A Finite Element Approximation of a Coupled Model for Two Turbulent Fluids

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joint works with

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Outline

- Introduction
- Continuous Scheme
- Finite Element Analysis
- Numerical results

Model equation

It includes:

- A one-equation simplified turbulent model for each fluid:
  Only one statistic of turbulent is considered, the turbulent kinetic energy (TKE).

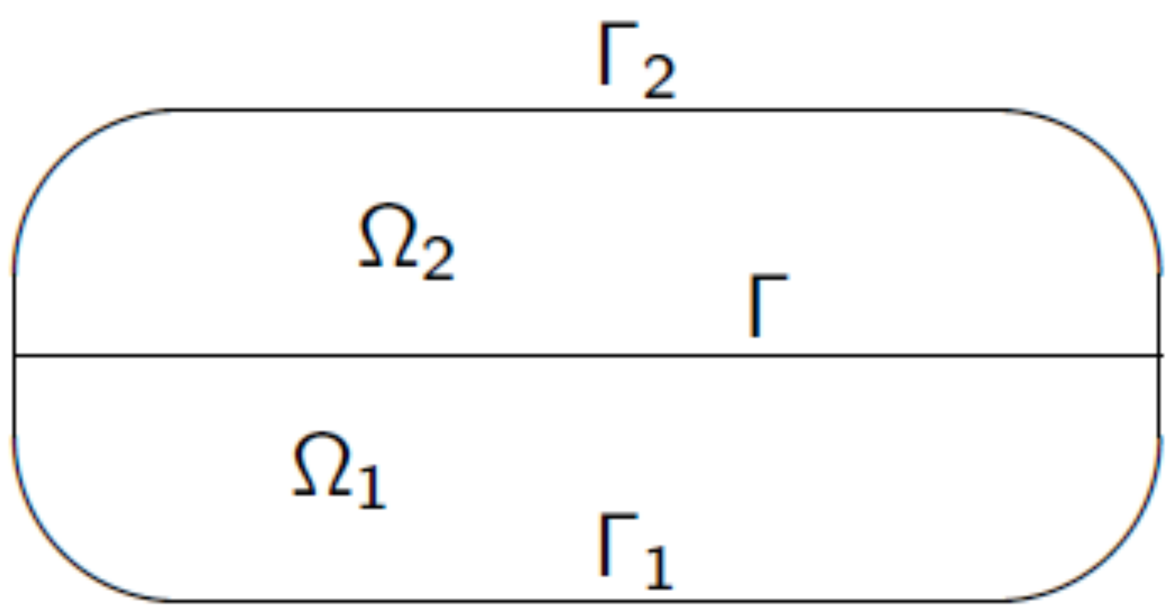
- Modeling of TKE generation of ocean-atmosphere interface by quadratic law

- Modeling of friction at interface by manning law

Model equation: RANS

Reynolds Averaged Navier-Stokes

$$
\begin{aligned}
-\nabla\cdot(\alpha_i(k_i)\nabla\mathbf{u}_i)+\nabla p_i &= \mathbf{f}_i \text{ in } \Omega_i,\\
\nabla\cdot\mathbf{u}_i &= 0 \text{ in } \Omega_i,\\
-\nabla\cdot(\gamma_i(k_i)\nabla k_i) &= \alpha_i(k_i)|\nabla\mathbf{u}_i|^2 \text{ in } \Omega_i,\\
\mathbf{u}_i &= \mathbf{0} \text{ on } \Gamma_i,\\
k_i &= 0 \text{ on } \Gamma_i,\\
\alpha_i(k_i)\partial_{\mathbf{n}_i}\mathbf{u}_i - p_i\mathbf{n}_i + \kappa_i(\mathbf{u}_i-\mathbf{u}_j)|\mathbf{u}_i-\mathbf{u}_j| &= \mathbf{0} \text{ on } \Gamma, 1\le i\ne j\le 2,\\
k_i &= \lambda|\mathbf{u}_1-\mathbf{u}_2|^2 \text{ on } \Gamma.
\end{aligned}
$$



Model equation

Data:

- Ω_i Open bounded sets of $\mathbb{R}^d$ either convex or with boundary $\partial\Omega_i = \Gamma_i \cup \Gamma \in \mathcal{C}^{1,1}$
- $\alpha_i,\ \gamma_i \in W^{1,\infty}(\mathbb{R})$ are the viscosity and diffusion coefficients, such that:

$$\begin{cases} \nu \le \alpha_i(\ell) \le \delta_1, \quad \nu \le \gamma_i(\ell) \le \delta_1, \quad \text{and} \\ |\alpha_i'(\ell)| \le \delta_2, \quad |\gamma_i'(\ell)| \le \delta_2, \end{cases}$$

- $\lambda > 0,\ \kappa_i > 0$ are the friction coefficients.
- $\mathbf{f}_i \in L^2(\Omega_i)^d$ are the source terms

Specific Purposes: Devise stable numerical methods to compute steady states, as simplified equilibra states of the ocean-atmosphere system.

Weak formulation of continuous model

Functional spaces:

- Velocity spaces: $$\mathbf{X}_i = \{\mathbf{v}_i \in \mathbf{H}^1(\Omega_i);\ \mathbf{v}_i = \mathbf{0} \quad \text{on} \quad \Gamma_i\}$$
- Pressure spaces $$L_0^2(\Omega_i) = \{q_i \in L^2(\Omega_i); \quad \text{such that} \int_{\Omega_i} q_i = 0\}$$
- TKE spaces: $$Y_i = \{k_i \in W^{1,r'}(\Omega_i); \quad k_i = 0 \quad \text{on} \quad \Gamma_i\}$$

$$\frac{1}{r} + \frac{1}{r'} = 1, \quad \text{and} \quad r > d$$

Weak formulation of continuous model

Find $(\mathbf{u}_i, p_i, k_i) \in \mathbf{X}_i \times L^2(\Omega_i) \times W^{1,r'}(\Omega_i)$ such that,

