An iterative procedure to solve a coupled atmosphere-ocean turbulence model.

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Motivation

General Purpose: Analysis of stability of discretization of turbulence models.

- Models of turbulence commonly used in industrial, environmental and (in this case) oceanographic applications.
- Many software safeguards needed to avoid overflows-underflows.
- There is a large need to perform a numerical analysis of stability of discretizations of turbulence models, and a subsequent improvement of discretization techniques.

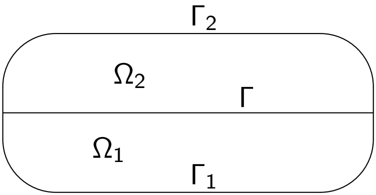
Model equations

We consider a coupled steady two-fluid (ocean-atmosphere) turbulence model. It includes

- A one-equation simplified turbulence model for each fluid: Only one statistic of turbulence is considered, the turbulent kinetic energy (TKE).
- Modeling of TKE generation at ocean-atmosphere interface by quadratic law.
- Modeling of friction at interface by Manning law.

Model equations

$$
\begin{aligned}
-\nabla\cdot(\alpha_i(k_i)\nabla\mathbf{u}_i) + \ \text{grad}\ p_i &= \mathbf{f}_i && \text{in} && \Omega_i,\\
\nabla\cdot\mathbf{u}_i &= 0 && \text{in} && \Omega_i,\\
-\nabla\cdot(\gamma_i(k_i)\nabla k_i) &= \alpha_i(k_i)|\nabla\mathbf{u}_i|^2 && \text{in} && \Omega_i,\\
\mathbf{u}_i &= \mathbf{0} && \text{on} && \Gamma_i,\\
k_i &= 0 && \text{on} && \Gamma_i,\\
\alpha_i(k_i)\partial_{\mathbf{n}_i}\mathbf{u}_i - p_i\mathbf{n}_i + c_i(\mathbf{u}_i-\mathbf{u}_j)|\mathbf{u}_i-\mathbf{u}_j| &= \mathbf{0} && \text{on} && \Gamma,\\
k_i &= \lambda|\mathbf{u}_1-\mathbf{u}_2|^2 && \text{on} && \Gamma.
\end{aligned}
\tag{1}
$$



Model equations

Data:

- Ω_i : Open bounded sets of $\mathbb{R}^d$ either convex or with boundary $\partial\Omega_i = \Gamma_i \cap \Gamma \in C^{1,1}$.
- $\alpha_i, \gamma_i \in W^{1,1}(\mathbb{R}), \quad \alpha_i, \gamma_i \geq \nu > 0$ uniformly in$\mathbb{R}$.
- $\lambda > 0$, $c_i > 0$: Friction coefficients.
- $f_i \in L^2(\Omega_i)$: Source terms.

Specific Purposes: Devise stable numerical methods to compute steady states, as simplified equilibria states of the ocean-atmosphere system.

Weak formulation of continuous model

- Model analyzed in Bernardi, Lewandowski, Murat, Chacon; Collège de France Seminar, Vol. XIV (2002).
- The solution for the TKE was defined by transposition:
- Essentially, this equation reads as

$$\begin{cases} -\Delta k &= \alpha \quad \text{in } \Omega, \\ k &= \delta \quad \text{on } \partial\Omega, \end{cases} \qquad \text{where} \tag{2}$$

  $\alpha \in L^1(\Omega)$ and $\delta \in L^4(\partial\Omega)$.

- The Laplace operator $\mathcal{L}$ which associates with data g in $H^{-1}(\Omega)$ the solution $\varphi = \mathcal{L}g \in H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta\varphi &= g \quad \text{in } \Omega, \\ \varphi &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

- is an isomorphism from $L^2(\Omega)$ into $H^2(\Omega) \cap H_0^1(\Omega)$ (Grisvard, 1985).

- The solution by transposition of (2) is now defined by :

$$\begin{cases} \text{Find } \; k \in L^2(\Omega) \text{ such that } \forall g \in L^2(\Omega), \\ \displaystyle\int_\omega k\, g\, d\mathbf{x} = -\int_{\partial\Omega} \delta\, \partial_n(\mathcal{L}g)\, d\tau + \int_\Omega \alpha\, (\mathcal{L}g)\, d\mathbf{x}. \end{cases} \tag{4}$$

- A direct numerical approximation of this formulation would need to explicitely know $\mathcal{L}$ acting on the elements of the discrete space, or some suitable approximation.
- BCLM prove that if the solution is smooth enough, this formulation for the TKE in model equations is equivalent to the more standard that follows

