A 3D Model for two coupled turbulent fluids: Numerical analysis of a finite element approximation

Tomas Chacon Rebollo[1], and Driss Yakoubi[2]

Abstract

This paper deals with the numerical analysis of a coupled two-fluid RANS turbulence model. We study a full finite element discretization of an iterative linearization procedure. We prove that the procedure is contracting for large enough eddy viscosities, whenever the discrete velocities are bounded in some $W^{1,p}$ norm for $p > 3$. We present some numerical tests where we study the accuracy of the procedure, and simulate a realistic flow in which an imposed wind in the upper atmosphere generates an upwelling in the oceanic flow.

1 Introduction

This paper deals with the numerical analysis of the approximation of RANS (Reynolds-Averaged Navier-Stokes) turbulence models, and more specifically on coupled two-fluid RANS models. Typically this model addresses the coupled atmosphere-ocean system, whose accurate numerical simulation is crucial to analyze the main issues related to climate change. From the practical point of view, RANS models are rather diffusive and provide overall predictions of many flows of engineering interest (Cf. Davidson [19]). However, from the mathematical point of view RANS models are more singular problems than the Navier-Stokes equations. The main mathematical difficulties of analysis of RANS models is that the source term (the production term) for the TKE (Turbulent Kinetic Energy) has just $L^1(\Omega)$ regularity, and then the equation for the TKE does not make sense in $H^{-1}(\Omega)$. Its solution must be understood in the renormalized –or entropy– sense (Cf. [2, 8, 7, 9, 10, 11]).

[1]Dpto. EDAN & IMUS, Universidad de Sevilla & LJLL, Université de Paris VI. Research partially funded by the Grant MTM2012-36124-C02-01 granted by the Spanish Ministerio de Economía y Competitividad and the FEDER EU Fund.

[2]GIREF. Dep. de mathématiques et de statistique. Université Laval. Québec. Canada
e-mails: chacon@us.es, yakoubi@giref.ulaval.ca

The numerical analysis of the finite element approximation of elliptic equations with r. h. s. in L^1 only has been performed for linear diffusion equations (Cf. Casado et al. [15]). The main reason is that the extension of the basic estimates of renormalized solutions (the Boccardo-Gallouet estimates) only hold if the numerical scheme satisfies a discrete maximum principle. For convection-diffusion equations there exist a few finite element schemes that satisfy this principle, based upon multi-dimensional unwinding techniques (Cf. [14, 18, 27]). However, the extension of the numerical analysis of these schemes to the solution of equations with L^1 r. h. s. has not been performed yet. For finite volume discretizations there exist more schemes satisfying the maximum principle, whose analysis has been performed in some cases.

Due to these difficulties, the mathematical and numerical analysis of RANS models performed up to date has applied to simplified equations. In [4] Bernardi et al. study a model for two-coupled turbulent fluids and its numerical approximation, where the TKE equation only contains eddy diffusion (with bounded eddy viscosities) and production term. This model also contains a non-linear friction term to model the interactions between the two fluids across the interface:

$$
(1.1)\quad \begin{cases}
-\nabla\cdot(\alpha_i(k_i)\nabla \boldsymbol{u}_i)+\nabla p_i &= \boldsymbol{f}_i \text{ in } \Omega_i,\\
\nabla\cdot\boldsymbol{u}_i &= 0 \quad \text{in } \Omega_i,\\
-\nabla\cdot(\gamma_i(k_i)\nabla k_i) &= \alpha_i(k_i)|\nabla\boldsymbol{u}_i|^2 \text{ in } \Omega_i,\\
\boldsymbol{u}_i &= \boldsymbol{0} \text{ on } \Gamma_i,\\
k_i &= 0 \text{ on } \Gamma_i,\\
\alpha_i(k_i)\partial_{\boldsymbol{n}_i}\boldsymbol{u}_i - p_i\boldsymbol{n}_i + \kappa_i(\boldsymbol{u}_i-\boldsymbol{u}_j)|\boldsymbol{u}_i-\boldsymbol{u}_j| &= \boldsymbol{0} \text{ on } \Gamma, 1\le i\ne j\le 2,\\
k_i &= \lambda|\boldsymbol{u}_1-\boldsymbol{u}_2|^2 \text{ on } \Gamma.
\end{cases}
$$

Here each triple $(\boldsymbol{u}_i, p_i, k_i)$ is defined in the domain Ω_i, $i = 1, 2$. The Ω_i ($i = 1, 2$) are bounded domains of $\mathbb{R}^d$, $d = 2, 3$, which are either convex or of class $\mathbb{C}^{1,1}$, with boundaries $\partial\Omega_i = \Gamma_i \cup \Gamma$, $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_1$ being the interface between the two fluids. Γ is assumed to be flat. This is the "rigid lid hypothesis" (introduced by K. Bryan in [13]), an hypothesis which is standard in oceanography for flows in large space scales. Each of the two turbulent fluids is modeled by a simplified one-equation turbulence model whose unknowns are the velocity $\boldsymbol{u}_i$, the pressure p_i and the turbulent kinetic energy (TKE) k_i. The generation of eddy viscosity in flow i is modeled by the term

$$-\nabla\cdot(\alpha_i(k_i)\nabla\boldsymbol{u}_i).$$

The (positive) quantity $\alpha_i(k_i)$ is the eddy viscosity. This is a simplification of the usual modeling of Reynolds Stress Tensor by

$$R_i \simeq -\alpha_i(k_i)\left(\nabla\boldsymbol{u}_i + \nabla^t\boldsymbol{u}_i\right)$$

to simplify the mathematical analysis. The equations for the TKE only contains the eddy dissipation and production terms, respectively given by

$$-\nabla \cdot (\gamma_i(k_i)\nabla k_i), \quad \text{and} \quad \alpha_i(k_i)|\nabla \boldsymbol{u}_i|^2,$$

where $\gamma_i(k_i)$ is the eddy diffusion for the TKE k_i. The interface terms model the friction between the two fluids (5th equation in (1.1)) and the generation of TKE (6th equation in (1.1)), both are wall laws that model the dissipation of turbulence in the boundary layers on both sides of the interface.

The analysis of model (1.1) reported in [4] is based upon the re-formulation of the equations for the TKEs by transposition combined with a compactness argument. The global system is proved to admit a solution $(\boldsymbol{u}_i, p_i, k_i) \in \boldsymbol{H}^1(\Omega_i) \times L^2(\Omega_i) \times H^{1/2}(\Omega_i)$ regularity. Several sub-sequent works have dealt with the numerical approximation of this model. The same authors perform in [5] an error analysis for spectral discretizations of (1.1) for smooth solutions. In [3] a finite element discretization by piecewise affine finite elements of this problem is studied by the compactness method. The convergence of a sub-sequence of discrete solutions to the continuous solution is proved. In [16] Chacón et al. introduce a linearization procedure to solve the same continuous model for large eddy viscosities. The procedure is proved to converge for large eddy viscosities, assuming that the velocities are smooth enough.

In the present paper we study a full finite element discretization of the iterative procedure introduced in [16]. We prove that the procedure converges under similar conditions, i. e., for large enough eddy viscosities, whenever the discrete velocities $\boldsymbol{u}_{ih}$ are bounded in some $W^{1,p}(\Omega_i)^d$ for $p > 3$. We prove that the sequence provided by the linearization procedure is contracting in H^1 norms for velocities and TKEs. The hardest technical point is the obtention of estimates of the interface quadratic terms. We treat them by specific discrete lifting operators that are compatible with the discretizations of velocity and TKE. We present some numerical tests where on one side we study the accuracy of the procedure. On another side we simulate a realistic flow in which an imposed wind in the upper atmosphere generates an upwelling in the oceanic flow.

The paper is structured as follows. Section 2 describes the iterative procedure introduced in [16], which is discretized by the finite element method in Section 3. Section 4 is devoted to the analysis of convergence of this discretization. Finally Section 5 reports the numerical tests.

2 Continuous iterative procedure

In this Section we recall the iterative procedure to solve problem (1.1) introduced in [16], that we shall discretize in the next section.

We start by recalling some standard notation that we use throughout the paper. We denote by $W^{s,p}(\Omega_i)$ the real Sobolev space, $0 \le s < \infty, 0 \le p \le \infty$, equipped with the norm $\| \cdot \|_{W^{s,p}(\Omega_i)}$. The space $W_0^{s,p}(\Omega_i)$ is the completion of the space of the smooth functions compactly supported in Ω_i with respect to the $\| \cdot \|_{W^{s,p}(\Omega_i)}$ norm. For $s = 1$ and $p = 2$, we denote the Hilbert spaces $W^{1,2}(\Omega_i)$ (resp. $W_0^{1,2}(\Omega_i)$) by $H^1(\Omega_i)$ (resp. $H_0^1(\Omega_i)$); The related norm is denoted by $\| \cdot \|_{1,\Omega_i}$. The case of $s = 0$ corresponds to the space $L^2(\Omega_i)$ equipped with its standard norm $\| \cdot \|_{0,\Omega_i}$. We finally denote by $| \cdot |_{1,\Omega}$ the semi norm in $H^1(\Omega_i)$ given by $|v|_{1,\Omega} = \|\nabla v\|_{0,\Omega}$.

To formulate the coupled problem (1.1) in variational form, we introduce the velocity space defined as follows:

$$\boldsymbol{X}_i = \{\boldsymbol{v}_i \in \boldsymbol{H}^1(\Omega_i), \quad \boldsymbol{v}_i = \boldsymbol{0} \,\text{on}\, \Gamma_i\}. \tag{2.1}$$

Thanks to Lions et al [25], the traces of functions in $\boldsymbol{X}_i$ on Γ belong to $\boldsymbol{H}_{00}^{\frac{1}{2}}(\Gamma)$. The special space $\boldsymbol{H}_{00}^{\frac{1}{2}}(\Gamma)$ is the subspace of $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ formed by the functions whose extension by $\boldsymbol{0}$ to $\partial\Omega_i$ belongs to $\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega_i)$. It is a Hilbert space endowed with the norm (see Adams [1], Theorem 7.48)

$$\begin{aligned}(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{H}_{00}^{\frac{1}{2}}(\Gamma)} &= \int_\Gamma \boldsymbol{u}(\boldsymbol{x})\, \boldsymbol{v}(\boldsymbol{x}) d\boldsymbol{x} + \int_\Gamma \frac{\boldsymbol{u}(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})}{d(\boldsymbol{x}, \partial\Gamma)} d\boldsymbol{x} \\ &\quad + \int_\Gamma \int_\Gamma \boldsymbol{u}(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})\, \frac{(\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}(\boldsymbol{y}))\,(\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{y}))}{|\boldsymbol{x} - \boldsymbol{y}|^d} d\boldsymbol{x} d\boldsymbol{y}.\end{aligned} \tag{2.2}$$

We next state a crucial hypothesis concerning the eddy viscosities and diffusions:

Hypothesis 1 The functions α_i and γ_i belong to $W^{1,\infty}(\mathbb{R}_+)$. Moreover there exist positive constants δ_1 and ν such that

$$\forall \ell \in \mathbb{R}_+, \qquad \nu \le \alpha_i^{(m)}(\ell) \le \delta_1, \quad \nu \le \gamma_i^{(m)}(\ell) \le \delta_1, \quad m = 0 \text{ or } 1.$$

This hypothesis is applied in practice, as some kind of numerical smoothing or truncation is applied to avoid too small or too large eddy viscosities that may lead to instabilities.

In order to write the weak formulation of the TKE, we are led to introduce two real numbers r and r' such that

$$\frac{1}{r} + \frac{1}{r'} = 1, \quad \text{where } r > d. \tag{2.3}$$

Now, we are in a position to write the weak formulation of problem (1.1). Let $\boldsymbol{f}_i \in L^2(\Omega_i)^d$.

