

AN ITERATIVE PROCEDURE TO SOLVE A COUPLED TWO-FLUIDS TURBULENCE MODEL*

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Abstract. This paper introduces a scheme for the numerical approximation of a model for two turbulent flows with coupling at an interface. We consider the variational formulation of the coupled model, where the turbulent kinetic energy equation is formulated by transposition. We prove the convergence of the approximation to this formulation for 3D flows for large turbulent viscosities and smooth enough flows, whenever bounded in $W^{1,p}$ Sobolev norms for p large enough. Under the same assumptions, we show that the limit is a solution of the initial problem. Finally, we give some numerical experiments to enlighten the theoretical work.

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1. INTRODUCTION

In this contribution we focus our attention on the modelling of the surface layer between the atmosphere and the ocean. We are interested in designing effective procedures to solve the following coupled model:

$$\left\{ \begin{array}{l} -\nabla \cdot (\alpha_i(k_i)\nabla \mathbf{u}_i) + \mathbf{grad} p_i = \mathbf{f}_i \text{ in } \Omega_i, \\ \nabla \cdot \mathbf{u}_i = 0 \text{ in } \Omega_i, \\ -\nabla \cdot (\gamma_i(k_i)\nabla k_i) = \alpha_i(k_i)|\nabla \mathbf{u}_i|^2 \text{ in } \Omega_i, \\ \mathbf{u}_i = \mathbf{0} \text{ on } \Gamma_i, \\ k_i = 0 \text{ on } \Gamma_i, \\ \alpha_i(k_i)\partial_{\mathbf{n}_i} \mathbf{u}_i - p_i \mathbf{n}_i + \kappa(\mathbf{u}_i - \mathbf{u}_j)|\mathbf{u}_i - \mathbf{u}_j| = \mathbf{0} \text{ on } \Gamma, 1 \leq i \neq j \leq 2, \\ k_i = \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2 \text{ on } \Gamma. \end{array} \right. \quad (1.1)$$

Where each triple (\mathbf{u}_i, p_i, k_i) is defined in the domain Ω_i , $1 \leq i \leq 2$. The generic point in \mathbb{R}^2 , resp. in \mathbb{R}^3 , is denoted by $\mathbf{x} = (x, z)$, resp. $\mathbf{x} = (x, y, z)$.

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System (1.1) is a simplified model for two stationary turbulent flows in adjacent domains, coupled by boundary conditions on the interface, such as the system atmosphere-ocean. Indeed, it is a simplified mathematical formulation of the RANS (Reynolds Averaged Navier-Stokes) model of order 1 used to simulate a stationary mean flow when convection is neglected. This kind of modelling is often used in engineering or geophysics, see for instance Bernardi *et al.* [6], Launder and Spalding [16], Mohammadi and Pironneau [21], Piquet [22], Wilcox [23].

In what follows, Ω_i ($i = 1, 2$) are bounded domains of \mathbb{R}^d , $d = 2, 3$, which are either convex or of class $\mathbb{C}^{1,1}$, with boundaries $\partial\Omega_i = \Gamma_i \cup \Gamma$, $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$ being the interface between the two fluids. Γ is assumed to be flat. Indeed, we assume that the so-called “rigid lid hypothesis” (introduced by Bryan in [9]) holds, an hypothesis which is standard in geophysics and oceanography. Each of the two turbulent fluids is modeled by a simplified one-equation turbulence model whose unknowns are the velocity \mathbf{u}_i and the turbulent kinetic energy (TKE) k_i .

In the first equation we model the generation of eddy viscosity in flow i by the term

$$-\alpha_i(k_i)\nabla\mathbf{u}_i.$$

The (positive) quantity $\alpha_i(k_i)$ is the eddy viscosity. This is a simplification of the usual modelling of Reynolds Stress Tensor by

$$R_i \simeq -\alpha_i(k_i) (\nabla\mathbf{u}_i + \nabla^t\mathbf{u}_i).$$

We prefer the first expression for simplicity of mathematical analysis, although our analysis still holds for the second one. We also neglect transport effects, we intend to analyze them in a forthcoming paper. The fluids are assumed to be incompressible (second equation). In the third equation we model the generation of TKE by means of a production source term

$$\alpha_i(k_i)|\nabla\mathbf{u}_i|^2,$$

although the physical one should be

$$\alpha_i(k_i)|\nabla\mathbf{u}_i + \nabla^t\mathbf{u}_i|^2.$$

Again, we prefer the first expression for simplicity of mathematical treatment. Also the turbulent diffusion of TKE is the function $\gamma_i(k_i)$. We neglect the viscous dissipation effects, to avoid to manage an additional statistic of the turbulence (a mixing length or the turbulent dissipation ε , for instance). We assume non-slipping boundary conditions in the boundary parts Γ_i for simplicity (fourth and fifth equations). These in practice are replaced by wall-laws to simulate the generation of turbulence on solid boundaries. The sixth equation globally models the interaction of the two boundary layers on one and another side of the interface Γ as friction effects, by means of a set of boundary conditions similar to Manning’s law. Finally, the last equation models the production of TKE in the interface. The coefficients κ_i and λ are positive.