for all $(\mathbf{v}_i, q_i, \varphi_i) \in \mathbf{X}_i \times L^2(\Omega_i) \times W_0^{1,r}(\Omega_i)$,

$$\int_{\Omega_i} \alpha_i(k_i)\nabla\mathbf{u}_i : \nabla\mathbf{v}_i - \int_{\Omega_i} (\nabla \cdot \mathbf{v}_i)\; p_i + \kappa_i \int_{\Gamma} |\mathbf{u}_i - \mathbf{u}_j|(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{v}_i \, d\tau = \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{v}_i$$

$$\int_{\Omega_i} (\nabla \cdot \mathbf{u}_i)\; q_i = 0,$$

$$k_i = 0 \text{ on } \Gamma_i, \qquad k_i = \lambda|\mathbf{u}_i - \mathbf{u}_j|^2 \quad \text{on } \Gamma, \qquad \text{and}$$

$$\int_{\Omega_i} \gamma_i(k_i)\nabla k_i \cdot \nabla\varphi_i = \int_{\Omega_i} \alpha_i(k_i)|\nabla\mathbf{u}_i|^2\; \varphi_i.$$

Weak formulation of continuous model

Since $\mathbf{u}_i \in \mathbf{X}_i$ then its trace on Γ belongs to $\mathbf{H}^{\frac{1}{2}}(\Gamma) \hookrightarrow L^3(\Gamma)^d$, Then

$$\int_\Gamma |\mathbf{u}_i - \mathbf{u}_j|\,(\mathbf{u}_i - \mathbf{u}_j)\cdot \mathbf{v}_i \, d\tau$$ is well defined.

- Analysis of this model by [Bernardi, Chacon, Lewandowski and Murat]

Difficulties:

1. $|\nabla \mathbf{u}_i|^2 \in L^1(\Omega_i)^d$,
2. Coupling fuilds by:

   $$\int_\Gamma |\mathbf{u}_i - \mathbf{u}_j|\,(\mathbf{u}_i - \mathbf{u}_j)\cdot \mathbf{v}_i, \text{ and } \quad k_i = \lambda |\mathbf{u}_i - \mathbf{u}_j|^2 \quad \text{on } \Gamma,$$

3. Coupling Eqs by terms: $|\nabla \mathbf{u}_i|^2, \ \alpha_i(\cdot), \ \text{and} \ \gamma_i(\cdot)$

Analysis of continuous scheme

1. $$a_i\left(k_i^n;\mathbf{u}_i^{n+1},\nabla\mathbf{v}_i\right)+b_i\left(\mathbf{v}_i,p_i^{n+1}\right)$$
$$+\kappa_i\int_\Gamma\left|\mathbf{u}_i^{n+1}-\mathbf{u}_j^{n+1}\right|\left(\mathbf{u}_i^{n+1}-\mathbf{u}_j^{n+1}\right)\cdot\mathbf{v}_i=\int_{\Omega_i}\mathbf{f}_i\cdot\mathbf{v}_i,$$

2. $$\int_{\Omega_i}\gamma_i(k_i^n)\nabla k_i^{n+1}\cdot\nabla\varphi_i=\int_{\Omega_i}\alpha_i(k_i^n)|\nabla\mathbf{u}_i^{n+1}|^2\varphi_i$$

With: $k_i^{n+1}=0$ on Γ_i, and $k_i^{n+1}=\lambda|\mathbf{u}_1^{n+1}-\mathbf{u}_2^{n+1}|^2$ on Γ.

The iterative scheme is contractive if

- The turbulent diffusion is large enough with respect to the data, and
- The iterates $\left(\mathbf{u}_i^{n+1}\right)_n$ and $\left(k_i^{n+1}\right)_n$ remains bounded in norms smooth enough:

Analysis of continuous scheme

Theorem [T. Chacon Rebollo, S. Del Pino & DY]

Assume that the sequences $(\mathbf{u}_i^n)_n$ and $(k_i^n)_n$ remain bounded in $W^{1,3+\varepsilon}(\Omega_i)^d$ and $W^{1,3}(\Omega_i)$ by M.
Then, there exists a constant C depending only on data such tha

if $K = \frac{C}{\nu} < 1$

the the iterative scheme is contractive in the sens that:

$$\sum_{i=1}^{2} \left|\mathbf{u}_i^{n+1} - \mathbf{u}_i^n\right|_{1,\Omega_i}^2 \leq K \sum_{i=1}^{2} \left|k_i^n - k_i^{n-1}\right|_{1,\Omega_i}^2, \quad \text{and}$$

$$\sum_{i=1}^{2} \left|k_i^{n+1} - k_i^n\right|_{1,\Omega_i}^2 \leq K \sum_{i=1}^{2} \left|k_i^n - k_i^{n-1}\right|_{1,\Omega_i}^2.$$

Finite Element Approximation

- Bernardi-Chacon-Lewandowski-Murat: (Numer. Math, 2004)

  analysis of 2D F.E solution

  - The velocity-pressure is discretized by the Mini-Element on both Ω_i
  - The velocity spaces $\mathbf{X}_{ih}$ are supposed to be compatible on interface Γ

  in the sens that the trace spaces

  $$Z_{ih} = \{\mathbf{v}_{ih}|_\Gamma, \quad \text{for } \mathbf{v}_{ih} \in \mathbf{X}_{ih}\}, \qquad i = 1, 2$$

  are equal.

Theorem [Bernardi-Chacon-Lewandowski-Murat]

In the 2D case, there exists a subsequence of the solutions $(\mathbf{u}_{1h}, p_{1h}, k_{1h}), (\mathbf{u}_{2h}, p_{2h}, k_{2h})$ that converge strongly in:

$$\left(H^1(\Omega_1)^2 \times L_0^2(\Omega_1) \times H^s(\Omega_1)\right) \times \left(H^1(\Omega_2)^2 \times L_0^2(\Omega_2) \times H^s(\Omega_2)\right), \quad \text{for } 0 \le s < 1/2$$

to a solution of model problem.