Find the triplet $(\boldsymbol{u}_i, p_i, k_i) \in \boldsymbol{X}_i \times L^2(\Omega_i) \times W^{1,r'}(\Omega_i)$ such that $\forall\, (\boldsymbol{v}_i, q_i, \varphi_i) \in \boldsymbol{X}_i \times L^2(\Omega_i) \times W_0^{1,r}(\Omega_i)$,

(2.4)

$$\mathbf{1}: \quad a_i(k_i; \boldsymbol{u}_i, \boldsymbol{v}_i) + b_i(\boldsymbol{v}_i, p_i) + \kappa_i \int_\Gamma |\boldsymbol{u}_i - \boldsymbol{u}_j|(\boldsymbol{u}_i - \boldsymbol{u}_j) \cdot \boldsymbol{v}_i d\tau = \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_i \, d\boldsymbol{x}$$

$$b_i(\boldsymbol{u}_i, q_i) = 0.$$

$$\mathbf{2}: \quad k_i = 0 \text{ on } \Gamma_i, \quad k_i = \lambda |\boldsymbol{u}_i - \boldsymbol{u}_j|^2 \quad \text{on } \Gamma, \quad \text{and}$$

$$\mathcal{N}_i(k_i; k_i, \varphi_i) = \int_{\Omega_i} \alpha_i(k_i) |\nabla \boldsymbol{u}_i|^2 \, \varphi_i \, d\boldsymbol{x}. \tag{2.5}$$

Where the forms $a_i(\cdot;\cdot,\cdot)$, $b_i(\cdot,\cdot)$ and $\mathcal{N}_i(\cdot;\cdot,\cdot)$ are defined by

$$a_i(\ell_i; \boldsymbol{u}_i, \boldsymbol{v}_i) = \int_{\Omega_i} \alpha(\ell_i) \nabla \boldsymbol{u}_i : \nabla \boldsymbol{v}_i \, d\boldsymbol{x}, \quad b_i(\boldsymbol{v}_i, q_i) = -\int_{\Omega_i} q_i \, \nabla \cdot \boldsymbol{v}_i \, d\boldsymbol{x},$$

$$\text{and} \quad \mathcal{N}_i(\ell_i; k_i, \varphi_i) = \int_{\Omega_i} \gamma_i(\ell_i) \nabla k_i \cdot \nabla \varphi_i \, d\boldsymbol{x}.$$

Since $\boldsymbol{u}_i \in \boldsymbol{X}_i$ then its trace on $\partial\Omega_i$ belongs to $\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega_i)$. By the Sobolev embedding from $\boldsymbol{H}^{\frac{1}{2}}(\partial\Omega_i)$ into $L^3(\partial\Omega_i)^d$, we conclude that the integral $\int_\Gamma |\boldsymbol{u}_i - \boldsymbol{u}_j| \, (\boldsymbol{u}_i - \boldsymbol{u}_j) \cdot \boldsymbol{v}_i \, d\tau$ is well defined. Also, as $\alpha_i(k_i) \, |\nabla \boldsymbol{u}_i|^2 \in W^{1,r'}(\Omega_i)$ when $\boldsymbol{u}_i \in \boldsymbol{X}_i$, then the term $\int_{\Omega_i} \alpha_i(k_i) |\nabla \boldsymbol{u}_i|^2 \, \varphi_i \, d\boldsymbol{x}$ is well defined.

In Lewandowski [24] it is proved that this formulation admits at least a solution. This proof is based upon a fixed point iteration, as follows:

Continuous algorithm: For $n \in \mathbb{N}$, and for a given $(\boldsymbol{u}_i^n, p_i^n, k_i^n) \in \boldsymbol{X}_i \times L^2(\Omega_i) \times W^{1,r'}(\Omega_i)$, find $\left(\boldsymbol{u}_i^{n+1}, p_i^{n+1}, k_i^{n+1}\right) \in \boldsymbol{X}_i \times L^2(\Omega_i) \times W^{1,r'}(\Omega_i)$ by:

1. For $n \geq 0$, knowing $(\boldsymbol{u}_i^n, p_i^n, k_i^n)$, find $\left(\boldsymbol{u}_i^{n+1}, p_i^{n+1}\right) \in \boldsymbol{X}_i \times L^2(\Omega_i)$ solution of

$$a_i\left(k_i^n; \boldsymbol{u}_i^{n+1}, \nabla \boldsymbol{v}_i\right) + b_i\left(\boldsymbol{v}_i, p_i^{n+1}\right) + \kappa_i \int_\Gamma \left|\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}\right| \left(\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}\right) \cdot \boldsymbol{v}_i d\tau = \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_i, \quad \forall \boldsymbol{v}_i \in \boldsymbol{x}_i. \tag{2.6}$$

$$\forall q_i \in L^2(\Omega_i), \quad b_i(\boldsymbol{u}_i^{n+1}, q_i) = 0, \tag{2.7}$$

2. Next, knowing $\left(\boldsymbol{u}_i^{n+1}, k_i^n\right)$, compute $k_i^{n+1} \in W^{1,r'}(\Omega_i)$ solution of

$$k_i^{n+1} = 0 \text{ on } \Gamma_i, \tag{2.8}$$

$$k_i^{n+1} = \lambda|\boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1}|^2 \text{ on } \Gamma, \quad \text{and} \tag{2.9}$$

$$\mathcal{N}_i(k_i^n; k_i^{n+1}, \varphi_i) = \int_{\Omega_i} \alpha_i(k_i^n)|\nabla \boldsymbol{u}_i^{n+1}|^2 \, \varphi_i \, d\boldsymbol{x}, \quad \forall \varphi_i \in W^{1,r'}(\Omega_i) \tag{2.10}$$

In [16], T. Chacon et al. it is proved that this continuous iterative scheme (2.6)–(2.10) is contracting for large enough eddy viscosities:

Theorem 2.1 (Convergence of the continuous scheme) *Assume that Hypothesis 1 holds and that $\boldsymbol{f}_i \in L^2(\Omega_i)^d$. Then if ν is large enough there exists a positive constant $K < 1$, depending only on Ω_i, M and on the data κ_i, α_i and λ, such that for all $n \in \mathbb{N}^*$,*

$$\begin{aligned} &\sum_{i=1}^{2} \left|\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^n\right|_{1,\Omega_i}^2 \leq K \sum_{i=1}^{2} \left|k_i^n - k_i^{n-1}\right|_{1,\Omega_i}^2, \quad \textit{and} \\ &\sum_{i=1}^{2} \left|k_i^{n+1} - k_i^n\right|_{1,\Omega_i}^2 \leq K \sum_{i=1}^{2} \left|k_i^n - k_i^{n-1}\right|_{1,\Omega_i}^2. \end{aligned} \tag{2.11}$$

3 Discrete iterative procedure

In view of the discretization of problem (1.1), we assume that the domains Ω_1 and Ω_2 are polygonal (when $d = 2$) or polyhedric (when $d = 3$). As a standard for Stokes problems, we consider pairs of finite element spaces $(\boldsymbol{X}_{i,h}, M_{i,h}) \subset \boldsymbol{X}_i \times M_i$ such that the familys $(\boldsymbol{X}_{i,h}, M_{i,h})_{h>0}$, for $i = 1, 2$ satisfy the discrete Babuska-Brezzi inf-sup condition on Ω_i, see for instance [12]:

There exists a constant $\beta_{i,h} > 0$, such that:

$$\forall q_{i,h} \in M_{i,h}, \qquad \sup_{\boldsymbol{v}_{i,h} \in \boldsymbol{X}_{i,h}} \frac{b_i\left(\boldsymbol{v}_{i,h}, q_{i,h}\right)}{|\boldsymbol{v}_{i,h}|} \geq \beta_{i,h} \|q_{i,h}\|_{0,\Omega_i} \tag{3.1}$$

We also consider the finite element discrete spaces of energies $K_{i,h} \subset W^{1,r'}(\Omega_i)$. We assume, as is standard in finite element approximation, that there exist interpolation operators

$$\Pi_{i,h}: \quad \boldsymbol{X}_i \cap C^0(\overline{\Omega_i}) \mapsto \boldsymbol{X}_{i,h}, \quad \text{and} \tag{3.2}$$

$$\mathcal{S}_{i,h}: \quad W^{1,r'}(\Omega_i) \mapsto K_{i,h}, \tag{3.3}$$

that satisfy the following approximation and stability properties

$$\|\boldsymbol{v} - \Pi_{i,h}\,\boldsymbol{v}\|_{W^{1,q}(\Omega)^d} \leq c h^{\ell_1 - 1 - \frac{d}{p} + \frac{d}{q}}\, |\boldsymbol{v}|_{W^{1,p}(\Omega)^d}, \quad \forall\, 0 \leq \ell_1 \leq s_1 + 1 \tag{3.4}$$

$$\|k - \mathcal{S}_{i,h}\,k\|_{W^{1,q}(\Omega)} \leq c h^{\ell_2 - 1 - \frac{d}{p} + \frac{d}{q}}\, |k|_{W^{1,p}(\Omega)}, \quad \forall\, 0 \leq \ell_2 \leq s_2 + 1, \tag{3.5}$$

for some nonnegative integers s_1 and s_2.

In addition we consider a pair of boundary finite element spaces $W_{i,h} \subset H^{1/2}_{00}(\Gamma)$ that we shall use to interpolate the boundary condition for the k_i on Γ. We assume that there exists an interpolation operator

$$\mathcal{L}_{i,h}: \quad H^{\frac{1}{2}}_{00}(\Gamma) \cap C^0(\overline{\Gamma}) \longrightarrow W_{i,h} \tag{3.6}$$

that satisfies the stability property

$$\|\mathcal{L}_{i,h} w_i\|_{H^{1/2}_{00}(\Gamma)} \leq C\, \|w_i\|_{H^{1/2}_{00}(\Gamma)}, \text{ for any } w_i \in H^{1/2}_{00}(\Gamma), \tag{3.7}$$

for some constant $C > 0$. We also assume that the operators $\mathcal{L}_{i,h}$ and $\mathcal{S}_{i,h}$ satisfy the following compatibility condition,

Hypothesis 2. For all $w_i \in W^{1,r'}(\Omega_i) \cap H^1(\Omega_i)$, the trace on Γ of the interpolate $\mathcal{S}_{i,h}\,(w_i)$ coincides with the interpolate of the trace of w_i:

$$\mathcal{L}_{i,h}\left(w_{i_{|\Gamma}}\right) = (\mathcal{S}_{i,h}\, w_i)\,|_{\Gamma}. \tag{3.8}$$

We finally assume that there exists a lifting operator

$$\mathcal{R}_{i,h} : W_{i,h} \mapsto K_{i,h}$$

such that,

$$\mathcal{R}_{i,h}(\varphi_{i,h})_{|\partial\Omega_i} = \varphi_{i,h}, \text{ for any } \varphi_{i,h} \in W_{i,h}. \tag{3.9}$$

Furthermore, this operator verifies the stability property: for all real number p, $1 < p < +\infty$,

$$\|\mathcal{R}_{i,h}(\varphi_{i,h})\|_{W^{1,p}(\Omega_i)} \;\leq\; C\, \|\varphi_{i,h}\|_{W^{1-1/p,p}(\partial\Omega_i)} \tag{3.10}$$

for some constant $C > 0$.