We assume that the turbulent diffusions α_i and γ_i belong to $W^{1,\infty}(\mathbb{R})$ and verify $\alpha_i \geq \nu$, $\gamma_i \geq \nu$, for some $\nu > 0$. The eddy diffusions usually are unbounded functions of the TKE of the form $a + b\sqrt{k}$, as we use in the numerical simulations reported in Section 5 (see for instance [8,17,21]). But this renders the analysis much more complex even for a one-fluid turbulence model (see [18]). So we consider a simplified model, that still includes several realistic non-linear interactions.

System (1.1) was studied in [3] where existence and uniqueness of small smooth solutions were proved. Spectral and Finite Element discretizations were studied in subsequent papers by the same authors and co-workers (see [4,5]). In these papers, the ability of these discretization techniques to approach the solution of model (1.1) was proved. However, in both cases the discretizations achieved consisted in fully non-linear sets of algebraic equations. Our purpose here is to derive iterative procedures to solve system (1.1) that decouples the interaction of the problem, leading to mildly non-linear problems.

Let us introduce the function spaces

$$\begin{aligned} \mathbf{X}_i &= \{ \mathbf{v}_i \in \mathbf{H}^1(\Omega_i); \quad \mathbf{v}_i = \mathbf{0} \quad \text{on } \Gamma_i \}, \\ L_0^2(\Omega_i) &= \left\{ q_i \in L^2(\Omega_i); \quad \int_{\Omega_i} q_i = 0 \right\}. \end{aligned} \tag{1.2}$$

Consider also two conjugate positive real numbers r and r' *i.e.*

$$\frac{1}{r} + \frac{1}{r'} = 1, \quad \text{such that } r > d.$$

We introduce the following iterative procedure to solve (1.1): once known $\mathbf{u}_i^n \in \mathbf{X}_i, p_i^n \in L_0^2(\Omega_i), k_i \in W^{1,r'}(\Omega_i), i = 1, 2$, solve:

Problem 1. Obtain $\mathbf{u}_i^{n+1} \in \mathbf{X}_i, i = 1, 2$, such that

$$\int_{\Omega_i} \alpha_i(k_i^n) \nabla \mathbf{u}_i^{n+1} : \nabla \mathbf{v}_i \, d\mathbf{x} - \int_{\Omega_i} (\nabla \cdot \mathbf{v}_i) p_i^{n+1} \, d\mathbf{x} + \kappa_i \int_{\Gamma} |\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}| (\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \cdot \mathbf{v}_i \, d\tau = \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{v}_i \, d\tau, \quad \forall \mathbf{v}_i \in \mathbf{X}_i, \quad \text{and}$$

$$\int_{\Omega_i} (\nabla \cdot \mathbf{u}_i^{n+1}) q_i \, d\mathbf{x} = 0, \quad \forall q_i \in L_0^2(\Omega_i),$$

and

Problem 2. Obtain $k_i^{n+1} \in W^{1,r'}(\Omega_i), i = 1, 2$, such that

$$k_i^{n+1} = 0 \quad \text{on } \Gamma_i, \quad k_i^{n+1} = \lambda |\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|^2 \quad \text{on } \Gamma, \\ \int_{\Omega_i} \gamma_i(k_i^n) k_i^{n+1} \varphi_i \, d\mathbf{x} = \int_{\Omega_i} \alpha_i(k_i^n) |\nabla \mathbf{u}_i^{n+1}|^2 \varphi_i \, d\mathbf{x}, \quad \forall \varphi_i \in W_0^{1,r}(\Omega_i).$$

Remark 1.1. We take $r > d$ to give a sense to the equation for the k_i in model (1.1). Indeed, the term $\alpha_i(k_i^n) |\nabla \mathbf{u}_i^{n+1}|^2$ belongs to $L^1(\Omega_i)$, and it follows from the Sobolev Imbedding Theorem that the test function φ_i in Problem 2 belongs to $L^\infty(\Omega_i)$, so that the right-hand member is well defined.

Observe that Problem 1 is in reality non-linear due to the presence of the Manning-like source term. This is a mild non-linearity due to the monotonic nature of this term, that may be made explicit in practice if mass-lumping techniques are used.