Weak formulation of model equations

Find

- $\mathbf{u}_i \in \mathbf{X}_i = \{\mathbf{v} \in \mathbf{H}^1(\Omega_i)^d;\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_i\}.$
- $p_i \in L^2(\Omega_i)\ ,\quad k_i \in \bigcap_{q<d'} W^{1,q}(\Omega_i)$

such that $\forall \mathbf{v}_i \in \mathbf{X}_i,\ \forall q_i \in L^2(\Omega_i),\ \forall \varphi_i \in W_0^{1,r}(\Omega_i),\ r > d,$

$$(\alpha_i(k_i)\nabla\mathbf{u}_i, \nabla\mathbf{v}_i)_{\Omega_i} + b_i(\mathbf{v}_i, p_i) + c_i(|\mathbf{u}_i - \mathbf{u}_j|(\mathbf{u}_i - \mathbf{u}_j), \mathbf{v}_i)_\Gamma = (\mathbf{f}_i, \mathbf{v}_i)_{\Omega_i},$$

$$b_i(\mathbf{u}_i, q_i) = 0,$$

$$k_i = 0 \quad \text{on } \Gamma_i, \qquad k_i = \lambda|\mathbf{u}_i - \mathbf{u}_j|^2 \qquad \text{on } \Gamma, \text{ and}$$

$$(\gamma_i(k_i)\nabla k_i, \nabla\varphi_i)_{\Omega_i} = (\alpha_i(k_i)|\nabla\mathbf{u}_i|^2, \varphi_i)_{\Omega_i},$$

where

$$b_i(\mathbf{u}_i, q_i) = -(div\mathbf{u}_i, q_i)_{\Omega_i}.$$

2D F. E. approximation

BCLM analyzed a 2D Finite Element solution (Numer. Math., 2004), as follows:

- The velocity - pressure is discretized by the Mini-Element $X_{ih} \times M_{ih} = (\mathbf{P_1} \bigoplus \text{Bubble}, \mathbf{P_1})$ on both Ω_1 and Ω_2.
- The velocity spaces X_{1h} and X_{2h} are supposed to be compatible on Γ, in the sense that the trace spaces

$$Z_{ih} = \{\mathbf{v}_{ih|_\Gamma}, \text{ for } \mathbf{v}_{ih} \in X_{ih}, \}, \quad i = 1, 2 \quad \text{are equal.}$$

- The TKEs are discretized by piecewise affine Finite Elements V_{ih}:

$$k_{ih} = k_{0ih} + D_{ih},$$

  where

  - $k_{0ih} \in K_{ih} = V_{ih} \cap H_0^1(\Omega_i)$, and
  - D_{ih} is a piecewise affine extrapolation of the boundary data $\lambda \, |\mathbf{u}_1 - \mathbf{u}_2|^2$ to the Ω_i.

2D F. E. approximation

The F. E. approximation is

$$\text{Obtain } (\mathbf{u}_{ih},\ p_{ih},\ k_{0ih}) \in X_{ih} \times M_{ih} \times K_{ih},\ i = 1, 2$$

$$\text{such that } \forall (\mathbf{v}_{ih}, q_{ih}, g_{ih}) \in X_{ih} \times M_{ih} \times K_{ih},$$

$$a_i(k_{ih}; \mathbf{u}_{ih}, \mathbf{v}_{ih}) + b_i(\mathbf{v}_{ih}, p_{ih}) - b_i(\mathbf{u}_{ih}, q_{ih}) + n(\mathbf{u}_{ih}, \mathbf{u}_{j_i h}, \mathbf{v}_{ih}) = \\ = \langle \mathbf{f}_i, \mathbf{v}_{ih} \rangle,$$

$$\int_{\Omega_i} \nabla \mathbf{k}_{ih} \cdot \nabla g_{ih} \, d\mathbf{x} = \int_{\Omega_i} \alpha_i(\mathbf{k}_{ih}) \, |\nabla \mathbf{u}_{ih}|^2 \, g_{ih} \, d\mathbf{x};$$

$$k_{ih} = k_{0ih} + D_{ih}, \text{ with } \quad D_{ih} = \lambda \, |P_{ih}(\mathbf{u}_{1h} - \mathbf{u}_{2h})|^2.$$

2D F. E. approximation

The following result is proved:

Theorem (Convergence of 2D F.E. approximation)

In the 2D case, there exists a subsequence of the solutions $(\mathbf{u}_{1h},\ p_{1h},\ k_{1h})$, $(\mathbf{u}_{2h},\ p_{2h},\ k_{2h})$ *provided by method (6)-(8) that converge strongly in*

$$(H^1(\Omega_1)^2 \times L^2_0(\Omega_1) \times H^s(\Omega_1)) \times (H^1(\Omega_2)^2 \times L^2_0(\Omega_2) \times H^s(\Omega_2)),$$

for $0 \le s < 1/2$, *to a solution of model problem, where the TKE equation is formulated by transposition.*

The problems we face now are

- **To build a more constructive** solution scheme.
- To analyze the 3D case.