The standard finite element spaces satisfy these properties. Indeed, assume that the Ω_i are polygonal (when $d = 2$) or polyhedric (when $d = 3$). Consider a family of triangular grids $(\mathcal{T}_{i,h})_h$ of $\overline{\Omega}_i$ that we assume to be regular, in the usual sense of the finite element method (see Ciarlet [17] or Girault-Raviart[20])

We make an additional assumption concerning the compatibility of the grids at the interface:

$$\mathcal{T}_{1,h}|_\Gamma = \mathcal{T}_{2,h}|_\Gamma.$$

For each nonnegative integer m and any element K in $\mathcal{T}_{i,h}$, let $\mathcal{P}_m(K)$ denote the space of restriction to K of polynomials with d variables and total degree $\leq m$. Thus we choose the following spaces (for $s_1, s_2 \in \mathbb{N}$)

$$(3.11)\quad \boldsymbol{X}_{i,h} = \left\{ \boldsymbol{v}_{i,h} \in C^0\left(\overline{\Omega}_i\right)^d,\ \forall K \in \mathcal{T}_{i,h},\ \boldsymbol{v}_{i,h\,|K} \in \mathcal{P}_{s_1+1}(K)^d \right\} \cap \boldsymbol{X}_i$$

$$(3.12)\quad M_{i,h} = \left\{ q_{i,h} \in L^2\left(\Omega_i\right),\ \forall K \in \mathcal{T}_{i,h},\ q_{i,h\,|K} \in \mathcal{P}_{s_1}(K) \right\} \text{ and}$$

$$(3.13)\quad K_{i,h} = \left\{ \ell_{i,h} \in C^0\left(\overline{\Omega}_i\right),\ \forall K \in \mathcal{T}_{i,h},\ \ell_{i,h}|_K \in \mathcal{P}_{s_2+1}(K),\ \ell_{i,h}|_{\Gamma_i} = 0 \right\},$$

$$(3.14)\quad W_{i,h} = \left\{ \varphi_{i,h} \in \mathcal{C}^0\left(\Gamma\right);\ \forall e \in \mathcal{E}_{i,h},\ \varphi_{i,h\,|e} \in \mathcal{P}_{s_2+1}(e),\ \varphi_{i,h\,|\partial\Gamma} = 0 \right\},$$

where $\left(\mathcal{E}_{i,h}\right)_{i,h}$ denotes all faces ($d = 3$) or edges ($d = 2$) of triangulation $\mathcal{T}_{i,h}$, which are contained in $\partial\Omega_i$. As interpolation operators $\Pi_{i,h}$, $\mathcal{S}_{i,h}$ and $\mathcal{L}_{i,h}$ we choose the standard Lagrange interpolation operators; i. e., $\forall K \in \mathcal{T}_{i,h}$,

$$\Pi_{i,h}\,\boldsymbol{v}|_K = \mathcal{I}_1|_K\,\boldsymbol{v},\ \forall \boldsymbol{v} \in C^0\left(\overline{\Omega}_i\right)^d,\quad \mathcal{S}_{i,h}\,\ell|_K = \mathcal{I}_2|_K\,\ell,\ \forall \ell \in C^0\left(\overline{\Omega}_i\right).$$

where $\mathcal{I}_1|_K\,\boldsymbol{v}$ (resp. $\mathcal{I}_2|_K\,\ell$) is the only polynomial of $\mathcal{P}_{s_1+1}(K)^d$ (resp., $\mathcal{P}_{s_2+1}(K)$) that has the same degrees of freedom as the function $\boldsymbol{v}$ (resp., ℓ). Furthermore, the discrete inf-sup condition (3.1) holds for the family space $(\boldsymbol{X}_{i,h}, M_{i,h})$. Similarly,

$$\mathcal{L}_{i,h}\,\omega|_e = \mathcal{I}_3|_e\,\omega,\ \forall \omega \in C^0\left(\overline{\Gamma}\right)^d,$$

where $\mathcal{I}_3|_e\,\omega$ is the only polynomial of $\mathcal{P}_{s_1+1}(K)^d$ (resp., $\mathcal{P}_{s_2+1}(e)$) that has the same degrees of freedom as the function ω.

Owing to [20] (Chap.2. Lemma 2.13), operators $\Pi_{i,h}$, $\mathcal{S}_{i,h}$ and $\mathcal{L}_{i,h}$ satisfy the estimates (3.4), (3.5) and (3.7).

Hypothesis 2 is trivially verified by these spaces and interpolation operators, as $W_{i,h}$ is the "trace space"on Γ of $K_{i,h}$. Finally, the existence of a lifting operator verifying (3.9) and (3.10) is proved in Bernardi *et al* [6], Theorem 4.1

We are now in a position to built the discrete problem from (2.6)-(2.10), by the Galerkin method. It reads:

For any $n \geq 0$, given $\left(\boldsymbol{u}_{i,h}^{n}, p_{i,h}^{n}, k_{i,h}^{n}\right) \in \boldsymbol{X}_{i,h} \times M_{i,h} \times K_{i,h}$ $i = 1, 2$,

1. $\forall\, (\boldsymbol{v}_{i,h}, q_{i,h}) \in \boldsymbol{X}_{i,h}$, compute $\left(\boldsymbol{u}_{i,h}^{n+1}, p_{i,h}^{n+1}\right)$ solution of

$$
\begin{aligned}
(3.15)\quad & a_i\left(k_{i,h}^{n}; \boldsymbol{u}_{i,h}^{n+1}, \nabla \boldsymbol{v}_{i,h}\right) + b_i\left(\boldsymbol{v}_{i,h}, p_{i,h}^{n+1}\right) \\
& + \kappa_i \int_{\Gamma} \left|\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}\right| \left(\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}\right) \cdot \boldsymbol{v}_{i,h}\, d\tau \quad = \int_{\Omega_i} \boldsymbol{f}_i \cdot \boldsymbol{v}_{i,h},
\end{aligned}
$$

$$
(3.16)\qquad b_i\left(\boldsymbol{u}_{i,h}^{n+1}, q_{i,h}\right) = 0,
$$

2. Knowing $\left(\boldsymbol{u}_{i,h}^{n+1}, k_{i,h}^{n}\right)$, $\forall \varphi_{i,h} \in K_{i,h}^{0} = K_{i,h} \cap W_0^{1,r}(\Omega_i)$, compute $k_{i,h}^{n+1}$ solution of

$$
(3.17)\qquad k_{i,h}^{n+1} = 0 \text{ on } \Gamma_i,
$$

$$
(3.18)\qquad k_{i,h}^{n+1} = \lambda\, \mathcal{L}_{i,h}\left(|\boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1}|^2\right) \text{ on } \Gamma, \quad \text{and}
$$

$$
(3.19)\qquad \mathcal{N}_i(k_{i,h}^{n}; k_{i,h}^{n+1}, \varphi_{i,h}) = \int_{\Omega_i} \alpha_i(k_{i,h}^{n}) |\nabla \boldsymbol{u}_{i,h}^{n+1}|^2\, \varphi_{i,h}\, d\boldsymbol{x}.
$$

We are now in position to derive the main result of the next section, namely the convergence of the discrete scheme (3.15)–(3.19) to the solution of the initial problem (1.1).

4 Numerical analysis of the discrete scheme

In this section we prove that discrete procedure (3.15)–(3.19) defined by finite element method converges. Its rather technical proof requires some preliminary results and based on several steps. The purpose is to recursively estimate the errors between the finite element sequence $\left(\boldsymbol{u}_{i,h}^{n}, k_{i,h}^{n},\right)$ and the continuous sequence $(\boldsymbol{u}_i^n, k_i^n,)$. These estimates prove that the discrete scheme is contracting, and that it converges as h tends to 0 and the iteration number n tends to infinity. The estimates for the pressure error are obtained as a consequence of the previous ones.

4.1 Analysis of the velocity sequence $\boldsymbol{u}_{i,h}^{n}$

It is readily checked that the velocities sequences $\boldsymbol{u}_i^n$ and $\boldsymbol{u}_{i,h}^n$ are bounded, see [16] in the continuous case

Proposition 4.1 *Assume that Hypothesis 1 holds and that $\boldsymbol{f}_i \in L^2(\Omega_i)^d$. Then, there exists a non negative constant c depending only on the domain Ω_i and the friction coefficient κ_i, such that far all $n \in \mathbb{N}$:*

$$\sum_{i=1}^{2} \left|\boldsymbol{u}_{i,h}^n\right|_{1,\Omega_i}^2 \leq \frac{c}{\nu^2} \sum_{i=1}^{2} \|\boldsymbol{f}_i\|^2 . \tag{4.1}$$

To prove the convergence of the discrete scheme, the idea is to estimate the difference between the continuous and discrete sequences, $\left|\boldsymbol{u}_{i,h}^n - \boldsymbol{u}_i^n\right|_{1,\Omega_i}$, $\left|k_{i,h}^n - k_i^n\right|_{1,\Omega_i}$ and $\left\|p_{i,h}^n - p_i^n\right\|$.

From now on, we assume the additional hypothesis on the velocities and TKE sequences $\boldsymbol{u}_i^n$, k_i^n:

Hypothesis 2: $\forall n \in \mathbb{N}$, we suppose that the velocity $\boldsymbol{u}_i^n$ belong to $W^{1,3+\varepsilon}(\Omega_i)^d$ and the energy sequence k_i^n belong to $W^{1,3}(\Omega_i)$, and one has

$$\|\boldsymbol{u}_i^n\|_{W^{1,3+\varepsilon}(\Omega_i)^d} \leq M, \quad \|k_i^n\|_{W^{1,3}(\Omega_i)} \leq M,$$

$$\|\boldsymbol{u}_{i,h}^n\|_{W^{1,3+\varepsilon}(\Omega_i)^d} \leq M, \quad \|k_{i,h}^n\|_{W^{1,3}(\Omega_i)} \leq M,$$

where M and ε are two positive numbers. Thus, applying the interpolation inequalities (3.4)-(3.5), we obtain for $\ell_1 = s_1 + 1$, $\ell_2 = s_2$, $d = 3, q = 2$ and $p = 3$ or $= 3 + \varepsilon$

$$|\boldsymbol{u}_i^n - \Pi_{i,h}\, \boldsymbol{u}_i^n|_{1,\Omega_i} \leq c h^{s_1 + \frac{3}{2} - \frac{3}{3+\varepsilon}}\, |\boldsymbol{u}_i^n|_{W^{1,3+\varepsilon}(\Omega_i)^d}, \tag{4.2}$$

$$|k_i^n - \mathcal{S}_{i,h}\, k_i^n|_{1,\Omega_i} \leq c h^{s_2 + \frac{1}{2}}\, |k_i^n|_{W^{1,3}(\Omega_i)}. \tag{4.3}$$

Remark 4.2 *Note that the natural estimates for velocities in model* (1.1) *are in H^1 norm, not in $W^{1,3}$ norm. Indeed the Proposition 4.1 and the result* (4.1) *shows that these norms are bounded in H^1. However we need a somewhat better regularity and bounds in our analysis.*

In what follows, for the sake of simplicity and clarity, we introduce a real number s equal to $min\{s_1, s_2\}$. Thus, all error estimates will be expressed by using this new parameter. The following Theorem shows the natural dependance between the norms of velocities and energies:

Theorem 4.3 *Assume that Hypotheses 1-2 hold and that $\boldsymbol{f}_i \in L^2(\Omega_i)^d, i = 1, 2$. Then there exists positive constants C_1, C_2 and C_3 depending only on*

$\Omega_i, \kappa_i, \alpha_i, \gamma_i$ *and* $\boldsymbol{f}_i$ *such that:*

$$\sum_{i+1}^{2} \left| \boldsymbol{u}_{i,h}^{n+1} - \Pi_{i,h}\, \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i}^2 \leq C_1 h^\sigma + C_2 \sum_{i+1}^{2} \left| k_{i,h}^n - k_i^n \right|_{1,\Omega_i}^2 , \tag{4.4}$$

$$\text{where} \quad \sigma = s + \frac{3}{2} - \frac{3}{3+\varepsilon} > s + \frac{1}{2}, \quad \forall \varepsilon > 0. \tag{4.5}$$

Proof. At step $n+1$, we take $\boldsymbol{v}_{i,h} = \boldsymbol{e}i, h^{n+1} = \frac{1}{\kappa_i} \left(\boldsymbol{u}_{i,h}^{n+1} - \Pi_{i,h}\, \boldsymbol{u}_i^{n+1} \right) \in \boldsymbol{X}_{i,h}$ as a test function in (2.6) and in (3.15). Then, calculating the difference between both obtained equations, and summing on $i = 1, 2$, yields

$$\underbrace{\sum_{i=1}^{2} \int_{\Omega_i} \alpha_i(k_{i,h}^n) \nabla \left(\boldsymbol{u}_{i,h}^{n+1} - \Pi_{i,h}\, \boldsymbol{u}_i^{n+1} \right) : \nabla \boldsymbol{e}_{i,h}^{n+1}}_{I_1}$$

$$+ \underbrace{\sum_{i=1}^{2} \frac{1}{\kappa_i} \int_{\Omega_i} \alpha_i(k_{i,h}^n) \nabla \left(\Pi_{i,h}\, \boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^{n+1} \right) : \nabla \boldsymbol{e}_{i,h}^{n+1}}_{I_2}$$

$$+ \underbrace{\sum_{i=1}^{2} \frac{1}{\kappa_i} \int_{\Omega_i} \left(\alpha_i(k_{i,h}^n) - \alpha_i(k_i^n) \right) \nabla \boldsymbol{u}_i^{n+1} : \nabla \boldsymbol{e}_{i,h}^{n+1}}_{I_3}$$

$$+ \int_\Gamma \left[\left| \boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1} \right| \left(\boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1} \right) - \left| \boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1} \right| \left(\boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1} \right) \right] \underbrace{\cdot \left[\left(\boldsymbol{u}_1^{n+1} - \Pi_{1,h}\, \boldsymbol{u}_1^{n+1} \right) - \left(\boldsymbol{u}_2^{n+1} - \Pi_{2,h}\, \boldsymbol{u}_2^{n+1} \right) \right]}_{I_4}$$

$$+ \int_\Gamma \left[\left| \boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1} \right| \left(\boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1} \right) - \left| \boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1} \right| \left(\boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1} \right) \right] \underbrace{\cdot \left[\left(\boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1} \right) - \left(\boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1} \right) \right]}_{I_5}$$

$$= 0 \ (I_1 + I_2 + I_3 + I_4 + I_5) .$$

Thanks to the inequality $(|\boldsymbol{b}|\boldsymbol{b} - |\boldsymbol{a}|\boldsymbol{a}) \cdot (\boldsymbol{b} - \boldsymbol{a}) \geq 0$, $\forall\, (\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^d$, we deduce that $I_5 \geq 0$. Consequently

$$|I_1| \leq |I_2| + |I_3| + |I_4|. \tag{4.6}$$

Estimation of I_1: It comes from Hypothesis 1 and from the definition of vector $\boldsymbol{e}_{i,h}^{n+1}$ that