Our main result states that if the sequences $(\mathbf{u}_i^n)_n$ and $(k_i^n)_n$ are respectively bounded in $W^{1,3+\varepsilon}(\Omega_i)^d$ and $W^{1,3}(\Omega_i)$, then, for small enough data (in a convenient sense), the iterative scheme is contracting. This regularity is realistic, as it is not far from the $W^{1,2}$ regularity that has been proved for problem (1.1) for general data. The main ingredients to show the convergence of our scheme are the convenient choices of test functions, and the use of the harmonic liftings R_i of Dirichlet boundary conditions on Γ on the Ω_i (see the proof of Thm. 3.4).

Our paper is organised as follows. In Section 2 we introduce a weak formulation of the above iterative procedure. Section 3 is devoted to prove the contractiveness of the TKE sequence. Due to the production term of the TKE on interface $\Gamma: k_i = \lambda |\mathbf{u}_1 - \mathbf{u}_2|^2$, it is necessary to estimate the expression

$$\| |\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|^2 - |\mathbf{u}_1^n - \mathbf{u}_2^n|^2 \|_{H_{00}^{\frac{1}{2}}(\Gamma)}, \tag{1.3}$$

where the special space $H_{00}^{\frac{1}{2}}(\Gamma)$ is the subspace of $H^{\frac{1}{2}}(\Gamma)$ whose extension by zero to $\partial\Omega_1$ (for instance, it could be also to $\partial\Omega_2$) belongs to $H^{\frac{1}{2}}(\partial\Omega_1)$. An intrinsic scalar product on $H_{00}^{\frac{1}{2}}(\Gamma)$ is defined as

$$((u, v))_{H_{00}^{\frac{1}{2}}(\Gamma)} = \int_{\Gamma} u(x) v(x) \, dx + \int_{\Gamma} \int_{\Gamma} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^d} \, dx \, dy + \int_{\Gamma} \frac{u(x) v(x)}{d(x, \partial\Gamma)}, \tag{1.4}$$

where the first two summands define the $H^{\frac{1}{2}}(\Gamma)$ scalar product (see Adams and Fournier [1], Thm. 7.48). Its expression involves the distance $d(x, \partial\Gamma)$ to the boundary of $\partial\Gamma$. It comes from the restriction to $H_{00}^{\frac{1}{2}}(\Gamma)$ of the scalar product in $H^{\frac{1}{2}}(\partial\Omega_1)$ for instance. It is given by Lions and Magenes in [20], Chapter 1, Theorem 11.7.

The estimation of (1.3) is done using a Grisvard’s result (Lem. 3.5 in the paper at hand, see [14] for the original reference) on estimates in $W^{s,p}$ of products of functions of W^{s_j,p_j} , for some real numbers s, s_j and some non-negative integers $p, p_j, j = 1, 2$. Concerning the sequence of pressure iterates $(p_i^n)_n$, we use a specific inf-sup condition (see Cor. 3.7) to show that it is a Cauchy sequence, see Theorem 3.8.

The convergence analysis is performed in Section 4. In Theorem 4.1 we prove that the triple $(\mathbf{u}_i^n, k_i^n, p_i^n)_n$ has a unique limit, which is a solution of the variational formulation (2.3)–(2.4).

We finally present some numerical tests in Section 5. These tests are realized with the software **FreeFEM3D** (see [12]) in meaningful situations, that agree with the expectations of our result.

2. ITERATIVE SCHEME

We shall at first describe the weak formulation of problem (1.1). We assume that α_i and γ_i are bounded functions from the set of nonnegative real numbers \mathbb{R}_+ onto \mathbb{R} , and belong to $W^{1,\infty}(\Omega_i)$, which satisfy

$$\forall \ell \in \mathbb{R}_+, \quad \delta_1 \geq \alpha_i(\ell) \geq \nu \quad \text{and} \quad \delta_1 \geq \gamma_i(\ell) \geq \nu, \tag{2.1}$$

and

$$\forall \ell \in \mathbb{R}_+, \quad |\alpha_i'(\ell)| \leq \delta_2 \quad \text{and} \quad |\gamma_i'(\ell)| \leq \delta_2, \tag{2.2}$$

where δ_1, δ_2 and ν are positive constants.