Finite Element Approximation

The problems we face now are:

- To build a more constructive solution schem
- To analyse the 3D.

Discrete spaces:

$$\mathbf{X}_{i,h} = \left\{ \mathbf{v}_{i,h} \in C^0\left(\overline{\Omega}_i\right)^d, \forall K \in \mathcal{T}_{i,h}, \in \mathcal{P}_2(K)^d \right\} \cap \mathbf{X}_i$$

$$M_{i,h} = \left\{ q_{i,h} \in L^2\left(\Omega_i\right), \forall K \in \mathcal{T}_{i,h},\ q_{i,h}|_K \in \mathcal{P}_1(K) \right\}$$

$$K_{i,h} = \left\{ \ell_{i,h} \in C^0\left(\overline{\Omega}_i\right), \forall K \in \mathcal{T}_{i,h},\ \ell_{i,h}|_K \in \mathcal{P}_2(K),\ \ell_{i,h}|_{\Gamma_i} = 0 \right\}.$$

The family $(\mathbf{X}_{i,h}, M_{i,h})_{h>0}$, for $i = 1, 2$ satisfy the Brezzi-Fortin:

$$\forall q_{i,h} \in M_{i,h}, \qquad \sup_{\mathbf{v}_{i,h} \in \mathbf{X}_{i,h}} \frac{b_i\left(\mathbf{v}_{i,h}, q_{i,h}\right)}{\left|\mathbf{v}_{i,h}\right|} \geq \beta_{i,h} \|q_{i,h}\|_{0,\Omega_i}$$

Finite Element Approximation

Standard interpolation:

$$\mathcal{S}_{i,h}: \quad H^1(\Omega_i) \times \mathcal{C}^0\left(\overline{\Omega}_i\right) \longrightarrow K_{i,h}$$
$$k_i \longrightarrow \mathcal{S}_{i,h}(k_i)\,.$$

$$\Pi_{i,h}: \quad \mathbf{X}_i \longrightarrow \mathbf{X}_{i,h} \quad \text{such that}$$
$$\mathbf{v}_i \longrightarrow \Pi_{i,h}(\mathbf{v}_i)\,.$$

$$\mathcal{L}_{i,h}: \quad H^{\frac{1}{2}}_{00}(\Gamma) \longrightarrow \mathbf{Z}_{i,h} \quad \text{such that}$$
$$\mathcal{L}_{i,h}(W|_\Gamma) = (\mathcal{S}_{i,h} W)|_\Gamma.$$

This is in the sens that for all $k_i \in H^1(\Omega_i) \times \mathcal{C}^0\left(\overline{\Omega}_i\right)$,

the trace on Γ of the interpolate $\mathcal{S}_{i,h}(k_i)|_\Gamma$

coincides with the interpolate of the trace $\mathcal{L}_{i,h}(k_i|_\Gamma)$.

Finite Element Approximation

Discrete algorithms:

1. $$a_i\left(k_{i,h}^n;\mathbf{u}_{i,h}^{n+1},\mathbf{v}_{i,h}\right)\qquad +b_i\left(\mathbf{v}_{i,h},p_{i,h}^{n+1}\right)$$
$$+\kappa_i\int_\Gamma\left|\mathbf{u}_{i,h}^{n+1}-\mathbf{u}_{j,h}^{n+1}\right|\left(\mathbf{u}_{i,h}^{n+1}-\mathbf{u}_{j,h}^{n+1}\right)\cdot\mathbf{v}_{i,h}\,d\tau=\int_{\Omega_i}\mathbf{f}_i\cdot\mathbf{v}_{i,h},$$

and $\forall q_{i,h}\in M_{i,h}$ $\qquad b_i\left(\mathbf{u}_{i,h}^{n+1},q_{i,h}\right)=0,$

2. $$k_{i,h}^{n+1}=0\text{ on }\Gamma_i,\qquad k_{i,h}^{n+1}=\lambda|\mathbf{u}_{1,h}^{n+1}-\mathbf{u}_{2,h}^{n+1}|^2\text{ on }\Gamma,$$

$$\int_{\Omega_i}\gamma_i(k_{i,h}^n)\,\nabla k_{ih}^{n+1}\cdot\nabla\varphi_{ih}=\int_{\Omega_i}\alpha_i(k_{i,h}^n)|\nabla\mathbf{u}_{i,h}^{n+1}|^2\,\varphi_{i,h},\quad\forall\varphi_{i,h}\in K_{i,h}^0,$$

where $\quad K_{i,h}^0=K_{i,h}\cap W_0^{1,r}(\Omega_i)$

Finite Element Approximation

Theorem [T. Chacon Rebollo & DY]

Assume that the sequences $(\mathbf{u}_i^n)_n$, $(\mathbf{u}_{ih}^n)_n$, $(k_i^n)_n$, and $(k_{ih}^n)_n$ remain bounded in $W^{1,3+\varepsilon}(\Omega_i)^d$ and $W^{1,3}(\Omega_i)$ by M.

Then there exists a constants depending only on data such that:

$$\sum_{i=1}^{2} \left[|k_{i,h}^n - k_i^n|_{1,\Omega_i}^2 + |\mathbf{u}_{i,h}^n - \mathbf{u}_i^n|_{1,\Omega_i}^2 + \|p_{i,h}^n - p_i^n\|^2 \right]$$

$$\leq \frac{c_1^n}{\nu^n} + c\left(h^{2\sigma} + h^{\sigma} + (\nu^2 + 1)\,h\right)$$

Were $\sigma = \frac{3}{2} - \frac{3}{3+\varepsilon} > \frac{1}{2}, \quad \forall \varepsilon > 0.$

Furthermore if $c_1 < \nu$, discrete scheme converges, and its limit i a solution of the continuous model.