(4.7)
$$|I_1| = \sum_{i=1}^{2} \int_{\Omega_i} \alpha_i(k_{i,h}^n) \nabla \left(\boldsymbol{u}_{i,h}^{n+1} - \Pi_{i,h}\, \boldsymbol{u}_i^{n+1} \right) : \nabla \boldsymbol{e}_{i,h}^{n+1} \;\geq\; \frac{\nu}{c_M} \sum_{i=1}^{2} |\boldsymbol{e}_{i,h}^{n+1}|^2_{1,\Omega_i}.$$

$$\begin{aligned}
\textbf{Estimation of } I_2 = \;& \sum_{i=1}^{2} \frac{1}{\kappa_i} \int_{\Omega_i} \alpha_i(k_{i,h}^n) \nabla \left(\Pi_{i,h}\, \boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^{n+1} \right) : \nabla \boldsymbol{e}_{i,h}^{n+1} \\
& \leq \frac{\delta}{c_m} \sum_{i=1}^{2} \left|\Pi_{i,h}\, \boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^{n+1}\right|_{1,\Omega_i} \left|\boldsymbol{e}_{i,h}^{n+1}\right|_{1,\Omega_i} \\
& \leq \frac{4\delta_1^2 c_M}{c_m^2 \nu} \sum_{i=1}^{2} \left|\Pi_{i,h}\, \boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^{n+1}\right|^2_{1,\Omega_i} + \frac{\nu}{4c_M} \sum_{i=1}^{2} \left|\boldsymbol{e}_{i,h}^{n+1}\right|^2_{1,\Omega_i}.
\end{aligned} \tag{4.8}$$

Estimation of I_3: It comes from Hypothesis 2 that $\nabla \boldsymbol{u}_i^{n+1} \in L^3\left(\Omega_i\right)^d$ and that $\|\nabla \boldsymbol{u}_i^{n+1}\|_{L^3(\Omega_i)^d} \leq M$. Furthermore, according to the second relation of Hypothesis 1 and the canonical injection from $H^1\left(\Omega_i\right)$ to $L^3\left(\Omega_i\right)$ and Hölder inequality, there exists a positive constant $c = c(\Omega_i)$ such that

$$\begin{aligned}
I_3 &= \sum_{i=1}^{2} \frac{1}{\kappa_i} \int_{\Omega_i} \left(\alpha_i(k_{i,h}^n) - \alpha_i(k_i^n) \right) \nabla \boldsymbol{u}_i^{n+1} : \nabla \boldsymbol{e}_{i,h}^{n+1} \\
&\leq \frac{\delta_2}{c_m} \sum_{i=1}^{2} \|k_{i,h}^n - k_i^n\|_{L^6} \, \|\nabla \boldsymbol{u}_i^{n+1}\|_{L^3} \left|\boldsymbol{e}_{i,h}^{n+1}\right|_{1,\Omega_i} \\
&\leq \frac{Mc\delta_2}{c_m} \sum_{i=1}^{2} \left|k_{i,h}^n - k_i^n\right|_{1,\Omega_i} \left|\boldsymbol{e}_{i,h}^{n+1}\right|_{1,\Omega_i} \\
&\leq \frac{2\, M^2\, c^2\, \delta_2^2\, c_M}{c_m^2\, \nu} \sum_{i=1}^{2} \left|k_{i,h}^n - k_i^n\right|^2_{1,\Omega_i} + \frac{\nu}{4c_M} \sum_{i=1}^{2} \left|\boldsymbol{e}_{i,h}^{n+1}\right|^2_{1,\Omega_i}.
\end{aligned} \tag{4.9}$$

Estimation of I_4: Using Hölder inequality, the continuity of the injection from $\boldsymbol{H}^{1/2}(\Gamma)$ to $L^3(\Gamma)^d$ and the trace operator from $\boldsymbol{H}^1(\Omega_i)$ to $\boldsymbol{H}^{1/2}(\Gamma)$, we deduce that there exists a positive constant $c = c(\Omega_i)$ such that:

$$
\begin{aligned}
I_4 = \int_\Gamma (& \underbrace{\left| \boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1} \right|}_{\in L^3(\Gamma)} \left(\boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1} \right) - \left| \boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1} \right| \overbrace{\left(\boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1} \right)}^{\in L^3(\Gamma)}) \\
& \cdot \underbrace{\left[\left(\boldsymbol{u}_1^{n+1} - \Pi_{1,h}\, \boldsymbol{u}_1^{n+1} \right) - \left(\boldsymbol{u}_2^{n+1} - \Pi_{2,h}\, \boldsymbol{u}_2^{n+1} \right) \right]}_{\in L^3(\Gamma)} \\
\le & \left(\left\| \boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1} \right\|_{L^3(\Gamma)}^2 + \left\| \boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1} \right\|_{L^3(\Gamma)}^2 \right) \\
& \left(\left\| \boldsymbol{u}_1^{n+1} - \Pi_{1,h}\, \boldsymbol{u}_1^{n+1} \right\|_{L^3(\Gamma)} + \left\| \boldsymbol{u}_2^{n+1} - \Pi_{2,h}\, \boldsymbol{u}_2^{n+1} \right\|_{L^3(\Gamma)} \right) \\
\le & \, c \left(\left\| \boldsymbol{u}_{1,h}^{n+1} - \boldsymbol{u}_{2,h}^{n+1} \right\|_{\boldsymbol{H}^{1/2}(\Gamma)}^2 + \left\| \boldsymbol{u}_1^{n+1} - \boldsymbol{u}_2^{n+1} \right\|_{\boldsymbol{H}^{1/2}(\Gamma)}^2 \right) \\
& \left(\left\| \boldsymbol{u}_1^{n+1} - \Pi_{1,h}\, \boldsymbol{u}_1^{n+1} \right\|_{\boldsymbol{H}^{1/2}(\Gamma)} + \left\| \boldsymbol{u}_2^{n+1} - \Pi_{2,h}\, \boldsymbol{u}_2^{n+1} \right\|_{\boldsymbol{H}^{1/2}(\Gamma)} \right) \\
\le & \, c \left(|\boldsymbol{u}_{1,h}^{n+1}|_{1,\Omega_i}^2 + |\boldsymbol{u}_{2,h}^{n+1}|_{1,\Omega_i}^2 + |\boldsymbol{u}_1^{n+1}|_{1,\Omega_i}^2 + |\boldsymbol{u}_2^{n+1}|_{1,\Omega_i}^2 \right) \\
& \left(\left| \boldsymbol{u}_1^{n+1} - \Pi_{1,h}\, \boldsymbol{u}_1^{n+1} \right|_{1,\Omega_i} + \left| \boldsymbol{u}_2^{n+1} - \Pi_{1,h}\, \boldsymbol{u}_2^{n+1} \right|_{1,\Omega_i} \right).
\end{aligned}
$$

Thanks to relation (4.1), we deduce

$$
I_4 \le \frac{c}{\nu^2} \sum_{i=1}^{2} \|\boldsymbol{f}_i\|^2 \sum_{i=1}^{2} \left| \boldsymbol{u}_i^{n+1} - \Pi_{i,h}\, \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i}. \tag{4.10}
$$

Combining (4.6) with estimates (4.7),(4.8),(4.9) and (4.10), yielding

$$
\sum_{i=1}^{2} |\boldsymbol{e}_{i,h}|_{1,\Omega_i}^2 \le \sum_{i=1}^{2} C_1 \left(\left| \boldsymbol{u}_i^{n+1} - \Pi_{i,h}\, \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i}^2 + \left| \boldsymbol{u}_i^{n+1} - \Pi_{i,h}\, \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i} \right) + C_2 \left| k_{i,h}^n - k_i^n \right|_{1,\Omega_i}^2.
$$

Thanks to the accuracy properties (4.2) of $\Pi_{i,h}$ and Hypothesis 2, we write

$$
\sum_{i=1}^{2} \left| \boldsymbol{u}_i^{n+1} - \Pi_{i,h}\, \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i} \le cM\, h^{s+\frac{3}{2}-\frac{3}{3+\varepsilon}}, \tag{4.11}
$$

where the $c > 0$ depend only on $\Omega_i, \alpha_i, \gamma_i$ and κ_i. Thus, (4.4) follows.
Theorem 4.3 and estimate (4.4) show that the convergence of the velocity sequence depends on that of the TKE sequence. So, the main purpose of the next paragraph is to prove that $\left(k_i^n - k_{i,h}^n \right)$ converges to 0 as h tends to 0 and n tends to ∞.

4.2 Analysis of the TKE sequence $k_{i,h}^n$

Theorem 4.4 *Under Hypothesis 2 and if $\boldsymbol{f}_i \in L^2(\Omega_i)^d$, there exists positive constants c_1 and c_2 depending only on Ω_i and on the data κ_i and λ, such that*

$$(4.12) \quad \forall n \in \mathbb{N}^*, \quad \sum_{i=1}^{2} \left| k_i^{n+1} - k_{i,h}^{n+1} \right|_{1,\Omega_i}^2 \leq \frac{c_1}{\nu} \sum_{i=1}^{2} \left| k_i^n - k_{i,h}^n \right|_{1,\Omega_i}^2 + \frac{c_2}{\nu} h^\sigma,$$

where σ is defined in (4.5).

The proof of this Theorem is made in several steps.

1- Proof of Th. 4.4: choice of the test function

To obtain an estimate for the error in the TKEs the standard choice for the test function in the second part (3.17)–(3.19) would be $\ell_{i,h}^{n+1} = k_{i,h}^{n+1} - \mathcal{S}_{i,h}(k_i^{n+1})$. However, in general this function is not equal to 0 on $\partial\Omega_i$. Due to this difficulty, we take the test function $\varphi_{i,h}$ equal to $\ell_{i,h}^{n+1} - \mathcal{R}_{i,h}(\ell_{i,h}^{n+1})$. Nevertheless, this choice requires to estimate the norm of the correction term $\mathcal{R}_{i,h}(\ell_{i,h}^{n+1})$.

2- Proof of Th. 4.4: Estimation of the correction term $\left| \mathcal{R}_{i,h}(\ell_{i,h}^{n+1}) \right|_{1,\Omega_i}$

We need the following result to estimate the energy sequence.