System (1.1) admits the following variational formulation:

Find $(\mathbf{u}_i, p_i, k_i) \in \mathbf{X}_i \times L^2(\Omega_i) \times W^{1,r'}(\Omega_i)$ such that, for all $(\mathbf{v}_i, q_i, \varphi_i) \in \mathbf{X}_i \times L^2(\Omega_i) \times W_0^{1,r}(\Omega_i)$,

$$\begin{aligned} a_i(k_i; \mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) + \kappa_i \int_{\Gamma} |\mathbf{u}_i - \mathbf{u}_j|(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{v}_i \, d\tau &= \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{v}_i \, d\mathbf{x} \\ b_i(\mathbf{u}_i, q_i) &= 0, \end{aligned} \tag{2.3}$$

and,

$$k_i = 0 \text{ on } \Gamma_i, \quad k_i = \lambda |\mathbf{u}_i - \mathbf{u}_j|^2 \text{ on } \Gamma, \quad \text{and}$$

$$C_i(k_i; k_i, \varphi_i) = \int_{\Omega_i} \alpha_i(k_i) |\nabla \mathbf{u}_i|^2 \varphi_i \, d\mathbf{x}, \tag{2.4}$$

where the forms $a_i(\cdot; \cdot, \cdot)$, $b_i(\cdot, \cdot)$ and $C_i(\cdot; \cdot, \cdot)$ are defined by

$$\begin{aligned} a_i(\ell_i; \mathbf{u}_i, \mathbf{v}_i) &= \int_{\Omega_i} \alpha(\ell_i) \nabla \mathbf{u}_i : \nabla \mathbf{v}_i \, d\mathbf{x}, \\ b_i(\mathbf{v}_i, q_i) &= - \int_{\Omega_i} q_i \nabla \cdot \mathbf{v}_i \, d\mathbf{x}, \\ C_i(\ell_i; k_i, \varphi) &= \int_{\Omega_i} \gamma_i(\ell_i) \nabla k_i \cdot \nabla \varphi_i \, d\mathbf{x}. \end{aligned}$$

Note that the bilinear forms a_i and C_i in (2.3) depend on k_i .

Remark 2.1. Since $\mathbf{u}_i \in \mathbf{X}_i$ then its trace on Γ belongs to $\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)$. Thus by using definition of this space $\mathbf{u}_i|_{\Gamma}$ belongs to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$ and applying the Sobolev embedding from $\mathbf{H}^{\frac{1}{2}}(\Gamma)$ into $L^3(\Gamma)^d$, we conclude that the integral $\int_{\Gamma} |\mathbf{u}_i - \mathbf{u}_j|(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{v}_i \, d\tau$ is well defined.

This formulation makes sense, as $\alpha_i(k_i)|\nabla\mathbf{u}_i|^2 \in W^{1,r'}(\Omega_i)$ when $\mathbf{u}_i \in \mathbf{X}_i$. In Lewandowski [19] it is proved that this formulation admits at least a solution.

We shall consider the following iterative procedure:

Given $(\mathbf{u}_i^n, p_i^n, k_i^n) \in \mathbf{X}_i \times L^2(\Omega_i) \times W^{1,r'}(\Omega_i)$, $i = 1, 2$, obtain $(\mathbf{u}_i^{n+1}, p_i^{n+1}, k_i^{n+1}) \in \mathbf{X}_i \times L^2(\Omega_i) \times W^{1,r'}(\Omega_i)$, such that $\forall(\mathbf{v}_i, q_i, \varphi_i) \in \mathbf{X}_i \times L^2(\Omega_i) \times W_0^{1,r}(\Omega_i)$,

$$a_i(k_i^n; \mathbf{u}_i^{n+1}, \nabla\mathbf{v}_i) + b_i(\mathbf{v}_i, p_i^{n+1}) + \kappa_i \int_{\Gamma} |\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \cdot \mathbf{v}_i d\tau = \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{v}_i d\tau, \tag{2.5}$$

$$\text{and } b_i(\mathbf{u}_i^{n+1}, q_i) = 0, \tag{2.6}$$

and

$$k_i^{n+1} = 0 \text{ on } \Gamma_i, \tag{2.7}$$

$$k_i^{n+1} = \lambda|\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|^2 \text{ on } \Gamma, \tag{2.8}$$

$$C_i(k_i^n; k_i^{n+1}, \varphi_i) = \int_{\Omega_i} \alpha_i(k_i^n)|\nabla\mathbf{u}_i^{n+1}|^2 \varphi_i d\mathbf{x}. \tag{2.9}$$

3. CONTRACTIVENESS

In this section we prove that the sequence of TKE $(k_i^n)_n$ is contracting, and that consequently the sequences of velocities $(\mathbf{u}_i^n)_n$ also is contracting, in the sense that

$$\left\{ \begin{array}{l} \sum_{i=1}^2 \|\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)}^2 \leq K \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2, \text{ and} \\ \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2 \leq K \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2. \end{array} \right. \tag{3.1}$$

We may interpret these inequalities in the sense that the sequence of pairs $(\mathbf{u}_i^n, k_i^n)_n$ is contracting in the Hilbert space $\mathbf{X}_i \times L^2(\Omega_i)$. However, to simplify our derivation, we shall not explicitly use this space. Finally, we show that the pressures $(p_i^n)_n$ is a Cauchy sequence.

