2.1585	2.1581
10^5	0.34462	0.34434	0.37597	0.37569	3.9150	3.9103
10^6	0.37305	0.37088	0.40758	0.40600	6.3094	6.3092

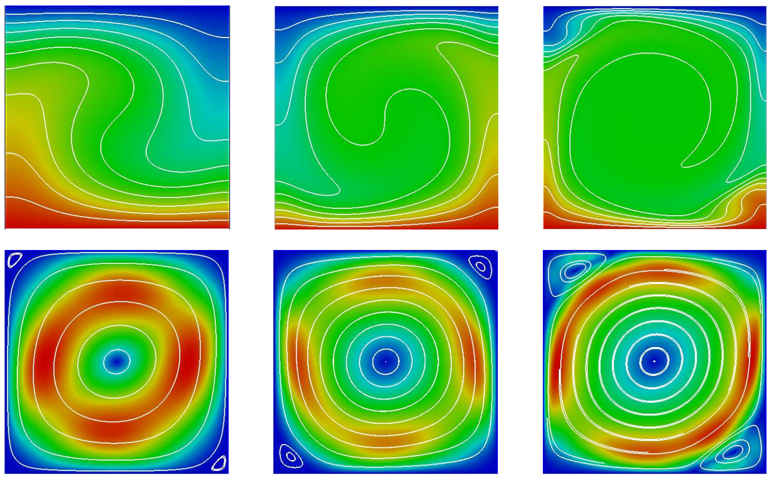

Fig. 6.6. *From left to right, on top the isotherms, at the bottom the velocity streamlines for* $Ra = 10^4, 10^5$ *and* 10^6.

7. Conclusion. In this paper, we have analyzed a model for the coupling of the convection-diffusion equation with the Boussinesq/Navier-Stokes equations for an incompressible fluid, where both the viscosity and conductivity depend on the temperature. We have proved that the time discrete problem admits at least a solution in suitable spaces. To approximate its solution, we have introduced an iterative scheme whose convergence was established under appropriate assumptions. We have also analyzed two schemes based on incremental projection methods. The basic projection scheme (BPS) is a first order “classical” approach that can be find in various form in the literature. We introduced a new method, the coupled prediction scheme, CPS, relying on an approximation of the temperature based on the non solenoidal velocity prediction produced by the projection. This new approach is flexible since the usual treatment of the non linear terms are still available. Moreover it gives a more efficient and consequently faster algorithm compared to the usual approaches. The analysis shows that the proposed CPS scheme is as accurate as the BPS, therefore, first order in time. Lastly, numerical tests confirm these theoretical findings and show its robustness with respect to the parameters (or Rayleigh number).

8. Future works. Three extensions of this work are ongoing. A review of the numerical effectiveness of the different form of the CPS (implicit, semi implicit and explicit). The development and analysis of a second order time-accurate scheme using the so-called ”rotational projection scheme” (Guermond *et al.* [23]) and a free surface algorithm in order to study the convection-diffusion equation combined with bi-fluids Navier-Stokes equations.

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