Lemma 4.5 *Under same assumptions of Theorem 4.4, there exists non negative constants c'_1 and c'_2 depending only on $\Omega_i, \alpha_i, \gamma_i$ and λ, such that the following inequality holds:*

$$(4.13) \qquad \sum_{1=1}^{2} \left| \mathcal{R}_{i,h}(\ell_{i,h}^{n+1}) \right|_{1,\Omega_i}^2 \leq c'_1 h^\sigma + c'_2 \sum_{1=1}^{2} \left| k_{i,h}^n - k_i^n \right|_{1,\Omega_i}^2 .$$

Proof. First, according to Hypothesis 2, for all non negative integer n, the sequences k_i^n are included in $W^{1,3}(\Omega_i)$. Then, its trace on Γ belongs to $W^{2/3,3}(\Gamma)$. Thanks to Sobolev's injections, it belongs to $H^{1/2}(\Gamma)$. As both traces on Γ_i vanish, *i.e.* $k_i^n|_{\Gamma_i} = k_{i,h}^n|_{\Gamma_i} = 0$. Then, they belong to $H_{00}^{1/2}(\Gamma)$. Furthermore, by (3.10), there exists a positive constant c_R such that:

$$(4.14) \qquad \left| \mathcal{R}_{i,h}(\ell_{i,h}^{n+1}) \right|_{1,\Omega_i} \leq c_R \, \|\ell_{i,h}^{n+1}\|_{H_{00}^{1/2}(\Gamma)}.$$

Owing Hypothesis 2, the trace on Γ of the function $\ell_{i,h}^{n+1}$ can be writen as

$$\begin{aligned}\ell_{i,h}^{n+1}|_\Gamma &= k_{i,h}^{n+1}|_\Gamma - \left(\mathcal{S}_{i,h}(k_i^{n+1})\right)|_\Gamma \\ &= \lambda\,\mathcal{L}_{i,h}\left(\left|\boldsymbol{u}_{1,h}^{n+1}-\boldsymbol{u}_{2,h}^{n+1}\right|^2\right) - \lambda\,\mathcal{L}_{i,h}\left(\left|\boldsymbol{u}_1^{n+1}-\boldsymbol{u}_2^{n+1}\right|^2\right) \\ &= \lambda\,\mathcal{L}_{i,h}\left(\left|\boldsymbol{u}_{1,h}^{n+1}-\boldsymbol{u}_{2,h}^{n+1}\right|^2 - \left|\boldsymbol{u}_1^{n+1}-\boldsymbol{u}_2^{n+1}\right|^2\right).\end{aligned}$$

We next use a result of continuity of the product of traces on Γ due to Grisvard, see for instance [20] and [21],

Lemma 4.6 *Assume that Ω is a bounded Lipschitz-continuous open subset of $\mathbb{R}^d$. Let r, r_1 and r_2 be three non negative reals and p, p_1, p_2 be three real numbers in $[1,+\infty)$ such that $r_1 \geq r$, $r_2 \geq r$ and either*

$$r_1 + r_2 - r \geq d\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right) \geq 0, \quad r_i - r > d\left(\frac{1}{p_i}-\frac{1}{p}\right), \quad \text{or} \tag{4.15}$$

$$r_1 + r_2 - r > d\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right) \geq 0, \quad r_i - r \geq d\left(\frac{1}{p_i}-\frac{1}{p}\right). \tag{4.16}$$

Then the mapping: $(u,v) \to uv$ is a continuous from $W^{r_1,p_1}(\Omega) \times W^{r_2,p_2}(\Omega)$ to $W^{r,p}(\Omega)$.

Using the results of Hebey [22], this Lemma also holds for Sobolev spaces defined on compact Riemannian manifolds. This is the case of Γ.
Due to the continuity of: $\mathcal{L}_{i,h}$, of the trace operators from $W^{1,3+\varepsilon}(\Omega_i)$ onto $W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma)$ and from $H^1(\Omega_i)$ onto $H_{00}^{1/2}(\Gamma)$ and using Hypothesis 2 and previous Lemma, there exists positive constants $c_{\mathcal{L}}$ and $c_i = c(\Omega_i)$, such that

$$\begin{aligned}\|\ell_{i,h}^{n+1}\|_{H_{00}^{1/2}(\Gamma)} &\leq \lambda c_{\mathcal{L}} \left\|\left|\boldsymbol{u}_{1,h}^{n+1}-\boldsymbol{u}_{2,h}^{n+1}\right|^2 - \left|\boldsymbol{u}_1^{n+1}-\boldsymbol{u}_2^{n+1}\right|^2\right\|_{H_{00}^{1/2}(\Gamma)} \\ &= \lambda c_{\mathcal{L}} \|[(\boldsymbol{u}_{1,h}^{n+1}-\boldsymbol{u}_1^{n+1}) - (\boldsymbol{u}_{2,h}^{n+1}-\boldsymbol{u}_2^{n+1})] \\ &\qquad\qquad [(\boldsymbol{u}_{1,h}^{n+1}+\boldsymbol{u}_1^{n+1}) - (\boldsymbol{u}_{2,h}^{n+1}+\boldsymbol{u}_2^{n+1})]\|_{H_{00}^{1/2}(\Gamma)} \\ &\leq \lambda c_{\mathcal{L}} \sum_{i=1}^2 c_i \left\|\boldsymbol{u}_{i,h}^{n+1}-\boldsymbol{u}_i^{n+1}\right\|_{H_{00}^{1/2}(\Gamma)} \sum_{i=1}^2 \left\|\boldsymbol{u}_{i,h}^{n+1}+\boldsymbol{u}_i^{n+1}\right\|_{W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma)} \\ &\leq \lambda c_{\mathcal{L}} c \sum_{i=1}^2 \left\|\boldsymbol{u}_{i,h}^{n+1}-\boldsymbol{u}_i^{n+1}\right\|_{\boldsymbol{H}^1(\Omega_i)} \sum_{i=1}^2 \left\|\boldsymbol{u}_{i,h}^{n+1}+\boldsymbol{u}_i^{n+1}\right\|_{W^{1,3+\varepsilon}(\Omega_i)}\end{aligned}$$

Using Poincaré-Friedrichs inequality yielding ($c = max\{c_1, c_2\}$.)

$$\|\ell_{i,h}^{n+1}\|_{H_{00}^{1/2}(\Gamma)} \leq 4\lambda M c_{\mathcal{L}} c \sum_{i=1}^{2} \left| \boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i}. \tag{4.17}$$

When adding and subtracting the quantity $\Pi_{i,h}\, \boldsymbol{u}_{i,h}^{n+1}$, we obtain

$$\|\ell_{i,h}^{n+1}\|_{H_{00}^{1/2}(\Gamma)} \leq 4\lambda M c_{\mathcal{L}} c \left(\sum_{i=1}^{2} |\boldsymbol{u}_{i,h}^{n+1} - \Pi_{i,h}\, \boldsymbol{u}_i^{n+1}|_{1,\Omega_i} + \sum_{i=1}^{2} \left|\Pi_{i,h}\, \boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^{n+1}\right|_{1,\Omega_i} \right)$$

Then, summing upon $i = 1, 2$ and using (4.14), we deduce that

$$\left|\mathcal{R}_{i,h}(\ell_{i,h}^{n+1})\right|_{1,\Omega_i}^2 \leq c \sum_{i=1}^{2} |\boldsymbol{e}_{i,h}^{n+1}|_{1,\Omega_i}^2 + c \sum_{i=1}^{2} \left|\Pi_{i,h}\, \boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^{n+1}\right|_{1,\Omega_i}^2,$$

where c denote some constant > 0 depending on data. Using estimate (4.4) in Theorem 4.3, we conclude estimate (4.13), whence Lemma 4.5.

3- Proof of Th. 4.4: Estimation of the test function $\ell_{i,h}^{n+1}$

Theorem 4.7 *Assume Hypotheses 1 and 2. If $\boldsymbol{f}_i \in L2(\Omega_i)^d$, $i = 1, 2$, there exist positive constants c_1 and c_2 depending on $\Omega_i, \kappa_i, \alpha_i$ and γ_i such that:*

$$\sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 \leq \frac{c_1}{\nu} \sum_{i=1}^{2} |\ell_{i,h}^{n}|_{1,\Omega_i}^2 + \frac{c_2}{\nu}\, h^{\sigma}, \quad \forall n \in \mathbb{N}^*. \tag{4.18}$$

Proof. To estimate $\ell_{i,h}^{n+1} = k_{i,h}^{n+1} - \mathcal{S}_{i,h}(k_i^{n+1})$, we use the difference between equations (2.10) and (3.19) taking as test function $\varphi_{i,h} = \ell_{i,h}^{n+1} - \mathcal{R}_{i,h}(\ell_{i,h}^{n+1})$:

$$\begin{aligned} A &:= \sum_{i=1}^{2} \int_{\Omega_i} \left[\gamma_i(k_{i,h}^n)\, \nabla\, k_{i,h}^{n+1} - \gamma_i(k_i^n)\, \nabla\, k_i^{n+1}\right] \cdot \nabla\, \varphi_{i,h} \\ &= B := \sum_{i=1}^{2} \int_{\Omega_i} \left[\alpha_i(k_{i,h}^n)\, \left|\nabla\, \boldsymbol{u}_{i,h}^{n+1}\right|^2 - \alpha_i(k_i^n)\, \left|\nabla\, \boldsymbol{u}_i^{n+1}\right|^2\right] \varphi_{i,h}. \end{aligned}$$

Adding and subtracting $\ell_{i,h}^{n+1}$ in the first factor, then A can be rewriting as

$$\begin{aligned}
A &= \sum_{i=1}^{2} \int_{\Omega_i} \Big[\gamma_i(k_{i,h}^n)\, \nabla\, k_{i,h}^{n+1} \;-\; \gamma_i(k_i^n)\, \nabla\, k_i^{n+1}\Big] \,\cdot\, \nabla(\ell_{i,h}^{n+1} \;-\; \mathcal{R}_{i,h}(\ell_{i,h}^{n+1})) \\
&= \sum_{i=1}^{2} \int_{\Omega_i} (\gamma_i(k_{i,h}^n) - \gamma_i(k_i^n)) \nabla k_i^{n+1} \cdot \nabla \ell_{i,h}^{n+1} \\
&+ \sum_{i=1}^{2} \int_{\Omega_i} \gamma_i(k_{i,h}^n) \nabla(\mathcal{S}_{i,h}\,(k_i^{n+1}) - k_i^{n+1}) \cdot \nabla \ell_{i,h}^{n+1} \\
&+ \sum_{i=1}^{2} \int_{\Omega_i} \gamma_i(k_{i,h}^n) \nabla |\ell_{i,h}^{n+1}|^2 \\
&- \sum_{i=1}^{2} \int_{\Omega_i} \bigg((\gamma_i(k_{i,h}^n) - \gamma_i(k_i^n)) \nabla\, k_i^{n+1} + \gamma_i(k_{i,h}^n) \nabla \ell_{i,h}^{n+1} + \\
&\qquad\qquad \gamma_i(k_{i,h}^n) \nabla(\mathcal{S}_{i,h}\,(k_i^{n+1}) - k_i^{n+1}) \bigg) \cdot \nabla \mathcal{R}_{i,h}(\ell_{i,h}^{n+1}).
\end{aligned}$$

The relation $A = B$ yields

(4.19)

$$\begin{aligned}
\sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 \leq \quad & |B| \\
(A_1 :=) \quad & + \sum_{i=1}^{2} \int_{\Omega_i} \Big| \big(\gamma_i(k_{i,h}^n) \;-\; \gamma_i(k_i^n)\big)\, \nabla\, k_i^{n+1} \,\cdot\, \nabla\, \ell_{i,h}^{n+1} \Big| \\
(A_2 :=) \quad & + \sum_{i=1}^{2} \int_{\Omega_i} \Big| \gamma_i(k_{i,h}^n)\, \nabla\, \big(\mathcal{S}_{i,h}\,(k_i^{n+1}) \;-\; k_i^{n+1}\big) \,\cdot\, \nabla\, \ell_{i,h}^{n+1} \Big| \\
(A_3 :=) \quad & + \sum_{i=1}^{2} \int_{\Omega_i} \Big| \big(\gamma_i(k_{i,h}^n) \;-\; \gamma_i(k_i^n)\big)\, \nabla\, k_i^{n+1} \,\cdot \nabla\, \mathcal{R}_{i,h} \left(\ell_{i,h}^{n+1}\right) \Big| \\
(A_4 :=) \quad & + \sum_{i=1}^{2} \int_{\Omega_i} \Big| \gamma_i(k_{i,h}^n)\, \nabla\, \ell_{i,h}^{n+1} \,\cdot\, \nabla\, \mathcal{R}_{i,h} \left(\ell_{i,h}^{n+1}\right) \Big| \\
(A_5 :=) \quad & + \sum_{i=1}^{2} \int_{\Omega_i} \Big| \gamma_i(k_{i,h}^n)\, \nabla\, \big(\mathcal{S}_{i,h}\,(k_i^{n+1}) \;-\; k_i^{n+1}\big) \,\cdot\, \nabla\, \mathcal{R}_{i,h} \left(\ell_{i,h}^{n+1}\right) \Big|.
\end{aligned}$$

The next step is to estimate $(A_j)_{1 \leq j \leq 5}$ and B.