We suppose from now on that the sequences $(\mathbf{u}_i^n)_n$ and $(k_i^n)_n$ verify the following hypothesis.

Hypothesis 3.1. $\forall n \in \mathbb{N}$, $\mathbf{u}_i^n \in W^{1,3+\varepsilon}(\Omega_i)^d$ and $k_i^n \in W^{1,3}(\Omega_i)$, and one has

$$\|\mathbf{u}_i^n\|_{W^{1,3+\varepsilon}(\Omega_i)^d} \leq M, \quad \|k_i^n\|_{W^{1,3}(\Omega_i)} \leq M,$$

where M and ε are two fixed positive numbers.

Remark 3.2. Note that the natural estimates for velocities in model (1.1) are in H^1 norm, not in $W^{1,3}$ norm. Indeed, choosing \mathbf{v}_i equal to $\frac{1}{\kappa_i}\mathbf{u}_i^{n+1} \in \mathbf{X}_i$ in equation (2.5), and summing upon $i = 1, 2$ gives

$$\sum_{i=1}^2 \frac{1}{\kappa_i} \left(\int_{\Omega_i} \alpha_i(k_i^n)|\nabla\mathbf{u}_i^{n+1}|^2 d\mathbf{x} \right) + \int_{\Gamma} |\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|^3 d\tau = \sum_{i=1}^2 \frac{1}{\kappa_i} \int_{\Omega_i} \mathbf{f}_i \mathbf{u}_i^{n+1} d\mathbf{x}.$$

Since the integrated term on Γ is nonnegative and thanks to (2.1), we deduce

$$\frac{\nu}{c_M} \sum_{i=1}^2 \|\nabla\mathbf{u}_i^{n+1}\|_{L^2(\Omega_i)}^2 \leq \frac{1}{c_m} \sum_{i=1}^2 \int_{\Omega_i} \mathbf{f}_i \mathbf{u}_i^{n+1} d\mathbf{x}.$$

Using the Cauchy-Schwarz and Poincaré-Friedrichs inequalities, we obtain

$$\sum_{i=1}^2 \|\nabla \mathbf{u}_i^{n+1}\|_{L^2(\Omega_i)^d}^2 \leq \frac{c}{\nu^2} \sum_{i=1}^2 \|\mathbf{f}_i\|_{L^2(\Omega_i)^d}^2, \tag{3.2}$$

where c is a positive constant, depending only on the domains Ω_i and the friction coefficients κ_i .

We next prove that the contractiveness of the TKE implies that of the velocities.

Lemma 3.3. *Assume that Hypothesis 3.1 holds and that $\mathbf{f}_i \in L^2(\Omega_i)^d$, $i = 1, 2$. Then there exists a positive constant c , depending only on Ω_i , such that*

$$\sum_{i=1}^2 \|\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)^d}^2 \leq \frac{c\delta_2^2 M^2}{\nu^2} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)^d}^2. \tag{3.3}$$

Proof. Let us take $\mathbf{v}_i = \frac{1}{\kappa_i}(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n) \in \mathbf{X}_i$ as a test function in (2.5) at iterations n and $n + 1$. Then, calculating the difference between both obtained equations, and summing on $i = 1, 2$, yields

$$\begin{aligned} & \sum_{i=1}^2 \frac{1}{\kappa_i} \int_{\Omega_i} \alpha_i(k^n) \nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n) : \nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n) \, dx \\ & \quad + \sum_{i=1}^2 \frac{1}{\kappa_i} \int_{\Omega_i} (\alpha_i(k^n) - \alpha_i(k_i^{n-1})) \nabla \mathbf{u}_i^n : \nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n) \, dx \\ & \quad + \int_{\Gamma} (|\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|(\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}) - |\mathbf{u}_1^n - \mathbf{u}_2^n|(\mathbf{u}_1^n - \mathbf{u}_2^n)) \cdot ((\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}) - (\mathbf{u}_1^n - \mathbf{u}_2^n)) \, d\tau = 0. \end{aligned}$$

The following inequality holds for all vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$,

$$(|\mathbf{b}|\mathbf{b} - |\mathbf{a}|\mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \geq 0. \tag{3.4}$$

To prove it, consider the function $J : \mathbb{R}^d \mapsto \mathbb{R}$ defined by $J(\mathbf{a}) = \frac{2}{3}|\mathbf{a}|^3$. J is convex and differentiable. Thus,

$$(\nabla J(\mathbf{a}) - \nabla J(\mathbf{b})) \cdot (\mathbf{b} - \mathbf{a}) \geq 0, \quad \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^d.$$

Then, (3.4) follows as $\nabla J(\mathbf{a}) = |\mathbf{a}|\mathbf{a} \cdot \mathbf{b}$.