Finite Element Approximation

Keys of the proof:

- Use convenient choices of test functions:
  - For instance, to obtain the estimate for $\sum_{i=1}^{2} \left| k_{ih}^{n+1} - k_i^n \right|_{1,\Omega_i}^2$

1. We introduce the following space:

$$W_{i,h} = \left\{ \varphi_{i,h} \in \mathcal{C}^0\left(\partial\Omega_i\right);\ \forall e \in \mathcal{E}_{i,h},\ \varphi_{i,h}|_e \in \mathbb{P}_2(e) \right\}.$$

2. We introduce the lifting operator:

$$\mathcal{R}_{ih} : W_{ih} \longmapsto K_{ih}, \qquad (R_{ih}(\varphi_{ih}))\,|_{\partial\Omega_i} = \varphi_{ih}, \qquad \forall \varphi_{ih} \in W_{ih}.$$

$$\|\mathcal{R}_{i,h}(\varphi_{i,h})\|_{W^{1,p}(\Omega_i)} \leq c\|\varphi_{i,h}\|_{W^{1-1/p,p}(\partial\Omega_i)}.$$

3. and set the test function TKE:

$$\varphi_{ih} = \ell_{i,h}^{n+1} - \mathcal{R}_{i,h}(\ell_{i,h}^{n+1}), \quad \text{where} \quad \ell_{i,h}^{n+1} = k_{i,h}^{n+1} - \mathcal{S}_{i,h}(k_i^{n+1}).$$

Finite Element Approximation

Keys of the proof:

- Due to the friction term, it is needed to estimate:

$$\left\| \left[\left(\mathbf{u}_{1,h}^{n+1} - \mathbf{u}_1^{n+1} \right) - \left(\mathbf{u}_{2,h}^{n+1} - \mathbf{u}_2^{n+1} \right) \right] \; \left[\left(\mathbf{u}_{1,h}^{n+1} + \mathbf{u}_1^{n+1} \right) - \left(\mathbf{u}_{2,h}^{n+1} + \mathbf{u}_2^{n+1} \right) \right] \right\|_{H_{00}^{1/2}(\Gamma)}$$

This is done using Grisvard's Theorem:

Assume that Ω is a bounded Lipschitz-continuous open subset of $\mathbb{R}^d$.
Let s, s_1, $s_2 \geq 0$ and p, p_1, $p_2 \in [1, +\infty)$ such that $s_1 \geq s$, $s_2 \geq s$ and either

$$s_1 + s_2 - s \geq d\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \geq 0, \quad s_i - s > d\left(\frac{1}{p_i} - \frac{1}{p}\right) \text{ or}$$

$$s_1 + s_2 - s > d\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \geq 0, \quad s_i - s \geq d\left(\frac{1}{p_i} - \frac{1}{p}\right)$$

Then the mapping $(u, v) \longmapsto uv$ is a continuous bilinear map from $W^{s_1,p_1}(\Omega) \times W^{s_2,p_2}(\Omega)$ to $W^{s,p}(\Omega)$.

Numerical tests by FreeFEM++

We have tested ou iterative scheme for the data:

$$(\text{Atmosphere:}) \quad \Omega_1 = [0,5]\times[0,1]\times[0,1], \qquad (\text{Ocean:}) \quad \Omega_2 = [0,5]\times[0,1]\times[-1,0].$$

Turbulent diffusions: (realistic values)

$$\alpha_i(\cdot) = \gamma_i(\cdot).$$

$$\gamma_1(k_1) = 3\cdot 10^{-3} + 0.277\cdot 10^{-4}\sqrt{k_1}, \qquad \gamma_1(k_2) = 3\cdot 10^{-2} + 0.185\cdot 10^{-5}\sqrt{k_2}$$

Friction coefficients: (realistic values)

$$\kappa_i = 10^{-3},\ \lambda = 5\cdot 10^{-2}.$$

Boundary data:

$$\mathbf{u}_1 = (1,0,0) \quad \text{on } y = 1$$

$$\mathbf{u}_i = (0,0,0) \quad \text{on the remaining of } \partial\Omega$$

$$k_i = 0 \quad \text{on } \partial\Omega/\Gamma.$$

Numerical tests by FreeFEM++

- Discretization P2-P1 for velocity-pressure
- Discretization P2 for TKE
- Computations with FreeFEM++ **http://www.freefem.org/ff++/**
- In order to verify the convergence order, we use different size meshes.

Numerical tests by FreeFEM++: Results

- The algorithm converges to steady state with a rate > 0.25, in agreement
  with theoritical analysis:

$$\sum_{i=1}^{2} |k_{i,h} - k_i|_{1,\Omega_i} + |\mathbf{u}_{i,h} - \mathbf{u}_i|_{1,\Omega_i} = \mathcal{E}_h \leq c\left(h^{\sigma} + h^{\sigma/2} + h^{1/2}\right)$$

Where $\sigma = \frac{3}{2} - \frac{3}{3+\varepsilon} > \frac{1}{2}, \quad \forall \varepsilon > 0.$

We set $\tau_h = \dfrac{log\left(\frac{\mathcal{E}_h}{\mathcal{E}_{h/2}}\right)}{log(2)} \approx \sigma$

Mesh size	τ_h
h	----
h/2	0.12
h/4	0.16
h/8	0.22
h/16	not yet! [in progress]

Numerical tests by FreeFEM++

Wind-induced flow on swimming pool

$$(\text{Atmosphere:}) \quad \Omega_1 = [0, 10^4] \times [0, 5 \cdot 10^3] \times [0, 500],$$