Estimation of A_1: Using Hypothesis 2, the Sobolev embedding of $H^1(\Omega_i)$ onto $L^6(\Omega_i)$, Hölder and Poincaré-Friedrichs inequalities, and the mean value

Theorem, there exists positive a constant c depending only on Ω_i such that

$$\begin{aligned} A_1 &\leq \delta_2 \sum_{i=1}^{2} \int_{\Omega_i} \left|(k_{i,h}^n - k_i^n) \nabla\, k_i^{n+1} \cdot \nabla \ell_{i,h}^{n+1}\right| \\ &\leq \delta_2 \sum_{i=1}^{2} \|k_{i,h}^n - k_i^n\|_{L^6(\Omega_i)} \|\nabla\, k_i^{n+1}\|_{L^3(\Omega_i)} |\ell_{i,h}^{n+1}|_{1,\Omega_i} \\ &\leq M\delta_2 c\, |k_{i,h}^n - k_i^n|_{1,\Omega_i}\, |\ell_{i,h}^{n+1}|_{1,\Omega_i}. \end{aligned}$$

To simplify the calculations, we introduce a positive number β which we shall fix later. According to Young's inequality

$$\frac{1}{\beta} a^2 + \beta b^2 \geq 2ab, \quad \forall a,\, b \in \mathbb{R}, \text{ and } \forall \beta > 0, \tag{4.20}$$

$$\text{hence,} \qquad A_1 \leq \frac{\nu}{\beta} \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 + \frac{M^2 \delta_2^2 c_2^2 \beta}{\nu} \sum_{i=1}^{2} |k_{i,h}^n - k_i^n|_{1,\Omega_i}^2. \tag{4.21}$$

<u>Estimation of A_2:</u> Using (4.3), (4.20) and Cauchy-Schwarz inequality

$$\begin{aligned} A_2 &= \sum_{i=1}^{2} \int_{\Omega_i} \left|\gamma_i(k_{i,h}^n)\, \nabla\, \left(\mathcal{S}_{i,h}\,(k_i^{n+1}) - k_i^{n+1}\right) \cdot \nabla\, \ell_{i,h}^{n+1}\right| \\ &\leq \frac{\nu}{\beta} \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 + \frac{\delta_1^2 \beta}{\nu} \sum_{i=1}^{2} \left|\mathcal{S}_{i,h}\,(k_i^{n+1}) - k_i^{n+1}\right|_{1,\Omega_i}^2 \\ &\leq \frac{\nu}{\beta} \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 + \frac{\delta_1^2 \beta c M^2}{\nu}\, h^{2\, s_2 + 1}. \end{aligned} \tag{4.22}$$

<u>Estimation of A_3:</u> Applying same arguments used to estimate A_1,

$$\begin{aligned} A_3 &\leq \delta_2 \sum_{i=1}^{2} \|k_{i,h}^n - k_i^n\|_{L^6(\Omega_i)}\, \|\nabla\, k_i^{n+1}\|_{L^3(\Omega_i)}\, |\mathcal{R}_{i,h}(\ell_{i,h}^{n+1})|_{1,\Omega_i} \\ &\leq M\delta_2 c \sum_{i=1}^{2} |k_{i,h}^n - k_i^n|_{1,\Omega_i}\, |\mathcal{R}_{i,h}(\ell_{i,h}^{n+1})|_{1,\Omega_i} \\ &\leq M\delta_2 c \sum_{i=1}^{2} |k_{i,h}^n - k_i^n|_{1,\Omega_i}\, |\ell_{i,h}^{n+1}|_{1,\Omega_i} \\ &\leq \frac{\nu}{\beta} \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 + \frac{M^2 \delta_2^2 c_2^2 \beta}{\nu} \sum_{i=1}^{2} |k_{i,h}^n - k_i^n|_{1,\Omega_i}^2. \end{aligned} \tag{4.23}$$

Estimation of A_4: Thanks to the Lemma 4.5 and relation (4.13), we obtain

$$
\begin{aligned}
A_4 \le\ & \delta_1 \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}\, |\mathcal{R}_{i,h}(\ell_{i,h}^{n+1})|_{1,\Omega_i} \\
\le\ & \frac{\nu}{\beta} \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 \,+\, \frac{\delta_1^2\beta}{\nu} \sum_{i=1}^{2} |\mathcal{R}_{i,h}(\ell_{i,h}^{n+1})|_{1,\Omega_i}^2 \\
\le\ & \frac{\nu}{\beta} \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 \,+\, \frac{c_1\beta}{\nu} \sum_{i=1}^{2} |k_{i,h}^{n} \,-\, k_i^n|_{1,\Omega_i}^2 \,+\, \frac{c_2\beta}{\nu}\, h^{\sigma},
\end{aligned} \tag{4.24}
$$

where c_1 and c_2 are a non negative constants depend only on $\Omega_i, \alpha_i, \gamma_i$ and M.

Estimation of A_5: Using the same techniques as previously and the error estimates 3.3 for operator $S_{i,h}$, we obtain

$$
\begin{aligned}
I_5 \le\ & \delta_1 \sum_{i=1}^{2} |\mathcal{S}_{i,h}\,(k_i^{n+1}) \,-\, k_i^{n+1}|_{1,\Omega_i}\, |\mathcal{R}_{i,h}(\ell_{i,h}^{n+1})|_{1,\Omega_i} \\
\le\ & \frac{\nu}{\beta} \sum_{i=1}^{2} |\mathcal{S}_{i,h}\,(k_i^{n+1}) \,-\, k_i^{n+1}|_{1,\Omega_i}^2 \,+\, \frac{\delta_1^2\beta}{\nu} \sum_{i=1}^{2} |\mathcal{R}_{i,h}(\ell_{i,h}^{n+1})|_{1,\Omega_i}^2 \\
\le\ & \frac{\beta c_1}{\nu} \sum_{i=1}^{2} |k_{i,h}^{n} \,-\, k_i^n|_{1,\Omega_i}^2 \,+\, \frac{\beta c_2}{\nu}\, h^{\sigma}.
\end{aligned} \tag{4.25}
$$

Summing up estimates for the $(A_j)_{1\le j\le 5}$, we can write

$$
\nu\left(1-\frac{4}{\beta}\right) \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 \,\le |B| \,+\, \frac{\beta c_1}{\nu} \sum_{i=1}^{2} |k_{i,h}^{n} \,-\, k_i^n|_{1,\Omega_i}^2 \,+\, \frac{c_2\beta}{\nu}\, h^{\sigma}.
$$

As $|k_{i,h}^n - k_i^n|_{1,\Omega_i}^2 \le 2|k_{i,h}^n - \mathcal{S}_{i,h}(k_i^n)| + 2|\mathcal{S}_{i,h}(k_i^n) - k_{i,h}^n|_{1,\Omega_i}^2$, we obtain by appliying the error estimates (4.3)

$$
\nu\left(1-\frac{4}{\beta}\right) \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 \le |B| \,+\, \frac{\beta c_1}{\nu} \sum_{i=1}^{2} |\ell_{i,h}^{n}|_{1,\Omega_i}^2 \,+\, \frac{c_2\beta}{\nu}\, h^{\sigma}. \tag{4.26}
$$

Estimation of B: It can be checked that the quantity B can be written as

$$
\begin{aligned}
B =\ & \sum_{i=1}^{2} \int_{\Omega_i} \left[\alpha_i(k_{i,h}^n)\, \left|\nabla\, \boldsymbol{u}_{i,h}^{n+1}\right|^2 \,-\, \alpha_i(k_i^n)\, \left|\nabla\, \boldsymbol{u}_i^{n+1}\right|^2\right] \varphi_{i,h} \\
=\ & \sum_{i=1}^{2} \int_{\Omega_i} \alpha_i(k_{i,h}^n)(\left|\nabla \boldsymbol{u}_{i,h}^{n+1}\right|^2 - \left|\nabla \boldsymbol{u}_i^{n+1}\right|^2)\varphi_{i,h} + \sum_{i=1}^{2} \int_{\Omega_i} (\alpha_i(k_{i,h}^n) - \alpha_i(k_i^n))\, \left|\nabla \boldsymbol{u}_i^{n+1}\right|^2 \varphi_{i,h}.
\end{aligned}
$$

Using the mean value Theorem, Hypotheses 1 and 2, Hölder inequality and Sobolev injections, we obtain

$$
\begin{aligned}
|B| \le \delta_1 \sum_{i=1}^{2} \int_{\Omega_i} |\nabla(\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_i^{n+1})||\nabla(\boldsymbol{u}_{i,h}^{n+1} + \boldsymbol{u}_i^{n+1})||\varphi_{i,h}| + \delta_2 \sum_{i=1}^{2} \int_{\Omega_i} |k_{i,h}^n - k_i^n||\nabla \boldsymbol{u}_i^{n+1}|^2 |\varphi_{i,h}| \\
\le \delta_1 \sum_{i=1}^{2} |\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_i^{n+1}|_{1,\Omega_i} \|\nabla(\boldsymbol{u}_{i,h}^{n+1} + \boldsymbol{u}_i^{n+1})\|_{L^3(\Omega_i)} \|\varphi_{i,h}\|_{L^6(\Omega_i)} \\
+ \delta_2 \sum_{i=1}^{2} \|k_{i,h}^n - k_i^n\|_{L^6(\Omega_i)} \|\nabla \boldsymbol{u}_i^{n+1}\|_{L^3(\Omega_i)}^2 \|\varphi_{i,h}\|_{L^6(\Omega_i)} \\
\le \left(Mc\delta_1 \sum_{i=1}^{2} \left| \boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i} + M^2 c \delta_2 \sum_{i=1}^{2} |k_{i,h}^n - k_i^n|_{1,\Omega_i} \right) |\varphi_{i,h}|_{1,\Omega_i}.
\end{aligned}
$$

As $\varphi_{i,h} = \ell_{i,h}^{n+1} - \mathcal{R}_{i,h}(\ell_{i,h}^{n+1})$, we can write $|\varphi_{i,h}|_{1,\Omega_i} \le |\ell_{i,h}^{n+1}|_{1,\Omega_i} + |\mathcal{R}_{i,h}(\ell_{i,h}^{n+1})|_{1,\Omega_i}$. From (4.14) we have $|\varphi_{i,h}|_{1,\Omega_i} \le c\,|\ell_{i,h}^{n+1}|_{1,\Omega_i}$. Then,

$$
\begin{aligned}
|B| \le c \left(M\delta_1 \sum_{i=1}^{2} \left| \boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i} + M^2 \delta_2 \sum_{i=1}^{2} |k_{i,h}^n - k_i^n|_{1,\Omega_i} \right) |\ell_{i,h}^{n+1}|_{1,\Omega_i} \\
\le \frac{2\nu}{\beta} \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 + \frac{c^2 M^4 \delta_2^2 \beta}{\nu} \sum_{i=1}^{2} |k_{i,h}^n - k_i^n|_{1,\Omega_i}^2 + \frac{c^2 M^2 \delta_1^2 \beta}{\nu} \sum_{i=1}^{2} \left| \boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i}^2 .
\end{aligned}
$$

Summing and subtracting $\Pi_{i,h}\, \boldsymbol{u}_i^{n+1}$ in the last inequality, and using (4.4), we obtain (to simplify we re-use the notation c_1, c_2 for positive constants)

$$
\begin{aligned}
|B| \le \frac{2\nu}{\beta} \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 + \frac{c_1 \beta}{\nu} \sum_{i=1}^{2} |k_{i,h}^n - k_i^n|_{1,\Omega_i}^2 \\
+ \frac{c_2 \beta}{\nu} \sum_{i=1}^{2} \left| \Pi_{i,h}\, \boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^{n+1} \right|_{1,\Omega_i}^2 + \frac{c_2 \beta}{\nu} h^\sigma
\end{aligned}
$$

Next, summing and subtracting $\mathcal{S}_{i,h}(k_i^{n+1})$, and using estimates (4.2)-(4.3)

$$
|B| \le \frac{2\nu}{\beta} \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 + \frac{c_1 \beta}{\nu} \sum_{i=1}^{2} |\ell_{i,h}^{n}|_{1,\Omega_i}^2 + \frac{c_2 \beta}{\nu} h^\sigma. \tag{4.27}
$$

Combining estimates (4.26) and (4.27), we obtain

$$
\nu \left(1 - \frac{6}{\beta} \right) \sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 \le \frac{\beta c_1}{\nu} \sum_{i=1}^{2} |\ell_{i,h}^{n}|_{1,\Omega_i}^2 + \frac{c_2 \beta}{\nu} h^\sigma .
$$

Finally, we choose $\beta = 7$ for instance to deduce (4.18), hence Theorem 4.7.