We deduce that

$$\sum_{i=1}^2 \frac{1}{\kappa_i} \int_{\Omega_i} \alpha_i(k_i^n) |\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)|^2 \, dx + \sum_{i=1}^2 \frac{1}{\kappa_i} \int_{\Omega_i} (\alpha_i(k_i^n) - \alpha_i(k_i^{n-1})) \nabla \mathbf{u}_i^n \cdot \nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n) \, dx \leq 0. \tag{3.5}$$

It comes from Hypothesis 3.1 that $\nabla \mathbf{u}_i^n$ belongs to $L^3(\Omega_i)^d$ and that $\|\nabla \mathbf{u}_i^n\|_{L^3(\Omega_i)^d} \leq M$. Furthermore, according to the relation (2.2) and the canonical injection from $H^1(\Omega_i)$ to $L^6(\Omega_i)$ and Hölder inequality, we obtain

$$\begin{aligned} \nu \sum_{i=1}^2 \|\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)^d}^2 &\leq \delta_2 \sum_{i=1}^2 \int_{\Omega_i} |k_i^n - k_i^{n-1}| |\nabla \mathbf{u}_i^n| |\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)| \, d\mathbf{x} \\ &\leq \delta_2 \sum_{i=1}^2 \|k_i^n - k_i^{n-1}\|_{L^6(\Omega_i)} \|\nabla \mathbf{u}_i^n\|_{L^3(\Omega_i)^d} \|\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)^d}, \\ &\leq \frac{\delta_2^2 M^2}{2\nu} \sum_{i=1}^2 \|k_i^n - k_i^{n-1}\|_{L^6(\Omega_i)}^2 + \frac{\nu}{2} \sum_{i=1}^2 \|\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)^d}^2. \end{aligned}$$

From this estimate we conclude relation (3.3). □

We next prove the contractiveness of the sequence of TKE $(k_i^n)_n$.

Theorem 3.4. *Assume that Hypothesis 3.1 holds and that $\mathbf{f}_i \in L^2(\Omega_i)^d$. Then there exists a positive constant c , depending only on Ω_i and on the data κ_i and λ , such that for all $n \in \mathbb{N}^*$,*

$$\sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2 \leq c \frac{(\delta_1^2 + 1)\delta_2^2}{\nu^3} M \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2. \tag{3.6}$$

Proof. The proof of this theorem is made in several steps.

First step. Choice of the test function

We first choose a particular test function φ_i in the equations (2.7)–(2.9). For that purpose, we need to introduce the special space $H_{00}^{\frac{1}{2}}(\Gamma)$ (see [20], Chap. 1, Thm. 11.7 for instance). We also need to introduce the following operator. Let R_i be a continuous harmonic lifting operator from $H_{00}^{\frac{1}{2}}(\Gamma)$ to $H^1(\Omega_i)$, defined as follows.

For any η in $H_{00}^{\frac{1}{2}}(\Gamma)$, $R_i\eta$ belongs to $H^1(\Omega_i)$, and satisfies

$$\begin{cases} -\Delta R_i\eta &= 0 \text{ in } \Omega_i, \\ R_i\eta &= \eta \text{ on } \Gamma, \\ R_i\eta &= 0 \text{ on } \Gamma_i. \end{cases} \quad \text{and}$$

Moreover, one has

$$\forall \eta \in H_{00}^{\frac{1}{2}}(\Gamma), \quad \|R_i\eta\|_{H^1(\Omega_i)} \leq c_R \|\eta\|_{H_{00}^{\frac{1}{2}}(\Gamma)}, \tag{3.7}$$

where $c_R > 0$ depends only on Ω_i .

According to Hypothesis 3.1, $\forall n \in \mathbb{N}^*$, $k_i^n \in W^{1,3}(\Omega_i)$, then its trace on Γ belongs to $W^{\frac{2}{3},3}(\Gamma)$. Thus, by Sobolev’s injections, it belongs to $H^{\frac{1}{2}}(\Gamma)$. Furthermore, $k_i^n = 0$ on Γ_i , then $k_i^n|_{\Gamma}$ belongs to $H_{00}^{\frac{1}{2}}(\Gamma)$.