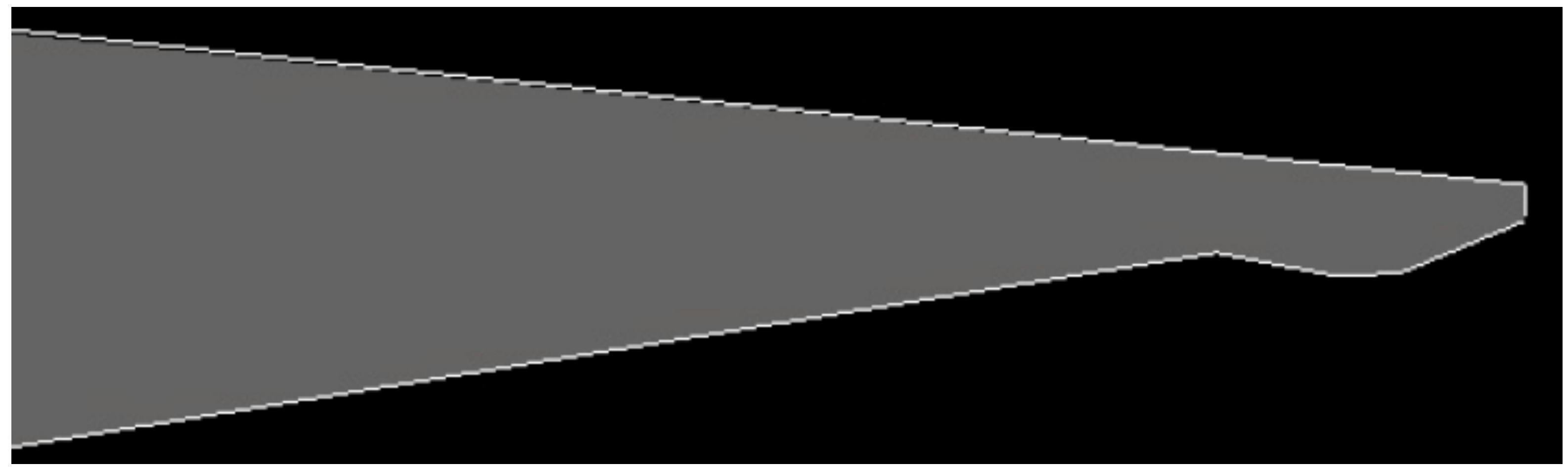
The ocean domain is defined by:

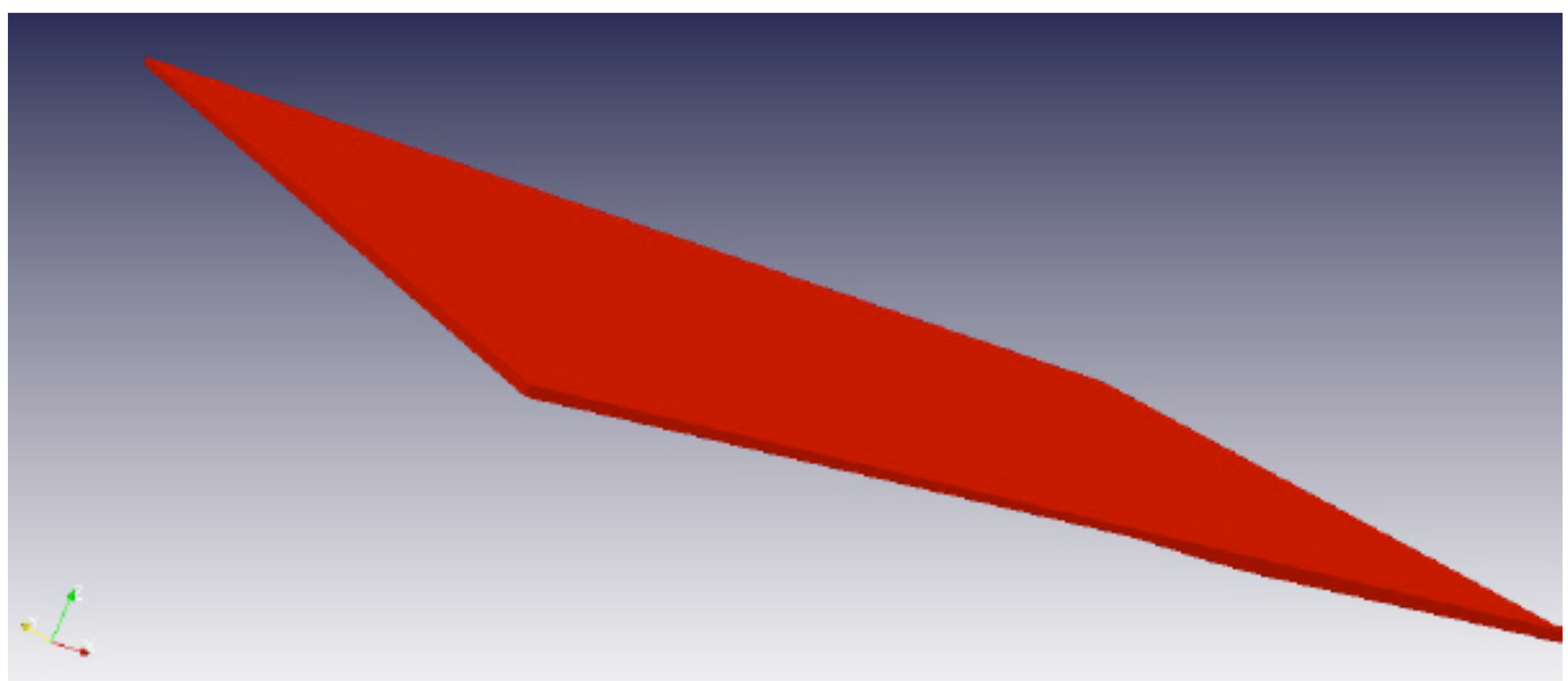
- Horizontal dimensions (m): $\omega = [0, 10^4] \times [0, 5 \cdot 10^3]$
- Bathemetry (m):

$$\begin{cases} 50 & \text{if} \quad 0 \leq x \leq 4 \cdot 10^3 \\ 50 \cdot \dfrac{5 \cdot 10^3 - x}{10^3} + 100 \cdot \dfrac{4 \cdot 10^3 - x}{10^3} & \text{if} \quad 4 \cdot 10^3 \leq x \leq 5 \cdot 10^3 \\ 100 & \text{if} \quad 5 \cdot 10^3 \leq x \leq 10^4 \end{cases}$$