4- Proof of Th. 4.4: Conclusion

To finish the proof of Theorem 4.4, we use the definition of $\ell_{i,h}^{n+1}$ and successively the error estimates (4.3) and (4.18)

$$\sum_{i=1}^{2} \Big|k_i^{n+1} - k_{i,h}^{n+1}\Big|_{1,\Omega_i}^2 \leq 2\sum_{i=1}^{2} \big|k_i^{n+1} - \mathcal{S}_{i,h}(k_i^{n+1})\big|_{1,\Omega_i}^2 + \Big|\mathcal{S}_{i,h}(k_i^{n+1}) - k_{i,h}^{n+1}\Big|_{1,\Omega_i}^2$$

$$\leq 2\sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 + c\, h^{2s_2+1}$$

$$\leq \frac{c_1}{\nu} \sum_{i=1}^{2} |\ell_{i,h}^{n}|_{1,\Omega_i}^2 + \frac{c_2}{\nu}\, h^{\sigma}. \tag{4.28}$$

As a consequence of Theorem 4.7 we obtain

Lemma 4.8 *Under same assumptions of the Theorem 4.7, the following estimate holds for all non negative integer* $n \geq 0$

$$\sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 \leq \Big(\frac{c_1}{\nu}\Big)^n \sum_{i=1}^{2} |\ell_{i,h}^{0}|_{1,\Omega_i}^2 + c_2\, \frac{1 - (c_1/\nu)^{n+1}}{\nu - c_1}\, h^{\sigma} \tag{4.29}$$

Proof. For all integer $m \leq n$, and since (4.18), we deduce that

$$\sum_{i=1}^{2} |\ell_{i,h}^{n+1}|_{1,\Omega_i}^2 \leq \frac{c_1}{\nu} \sum_{i=1}^{2} |\ell_{i,h}^{n}|_{1,\Omega_i}^2 + \frac{c_2}{\nu} h^{\sigma} \leq (\frac{c_1}{\nu})^m \sum_{i=1}^{2} |\ell_{i,h}^{n+1-m}|_{1,\Omega_i}^2 + \frac{c_2}{\nu}\, h^{\sigma} \sum_{j=0}^{m-1} (\frac{c_1}{\nu})^j.$$

Now we may prove that the discrete scheme converges, and that its limit is a solution of the continuous problem (1.1). For $\nu > c_1$, estimate (4.29) follows.

Theorem 4.9 (Convergence) *Under the Hypothesis of Theorem 4.7, there exists non negative constants* c, c_1 *and* c_2 *depending only on* $\Omega_i, \kappa_i, \lambda, \alpha_i, \gamma_i$ *and* $\boldsymbol{f}_i$*, such that the following estimates hold for all integer* $n \geq 0$ *and for all positive real* $h > 0$*, for* ν *small enough:*

$$\sum_{i=1}^{2} \big(|k_{i,h}^n - k_i^n|_{1,\Omega_i}^2 + |\boldsymbol{u}_{i,h}^n - \boldsymbol{u}_i^n|_{1,\Omega_i}^2\big) \leq \frac{c_1^n}{\nu^n} + \frac{c}{\nu - c_1} h^{\sigma}. \tag{4.30}$$

and

$$\beta_h \sum_{i=1}^{2} \|p_i^{n+1} - p_{i,h}^{n+1}\|^2 \leq \frac{c_1^n}{\nu^n} + \frac{c}{\nu - c_1} h^\sigma + c_2 \left\| \left|\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}\right| \left(\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}\right) - \left|\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}\right| \left(\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}\right)\right\|^2_{L^{\frac{3}{2}}(\Gamma)^d}. \tag{4.31}$$

If $\nu > c_1$, the discrete scheme converges, and its limit is the solution of the initial coupled problem 1.1.

Proof. Estimate (4.30) is obviously can be obtained using Theorems 4.3, 4.4 and Lemma 4.8. Furthermore, if $\nu > c_1$, it can be checked that the sequence $((\boldsymbol{u}_{i,h}^n, k_{i,h}^n))_{n\in\mathbb{N}}^{h>0}$ converges when the pair (h, n) tends to $(0, \infty)$. Lets denote by $(\boldsymbol{u}_i, k_i)$ its limit.

Next, to prove estimate (4.31) and then the convergence of the discrete pressure sequence, we make the difference between continuous and discrete equations (2.6) and (3.15), where we consider the test function $\boldsymbol{v}_{i,h}$ belongs to the discrete space $\boldsymbol{X}_{i,h}$ for both of them, such that $|\boldsymbol{v}_{i,h}|_{1,\Omega_i} = 1$. This yields

$$\begin{aligned} & b_i(\boldsymbol{v}_{i,h}, p_i^{n+1} - p_{i,h}^{n+1}) = \\ & - \kappa_i \int_\Gamma \Big(|\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}|(\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}) - |\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}|(\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}) \Big) \cdot \boldsymbol{v}_{i,h} \\ & - \int_{\Omega_i} \alpha_i(k_i^n) \nabla(\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_{i,h}^{n+1}) : \nabla \boldsymbol{v}_{i,h} + \int_{\Omega_i} \big(\alpha_i(k_{i,h}^n) - \alpha_i(k_i^n)\big) \nabla \boldsymbol{u}_{i,h}^{n+1} : \nabla \boldsymbol{v}_{i,h}. \end{aligned}$$

We know that both velocities $\boldsymbol{u}_i^{n+1}$ and $\boldsymbol{u}_{i,h}^{n+1}$ belong to $\boldsymbol{H}^1(\Omega_i)$, then their traces on Γ belong to $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, thus to $L^3(\Gamma)^d$ by Sobolev injection. Then

$$\left| \left|\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}\right| \left(\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}\right) - \left|\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}\right| \left(\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}\right) \right| \in L^{\frac{3}{2}}(\Gamma)^d.$$

Using the mean value Theorem, (4.1) and Hôlder inequality, we obtain

$$\begin{aligned} & b_i(\boldsymbol{v}_{i,h}, p_i^{n+1} - p_{i,h}^{n+1}) \leq \\ & \left\| |\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}|(\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}) - |\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}|(\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}) \right\|_{L^{\frac{3}{2}}(\Gamma)^d} \|\boldsymbol{v}_{i,h}\|_{L^3(\Gamma)^d} \\ & + \Big(\delta_1 |\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_{i,h}^{n+1}|_{1,\Omega_i} + \delta_2 \|k_i^n - k_{i,h}^n\|_{L^6(\Omega_i)} \|\nabla \boldsymbol{u}_{i,h}^{n+1}\|_{L^3(\Omega_i)^d} \Big) |\boldsymbol{v}_{i,h}|_{1,\Omega_i} \\ & \leq c \Bigg[\left\| \left|\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}\right| \left(\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1}\right) - \left|\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}\right| \left(\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1}\right) \right\|_{L^{\frac{3}{2}}(\Gamma)^d} \\ & \qquad + |\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_{i,h}^{n+1}|_{1,\Omega_i} + \|k_i^n - k_{i,h}^n\|_{1,\Omega_i} \Bigg]. \end{aligned}$$

Since estimate (4.30), the inf-sup condition (3.1) and summing on $i = 1, 2$

$$\beta_h \sum_{i=1}^{2} \|p_i^{n+1} - p_{i,h}^{n+1}\|^2 \leq \frac{c_1^n}{\nu^n} + \frac{c}{\nu - c_1} h^{\sigma}$$
$$+c_2 \left\| \left| \boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1} \right| \left(\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_j^{n+1} \right) - \left| \boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1} \right| \left(\boldsymbol{u}_{i,h}^{n+1} - \boldsymbol{u}_{j,h}^{n+1} \right) \right\|^2_{L^{\frac{3}{2}}(\Gamma)^d}.$$

Thanks to the continuity of the trace operator from $\boldsymbol{H}^1(\Omega_i)$ to $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, and the injection from $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$ into $L^3(\Gamma)^d$. We deduce that when (h, n) tends to $(0, \infty)$, the sequences $\Big((\boldsymbol{u}_{1,h}^n - \boldsymbol{u}_{2,h}^n) \,|\boldsymbol{u}_{1,h}^n - \boldsymbol{u}_{2,h}^n| \Big)_{n\in\mathbb{N}}^{h>0}$ and $((\boldsymbol{u}_1^n - \boldsymbol{u}_2^n)\,|\boldsymbol{u}_1^n - \boldsymbol{u}_2^n|)_{n\in\mathbb{N}}$ tend to the same limit $(\boldsymbol{u}_1 - \boldsymbol{u}_2)\,|\boldsymbol{u}_1 - \boldsymbol{u}_2|$ strongly in $L^{\frac{3}{2}}(\Gamma)^d$. We conclude that the sequence $(p_{i,h}^n)_n^h$ converges in $L^2(\Omega_i)$ strong to p_i. We recall that the limit $(\boldsymbol{u}_i, p_i, k_i)$ is solution of problem 1.1, see [16].

5 Numerical experiments

The aim of this section is twofold: We intend to test the convergence order (in h) of the algorithm (3.15)–(3.19), and also the ability of the discretization to reproduce overall qualitative features of a realistic flow. Actually, we simulate an upwelling oceanic flow generated by an atmospheric cavity flow.

In both cases, we use the with FreeFEM++ code to perform our tests (Cf. [23]). The solver uses a Taylor-Hood $\mathcal{P}_2 - \mathcal{P}_1$ finite element method (FEM) for the space discretization of velocity-pressure, and $\mathcal{P}_2$ (FEM) for the TKEs equation. This corresponds to the choices $s_1 = s_2 = 1$ in the finite element spaces stated in (3.11), (3.12) and (3.13).

At each step, linear systems are obtained and solved using a preconditioned GMRES iterative routine, see for instance Saad [26].

5.1 Accuracy test

The first test is used to validate the space accuracy. In order to evaluate the convergence rates, we have used a time-stepping strategy which consists in looking at the solutions problem (1.1) as steady states of the corresponding evolution problem. We have chooses an implicit backward Euler time discretization with time step $\delta t = 0.1$:

$$\frac{d\boldsymbol{u}_i}{dt} = \frac{\boldsymbol{u}_i^{n+1} - \boldsymbol{u}_i^n}{\delta t}, \quad \frac{dk_i}{dt} = \frac{k_i^{n+1} - k_i^n}{\delta t}.$$

The computational domains are $\Omega_1 =]0,2[\times]0,1[\times]0,1[$ for atmosphere and $\Omega_2 =]0,2[\times]0,1[\times]-1,0[$ for ocean. We built the solution of our iterative scheme by taking the data which are used in [16]:

$$\gamma_1(k_1) = 3\times 10^{-3} + 0.277\times 10^{-4}\sqrt{k_1}; \quad \gamma_1(k_2) = 3\times 10^{-2} + 0.185\times 10^{-5}\sqrt{k_2}.$$
$$\alpha_i(\cdot) = \gamma_i(\cdot), \qquad \kappa_i = 10^{-3}, \text{ and } \lambda = 5\times 10^{-2}.$$

The velocity boundary conditions imposes that: $\boldsymbol{u}_1 = \mathbf{0}$ on $\Gamma_1/\tilde{\Gamma}_1$, $\boldsymbol{u}_1 = (1,0,0)$ on $\tilde{\Gamma}_1$ and $\boldsymbol{u}_2 = \mathbf{0}$ on Γ_2, where $\tilde{\Gamma}_1$ is the upper face $z = 1$ of Ω_1. Whereas we consider the homogenous Dirichlet boundary conditions on all border $\partial\Omega_1 \cup \partial\Omega_2$, and equal to $\lambda|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2$ on the interface Γ.

To estimate the convergence rate, we have used several mesh sizes h. We recover a convergence order for velocity-energy that decreases as h tends to 0, which is confirm theoretical estimate obtained by passing in (4.30) to the limit when n tends to ∞:

$$\mathcal{E}_h := \sum_{i=1}^{2} (|k_{i,h} - k_i|_{1,\Omega_i} + |\boldsymbol{u}_{i,h} - \boldsymbol{u}_i|_{1,\Omega_i}) \le c\, h^p. \tag{5.1}$$

with $p = \sigma/2$. Recall that in this simulation, the discrete spaces correspond to take $s_1 = s_2 = 1$. Then from (4.5), our theoretic estimate for the convergence order is $p > 1/4$. We set

$$\mathcal{O}_h = \frac{log(\frac{\mathcal{E}_h}{\mathcal{E}_{\frac{h}{2}}})}{log(2)} \approx p$$

Table 1 shows that the convergence order is ≈ 0.25 We roughly recover the

Table 1: Estimated convergence order

Mesh size	Order $\mathcal{O}_h$
h	——
$h/2$	0.12
$h/4$	0.16
$h/8$	0.22
$h/16$	0.23

theoretically expected accuracy, which however is small due to the non-linear interface term. Improved accuracy may be obtained by taking finite element spaces with higher order interpolation.