Solution by fixed point iteration

- To solve model problem we propose the following fixed point iterative method:

$$\text{Find } (\mathbf{u}_i^{n+1}, p_i^{n+1}, k_i^{n+1}) \in \mathbf{X}_i \times L^2(\Omega_i) \times \bigcap_{q<d'} W^{1,q}(\Omega_i), \text{ s. t.}$$

$$\forall (\mathbf{v}_i, q_i, \varphi_i) \in \mathbf{X}_i \times L^2(\Omega_i) \times W^{1,r}(\Omega_i),\ r > d,$$

$$\begin{aligned} (\alpha_i(k_i^n)\nabla \mathbf{u}_i^{n+1}, \nabla \mathbf{v}_i)_{\Omega_i} &+ b_i(\mathbf{v}_i, p_i^{n+1}) + \\ +c_i(|\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}), \mathbf{v}_i)_\Gamma &= (\mathbf{f}_i, \mathbf{v}_i)_{\Omega_i}, \\ b_i(\mathbf{u}_i^{n+1}, q_i) &= 0; \end{aligned}$$

$$\mathbf{k}_i^{n+1} = 0, \quad \text{on} \quad \Gamma_i, \qquad k_i^{n+1} = \lambda |\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|^2, \quad \text{on } \Gamma,$$

$$(\gamma_i(k_i^n)\nabla k_i^{n+1}, \nabla \varphi_i)_{\Omega_i} = (\alpha_i(k_i^n)|\nabla \mathbf{u}_i^{n+1}|^2, \varphi_i)_{\Omega_i}.$$

Solution by fixed point iteration

- The friction boundary term, although non-linear, is monotone and has a smoothing effect.
- In practice it can be mass-lumped and acts re-enforcing the diagonal-dominance of the mass matrix.
- The b. c. for the k_i must be discretized with care.
- In practice, this semi-implicit discretization does not set any implementation problem.

Analysis if iterative scheme

The iterative scheme is contractive if

- The turbulent diffusion is large enough with respect to the data,and
- The iterates $\{\mathbf{u}^n\}_n$ and $\{k_i^n\}_n$ remain bounded in norms smooth enough:

Analysis if iterative scheme

Theorem (Contractiveness)

Assume that the sequences $\{\mathbf{u}^n\}_n$ and $\{k_i^n\}_n$ remain bounded in $W^{1,3+\epsilon}(\Omega_i)^d$ and $W^{1,3}(\Omega_i)$ by M.
Then there exists a constant C depending only on Ω_i, α_i, Γ_i, c_i, λ, $\mathbf{f}_i$ and M such that if $K = \dfrac{C}{\nu} < 1$, then, the iterative scheme is contractive, in the sense that

$$\sum_{i=1}^{2} ||\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)||_{0,\Omega_i} \leq K \sum_{i=1}^{2} ||\nabla(k_i^n - k_i^{n-1})||_{0,\Omega_i},$$

and

$$\sum_{i=1}^{2} ||\nabla(k_i^{n+1} - k_i^n)||_{0,\Omega_i} \leq K \sum_{i=1}^{2} ||\nabla(k_i^n - k_i^{n-1})||_{0,\Omega_i}.$$

Iterative scheme

Keys of the proof:

- Use convenient choices of test functions:
  - For instance, to obtain the estimate for

$$\sum_{i=1}^{2} ||\nabla(k_i^{n+1} - k_i^n)||_{0,\Omega_i},$$

introduce the harmonic lifting $R_i : H_{00}^{1/2}(\Gamma) \mapsto H^1(\Omega_i)$ that brings a $\eta \in H_{00}^{1/2}(\Gamma)$ into $R_i(\eta) \in H^1(\Omega_i)$ given by

$$-\Delta R_i = 0 \text{ in } \Omega_i, \quad R_i = \eta \text{ on } \Gamma_i, \quad R_i = 0 \text{ on } \Gamma$$

and set the test function for k_i
$\varphi_i = (k_i^{n+1} - k_i^n) - R_i(k_i^{n+1} - k_i^n)$. Then proceed.