The idea consists in choosing the test function φ_i equal to $(k_i^{n+1} - k_i^n) - R_i(k_i^{n+1} - k_i^n)$ in equation (2.9) at steps n and $n + 1$. Then, we make the difference between both obtained equations, and sum upon $i = 1, 2$. We find

$$\nu \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{0,\Omega_i}^2 \leq \sum_{j=1}^7 I_j, \tag{3.8}$$

where

$$\begin{aligned}
I_1 &= \sum_{i=1}^2 \left| \int_{\Omega_i} \alpha_i(k_i^n) (|\nabla \mathbf{u}_i^{n+1}|^2 - |\nabla \mathbf{u}_i^n|^2) (k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|, \\
I_2 &= \sum_{i=1}^2 \left| \int_{\Omega_i} (\alpha_i(k_i^n) - \alpha_i(k_i^{n-1})) |\nabla \mathbf{u}_i^n|^2 (k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|, \\
I_3 &= \sum_{i=1}^2 \left| \int_{\Omega_i} (\gamma_i(k_i^n) - \gamma_i(k_i^{n-1})) \nabla k_i^n \cdot \nabla (k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|, \\
I_4 &= \sum_{i=1}^2 \left| \int_{\Omega_i} (\gamma_i(k_i^n) - \gamma_i(k_i^{n-1})) \nabla k_i^n \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|, \\
I_5 &= \sum_{i=1}^2 \left| \int_{\Omega_i} \alpha_i(k_i^n) (|\nabla \mathbf{u}_i^{n+1}|^2 - |\nabla \mathbf{u}_i^n|^2) R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|, \\
I_6 &= \sum_{i=1}^2 \left| \int_{\Omega_i} (\alpha_i(k_i^n) - \alpha_i(k_i^{n-1})) |\nabla \mathbf{u}_i^n|^2 R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|, \quad \text{and} \\
I_7 &= \sum_{i=1}^2 \left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla (k_i^{n+1} - k_i^n) \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|.
\end{aligned}$$

Second step. Estimates of I_j , $1 \leq j \leq 7$

Estimation of I_1 . We write $|\nabla \mathbf{u}_i^{n+1}|^2 - |\nabla \mathbf{u}_i^n|^2 = \nabla (\mathbf{u}_i^{n+1} - \mathbf{u}_i^n) \cdot \nabla (\mathbf{u}_i^{n+1} + \mathbf{u}_i^n)$, and use Hypothesis 3.1, and relation (2.1). Thanks to the Sobolev embedding of $H^1(\Omega_i)$ into $L^6(\Omega_i)$ and from Hölder and Poincaré-Friedrichs inequalities, we obtain

$$\begin{aligned}
& \left| \int_{\Omega_i} \alpha_i(k_i^n) (|\nabla \mathbf{u}_i^{n+1}|^2 - |\nabla \mathbf{u}_i^n|^2) (k_i^{n+1} - k_i^n) \, d\mathbf{x} \right| \\
& \leq \delta_1 \|\nabla (\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)^d} \left(\|\mathbf{u}_i^{n+1}\|_{W^{1,3}(\Omega_i)^d} + \|\mathbf{u}_i^n\|_{W^{1,3}(\Omega_i)^d} \right) \|k_i^{n+1} - k_i^n\|_{L^6(\Omega_i)} \\
& \leq Mc\delta_1 \|\nabla (\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)^d} \|\nabla (k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)},
\end{aligned}$$

where c is a positive constant, depending only on domains Ω_i . To simplify the calculations, we introduce a positive number β which we shall fix later.

According to Young's inequality

$$\frac{1}{\beta} a^2 + \beta b^2 \geq 2ab, \quad \forall a, b \in \mathbb{R}, \text{ and } \forall \beta > 0, \tag{3.9}$$

we obtain

$$\left| \int_{\Omega_i} \alpha_i(k_i^n) (|\nabla \mathbf{u}_i^{n+1}|^2 - |\nabla \mathbf{u}_i^n|^2) (k_i^{n+1} - k_i^n) \, d\mathbf{x} \right| \leq \frac{\nu}{\beta} \|\nabla (k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2 + \frac{\beta M^2 \delta_1^2 c^2}{\nu} \|\nabla (\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)^d}^2.$$

Summing upon $i = 1, 2$, and due to the relation (3.3) from Lemma 3.3, there exists a positive constant c_1 , depending only on Ω_i, α_i and M , such that

$$I_1 \leq \frac{c_1 \beta \delta_1^2 \delta_2^2 M^2}{\nu^3} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2 + \frac{\nu}{\beta} \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2. \tag{3.10}$$