5.2 Up-welling effects near the shores

This test is designed to test the ability of our iterative algorithm presented above to simulate genuine 3D effects that arise in geophysical flows. Concretely, we test the formation of the up-welling effect, due to the interaction between wind-tension and Coriolis forces. In fact, we have considered the following problem, that includes transport and Coriolis terms, instead of system (1.1),

$$(5.2)\quad \left\{ \begin{array}{rcl} \partial_t \boldsymbol{u}_i + (\boldsymbol{u}_i \cdot \nabla)\, \boldsymbol{u}_i + \tau\,(-u_{i,y}, u_{i,x}, 0) & & \\ -\nabla \cdot (\alpha_i(k_i) \nabla \boldsymbol{u}_i) + \nabla p_i & = & \boldsymbol{f}_i \\ \nabla \cdot \boldsymbol{u}_i & = & 0 \\ \partial_t k_i + \boldsymbol{u}_i \nabla k_i - \nabla \cdot (\gamma_i(k_i) \nabla k_i) & = & \alpha_i(k_i) |\nabla \boldsymbol{u}_i|^2. \end{array} \right.$$

where the component of the velocity fields are denoted by $\boldsymbol{u}_i = (u_{i,x}, u_{i,y}, u_{i,z})$. The term $\tau\,(-u_{i,y}, u_{i,x}, 0)$ models influence of the Coriolis forces, where the real τ depends on the angular velocity of the earth and the latitude. Furthermore, in order to generate several remarkable patterns in the flows, we have added the convection term $(\boldsymbol{u}_i \cdot \nabla)\, \boldsymbol{u}_i$ in the first equation and the transport term $\boldsymbol{u}_i \nabla k_i$ in the TKEs equation. The boundary conditions are the same as for system (1.1).

We have considered the computational domains $\Omega_1 =]0,\, 10^4[\times]0,\, 5\cdot 10^3[\times]0,\, 500[\,(m)$ for atmosphere and a swimming-pool to model the geometry of the ocean (see Figure 1) $\Omega_2 = \omega \times \{z = D(x,y); \quad (x,y) \in \omega\}$, such that

- Horizontal dimensions (m): $\quad \omega =]0,\, 10^4[\times]0,\, 5\cdot 10^3[$
- Bathemetry (m):

$$D(x,y) = \left\{ \begin{array}{ll} -50 & \text{if } \ 0 \le x \le 4\cdot 10^3 \\ -50 \cdot \dfrac{5\cdot 10^3 - x}{10^3} - 100 \cdot \dfrac{4\cdot 10^3 - x}{10^3} & \text{if } \ 4\cdot 10^3 \le x \le 5\cdot 10^3 \\ -100 & \text{if } \ 5\cdot 10^3 \le x \le 10^4 \end{array} \right.$$

We use the following set of data:

- $\alpha_1(k_1) = \gamma_1(k_1) = 3\cdot 10^{-}2 + 0,277\cdot 10^{-4}\sqrt{k_1}\, m^2/s$
- $\alpha_2(k_2) = \gamma_2(k_2) = \nu + 0,185\cdot 10^{-5}\sqrt{k_2}$, where $\nu = (10^{-1}, 10^{-2}, 10^{-4})\, m^2/s$
- $\lambda = 5\cdot 10^{-2}$, $\kappa_i = 10^{-3}$, $\boldsymbol{f}_i = \boldsymbol{0}$,

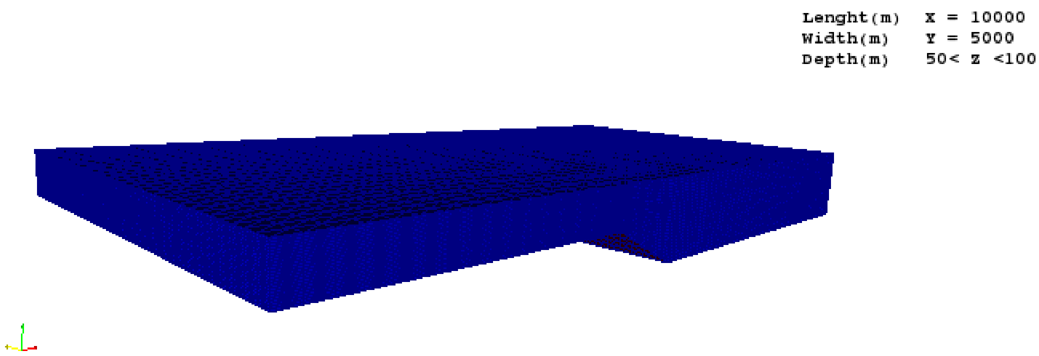

Figure 1: Computational domain of ocean

- $\tau = 2\theta \, sin(\varphi)$, where $\theta = 7,3 \cdot 10^{-5} \, s^{-1}$, $\varphi = 45^0 \, N$.

The viscosity ν is constant in each space direction, but it is assumed to model the eddy viscosity terms. Then it is assumed to be isotropic and is scaled by the different diameters of ω in the directions OX, OY and OZ. Concerning boundary conditions, we have imposed $\boldsymbol{u}_1 = \mathbf{0}$ on $\Gamma_1/\tilde{\Gamma}_1$, $\boldsymbol{u}_1 = (-1, 0, 0)$ on $\tilde{\Gamma}_1$ and $\boldsymbol{u}_2 = \mathbf{0}$ on Γ_2, where $\tilde{\Gamma}_1$ is the upper face $z = 500$ of Ω_1. Whereas we consider the homogenous Dirichlet boundary conditions on all the border $\partial\Omega_1 \cup \partial\Omega_2$, and equal to $\lambda|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2$ on the interface Γ. These settings are chosen in order to create a driven cavity-like flow in atmosphere domain Ω_1.

The atmospheric flow generates a wind at the top of the pool, i.e the interface air-water, and subsequently the formation of the up-welling flow besides a lateral wall of the pool. The relatively short dimension of the domain in the cross-wind direction also originates a down-welling flow in the vertical face of the pool opposite to the up-welling. On another hand, the relatively short dimension of the domain in the wind direction originates a longitudinal recirculation, that accelerates due to the bottom ramp, in a direction opposite to the wind.

Note that in the representation of the numerical results, the depth and the vertical velocity have been increased by a factor 10 in order to provide a good visualization. Figure 2 shows the vertical velocity, where Coriolis acceleration effects are apparent. In the Northern Hemisphere, the Earth rotation deviates the flow to its right. Figure 3 represents the velocity profile along a vertical cut of the domain (the plane $y = 2500$). We observe a global recirculation of the flow, that produces an acceleration along the ramp, in the direction opposite to the wind. The flow presents a quasi-parabolic vertical profile in the less depth part of the domain. Figures 4 and 5 show

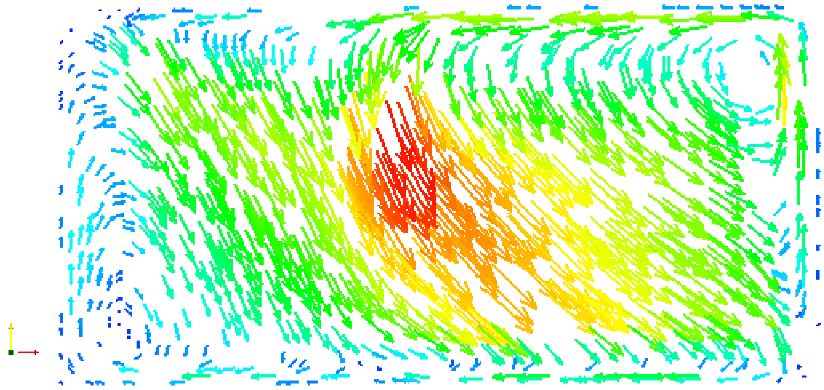

Figure 2: Surface velocity in the ocean

the projection of the 3D velocity on two planes orthogonal to the wind direction ($x = 5000$, $x = 8000$ respectively). We observe the up-welling effect on the left of the domain, but also a down-welling effect on the right of the domain. The overall flow is a recirculation transversal to the wind direction. The trajectory of a flow particle describes spirals around an axis parallel to the wind direction.

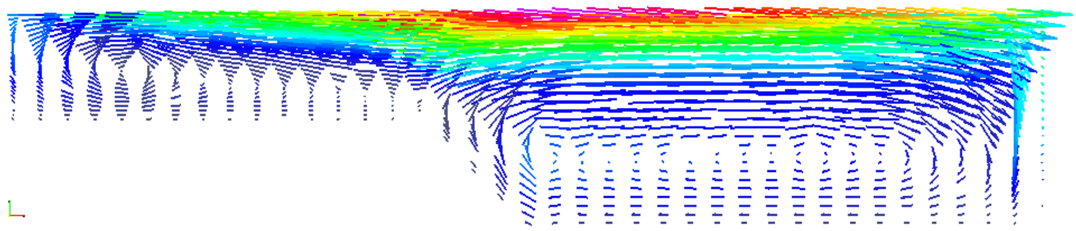

Figure 3: Projection of $3D$ velocity in the ocean onto plane y = 2500

6 Conclusion

In this paper, we have analyzed a numerical model for the coupling two steady turbulent fluids. The finite element method is used to approximate its solution. Both the viscosity and diffusion depend on the turbulent kinetic energy but Coriolis forces and buoyancy effects are neglected in the theoretical analysis. However, we have kept several non-linear interactions

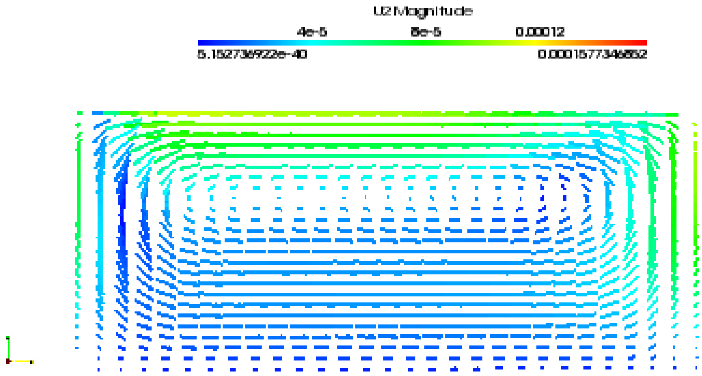

Figure 4: Projection of $3D$ velocity in the ocean onto plane x = 8000

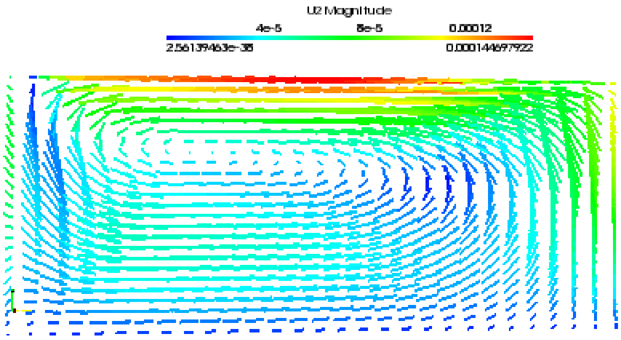

Figure 5: Projection of $3D$ velocity in the ocean onto plane x = 5000

across the common boundary. We have proved that the discrete scheme introduced converges for large enough eddy viscosities to the solution of continuous problem. We have established that the space-accuracy in H^1-norm strongly depends on the regularity of the solution i.e $W^{1,p}, p > d$.

Furthermore, numerical tests confirm our theoretical results, in the sense that the first one shows that the convergence order on h is equal to 0.25 when the finite element space is $\mathcal{P}_2 \times \mathcal{P}_1 \times \mathcal{P}_2$ for the (velocity, pressure, energy) triplet. Finally, in the second test in paragraph 5.2, and in order to perform a somewhat realistic computation of the model we used, including Coriolis forces, convection and transport terms, with a shallow oceanic domain. We conclude that we correctly reproduce the 3D features associated to the interaction between wind-stress induced by atmosphere and Coriolis forces.

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