Iterative scheme

Keys of the proof:

- Due to the friction term, it is needed to estimate the expression

$$\left\| \, |\mathbf{u}_1^{n+1} - \mathbf{u}_2^n|^2 - |\mathbf{u}_1^{n+1} - \mathbf{u}_2^n|^2 \, \right\|_{H_{00}^{1/2}(\Gamma)}$$

  This is done using Grisvard's theorem on estimates in $W^{s,p}$ of products of functions of W^{s_j,p_j}:

Iterative scheme

Theorem

Let $s_1 \geq s$ *and* $s_2 \geq s$ *satisfying either:*

$s_1 + s_2 - s \geq d(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0$ *and* $s_j - s > d(\frac{1}{p_j} - \frac{1}{p}),\ j = 1, 2$

or

$s_1 + s_2 - s > d(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}) \geq 0$ *and* $s_j - s \geq d(\frac{1}{p_j} - \frac{1}{p}),\ j = 1, 2,$

Then the bilinear form

$(u, v) \in W^{s_1,p_1}(\mathbb{R}^d) \times W^{s_2,p_2}(\mathbb{R}^d) \mapsto u.v \in W^{s,p}(\mathbb{R}^d)$ *is continuous .*

Iterative scheme

Keys of the proof
The pressure iterates form a Cauchy sequence :

$$\beta \, ||p_j^{m+1} - p_j^{n+1}||_{0,\Omega_j} \leq \sup_{\mathbf{v}_j \in \mathbf{X}_j} \frac{b_j(\mathbf{v}_j, p_j^{m+1} - p_j^{n+1})}{||\nabla \mathbf{v}||_{0,\Omega_j}} \leq$$

$$\leq C \sum_i \Big[||\nabla(\mathbf{u}_i^{m+1} - \mathbf{u}_i^{n+1})||_{0,\Omega_i} + ||\nabla(k_i^{m+1} - k_i^{n+1})||_{0,\Omega_i} +$$

$$+ \Big\| \, |\mathbf{u}_i^{m+1}|\mathbf{u}_i^{m+1} - |\mathbf{u}_i^{n+1}|\mathbf{u}_i^{n+1} \, \Big\|^2_{L^{3/2}(\Gamma)} \Big]$$

- So, $p_j^n \to p_j$ in $L^2(\Omega_j)$
- Then, we may pass to the limit in the equation of iterates.

Convergence theorem

Theorem (Convergence)

Under the conditions of the Contractiveness Theorem (ν large enough), the sequence

$$\{(\mathbf{u}_i^n, p_i^n, k_i^n)\}_n, \quad \textit{converges in} \quad \mathbf{X}_i \times L^2(\Omega_i) \times \bigcap_{q<d'} W^{1,q}(\Omega_i)$$

to a solution $(\mathbf{u}_i, p_i, k_i)$ of model problem.
In addition this problem admits a unique solution under these conditions.

Numerical tests

We have tested our iterative scheme for the data

- **Domains**
  - $\Omega_1 = [0,5] \times [0,1] \times [0,1]$ (Atmosphere),
  - $\Omega_2 = [0,5] \times [0,1] \times [0,-11]$ (Ocean).
- **Turbulent diffusions:** (Realistic values)
  - $\gamma_1(k_1) = 3 \cdot 10^{-3} + 0,277 \cdot 10^{-4}\sqrt{k_1}$,
  - $\gamma_2(k_2) = 3 \cdot 10^{-2} + 0,185 \cdot 10^{-5}\sqrt{k_2}$,
  - $\alpha_i = \gamma_i$.
- **Friction coefficients:** (Realistic values)
  - $c_i = 10^{-3}$, $\lambda = 5 \cdot 10^{-2}$.
- **Boundary data:**
  - $\mathbf{u} = (1,0,0)$ on the "lid" $y = 1$, $\mathbf{u} = (0,0,0)$ on the remaining of $\partial\Omega$.
  - $k_i = 0$ on $\partial\Omega$ (Only creation of TKE at interface Γ)

Numerical tests

- Discretization $Q1 - Q1$ for velocity-pressure stabilized by penalty on the pressure.
- Discretization $Q1$ for TKE.
- Computations with FreeFem3D (http://www.freefem.org/ff3d).
- 50x20x20 grid on each Ω_i

Results

- The algorithm converges to steady state with a linear rate, in agreement with theoretical expectations.
- Ocean at atmosphere flows behave as driven cavities.
- The TKE attempts its maximum values in a neighborhood of interface. Larger values in ocean.