Numerical tests by FreeFEM++

Ocean Domain





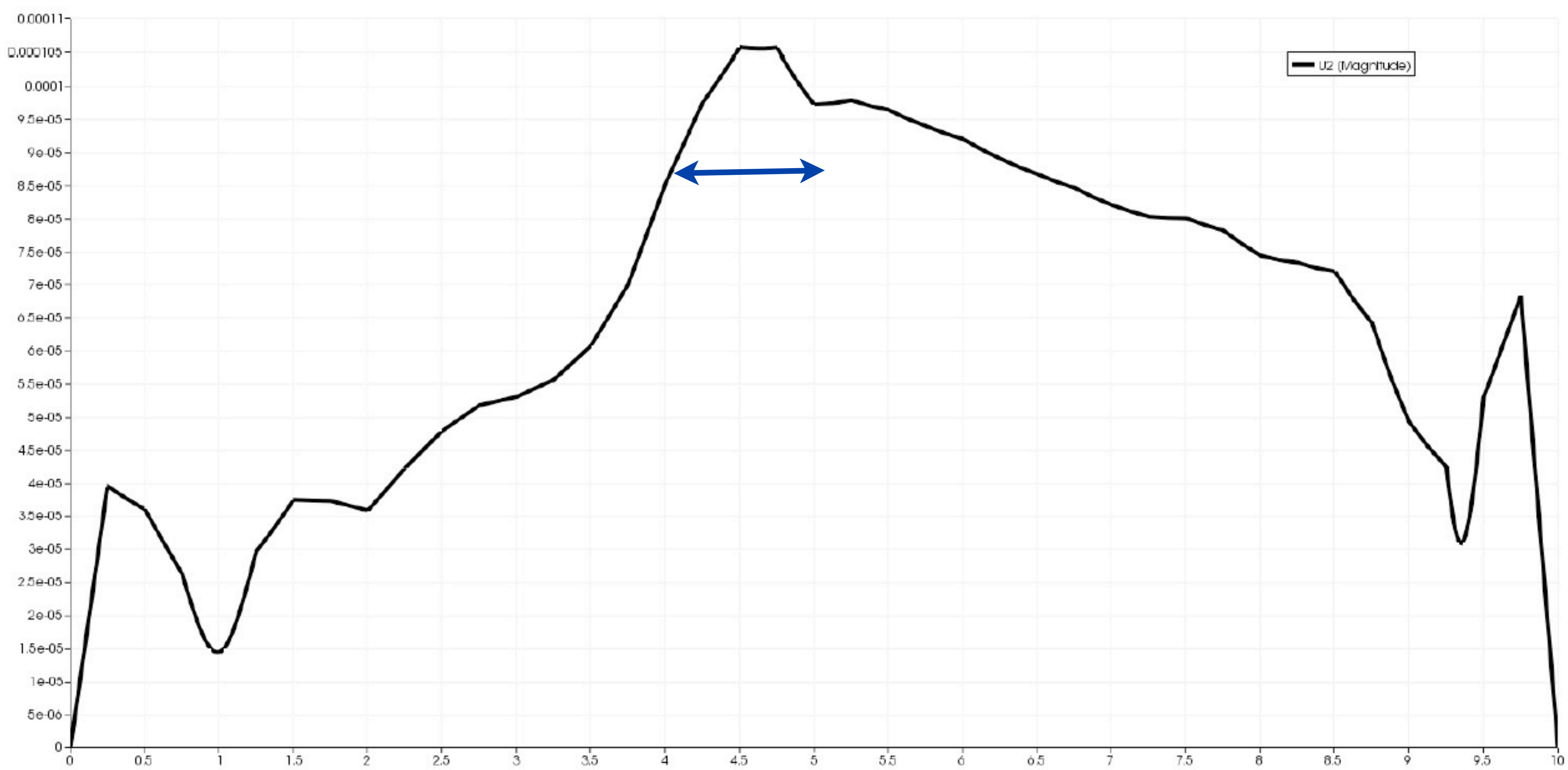
We test the formation of the up-welling effect, due to the interaction
between wind-tension and Coriolis forces

Numerical tests by FreeFEM++
Methodology

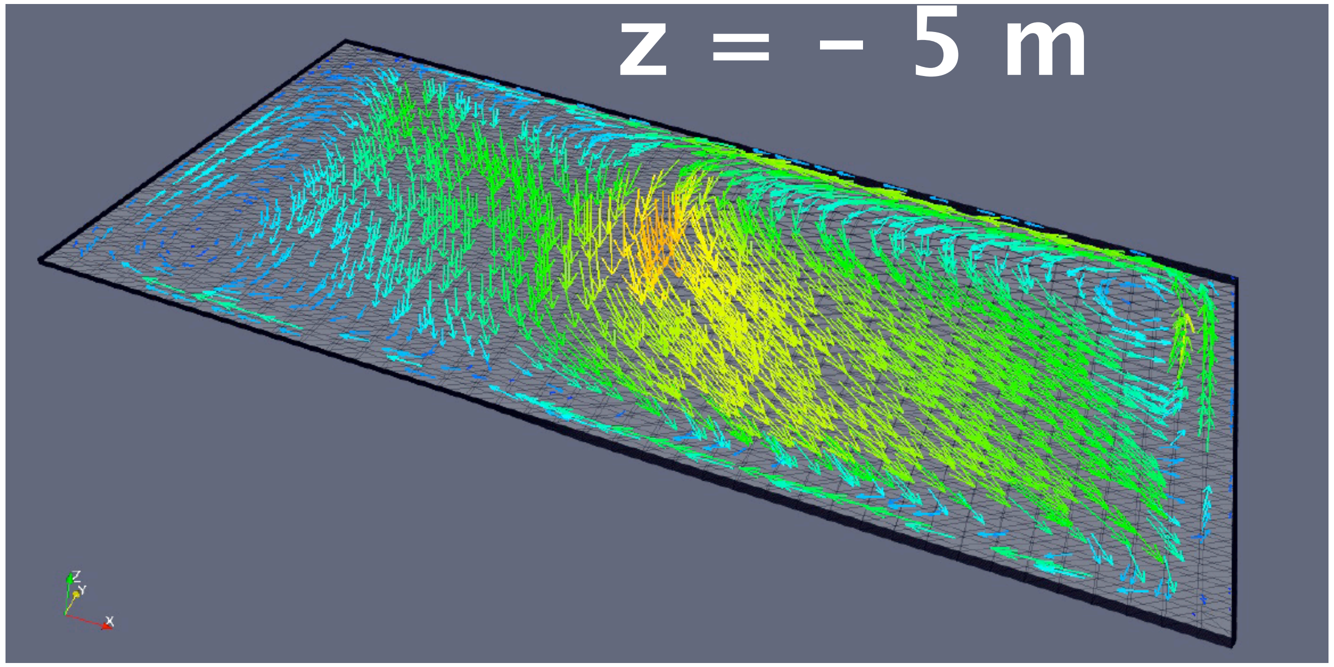
The system of equation we solve in FreeFEM++:

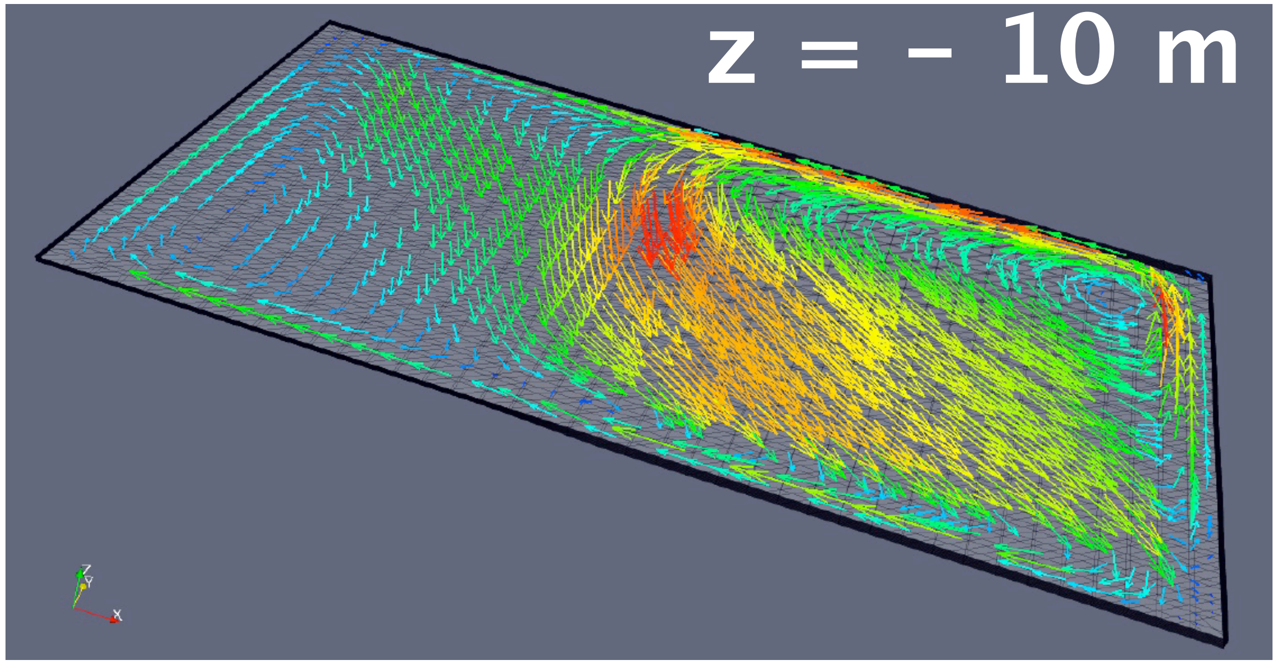
$$\begin{aligned}
\partial_t \mathbf{u}_i + (\mathbf{u}_i \cdot \nabla)\, \mathbf{u}_i + \tau\,(-u_{i,y}, u_{i,x}, 0) - \nabla \cdot (\alpha_i(k_i)\nabla \mathbf{u}_i) + \nabla p_i &= \mathbf{f}_i \text{ in } \Omega_i, \\
\nabla \cdot \mathbf{u}_i &= 0 \text{ in } \Omega_i, \\
\partial_t k_i + \mathbf{u}_i \nabla k_i - \nabla \cdot (\gamma_i(k_i)\nabla k_i) &= \alpha_i(k_i)|\nabla \mathbf{u}_i|^2 \text{ in } \Omega_i, \\
\mathbf{u}_i &= \mathbf{0} \text{ on } \Gamma_i, \\
k_i &= 0 \text{ on } \Gamma_i, \\
\alpha_i(k_i)\partial_{\mathbf{n}_i}\mathbf{u}_i - p_i \mathbf{n}_i + \kappa_i(\mathbf{u}_i - \mathbf{u}_j)|\mathbf{u}_i - \mathbf{u}_j| &= \mathbf{0} \text{ on } \Gamma, 1 \le i \ne j \le 2, \\
k_i &= \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2 \text{ on } \Gamma.
\end{aligned}$$

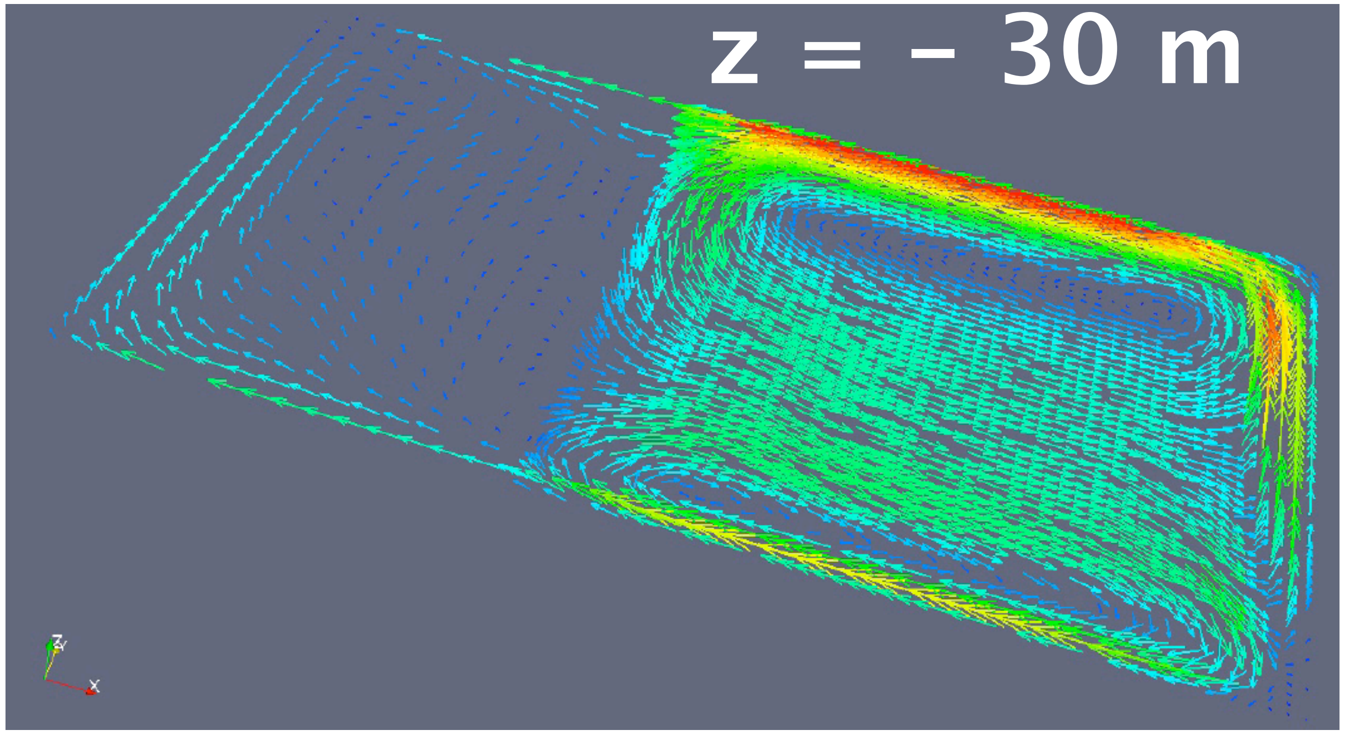
Numerical tests by FreeFEM++
Results:

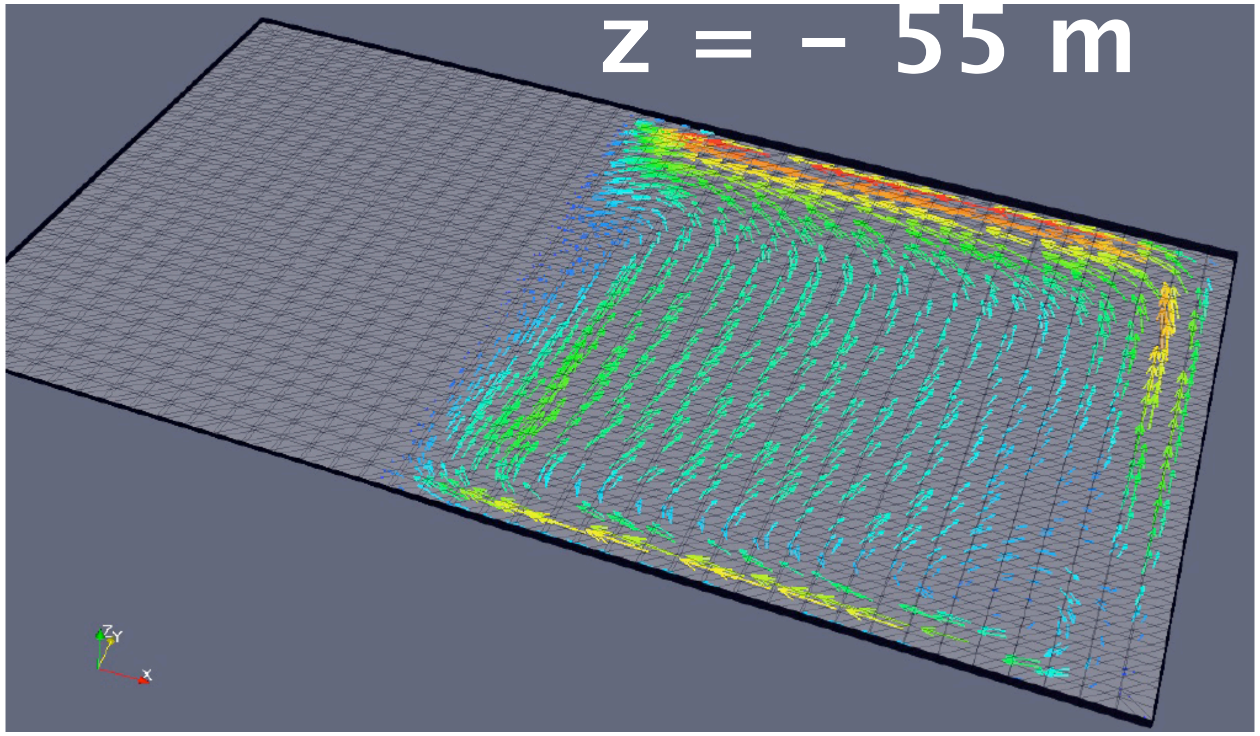


Numerical tests by FreeFEM++
Velocity fields:









Numerical tests by FreeFEM++
Velocity fields:

Movies