Estimation of I_2 and I_3 . Using the same arguments we used for estimation of I_1 , there exists two positive constants, depending only on Ω_i, γ_i and M , such that

$$I_2 \leq \frac{c_2 \beta \delta_2^2 M^2}{\nu} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2 + \frac{\nu}{\beta} \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2, \tag{3.11}$$

and

$$I_3 \leq \frac{c_3 \beta \delta_2^2 M^2}{\nu} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2 + \frac{\nu}{\beta} \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2. \tag{3.12}$$

Estimation of I_4 . We recall that

$$I_4 = \sum_{i=1}^2 \left| \int_{\Gamma_i} (\gamma_i(k_i^n) - \gamma_i(k_i^{n-1})) \nabla k_i^n \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|.$$

Let us apply the Mean Value Theorem to the function γ_i , use relation (2.2) and Hölder inequality. We find

$$\left| \int_{\Gamma_i} (\gamma_i(k_i^n) - \gamma_i(k_i^{n-1})) \nabla k_i^n \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right| \leq \delta_2 \|k_i^n - k_i^{n-1}\|_{L^6(\Omega_i)} \|\nabla k_i^n\|_{L^3(\Omega_i)} \|\nabla R_i(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}.$$

The continuity of the lifting operator (3.7) and Hypothesis 3.1 imply

$$\left| \int_{\Gamma_i} (\gamma_i(k_i^n) - \gamma_i(k_i^{n-1})) \nabla k_i^n \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right| \leq c_R \delta_2 M \|k_i^n - k_i^{n-1}\|_{L^6(\Omega_i)} \|k_i^{n+1} - k_i^n\|_{H_{00}^{\frac{1}{2}}(\Gamma)}.$$

According to the continuity of the canonical injection from $H^1(\Omega_i)$ to $L^6(\Omega_i)$, the continuity of the trace operator from \mathbf{X}_i to $H_{00}^{\frac{1}{2}}(\Gamma)$, and using Young’s inequality (3.9), there exists a positive constant $c_4 > 0$, depending only on Ω_i, γ_i , and M , such that

$$\left| \int_{\Gamma_i} (\gamma_i(k_i^n) - \gamma_i(k_i^{n-1})) \nabla k_i^n \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right| \leq \frac{c_4 \beta \delta_2^2 M^2}{\nu} \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2 + \frac{\nu}{\beta} \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2.$$

Summing on $i = 1, 2$, we find

$$I_4 \leq \frac{c_4 \beta \delta_2^2 M^2}{\nu} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2 + \frac{\nu}{\beta} \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2. \tag{3.13}$$

Estimation of I_5 . We have

$$I_5 = \sum_{i=1}^2 \left| \int_{\Omega_i} \alpha_i(k_i^n) (|\nabla \mathbf{u}_i^{n+1}|^2 - |\nabla \mathbf{u}_i^n|^2) R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|.$$

Replacing $|\nabla \mathbf{u}_i^{n+1}|^2 - |\nabla \mathbf{u}_i^n|^2$ by $\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n) \cdot \nabla(\mathbf{u}_i^{n+1} + \mathbf{u}_i^n)$, and using Hypothesis 3.1, Hölder inequality and formula (3.7), there exists a positive constant $\theta > 0$, depending only on Ω_i , such that

$$\left| \int_{\Omega_i} \alpha_i(k_i^n) (|\nabla \mathbf{u}_i^{n+1}|^2 - |\nabla \mathbf{u}_i^n|^2) R_i(k_i^{n+1} - k_i^n) \, dx \right| \leq \theta \delta_1 M \|\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)^d} \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}.$$

Then, using relations (3.3) from Lemma 3.3 and (3.9), and summing on $i = 1, 2$, we obtain the following estimation of I_5 :

$$I_5 \leq \frac{c_5 \beta \delta_1^2 \delta_2^2 M^2}{\nu^3} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2 + \frac{\nu}{\beta} \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2. \tag{3.14}$$

Estimation of I_6 . Applying the same techniques, we have

$$\left| \int_{\Omega_i} (\alpha_i(k_i^n) - \alpha_i(k_i^{n-1})) |\nabla \mathbf{u}_i^n|^2 R_i(k_i^{n+1} - k_i^n) \, dx \right| \leq \delta_2 \|k_i^n - k_i^{n-1}\|_{L^6(\Omega_i)} \|\nabla \mathbf{u}_i^n\|_{L^3(\Omega_i)^d} \|R_i(k_i^{n+1} - k_i^n)\|_{L^6(\Omega_i)}.$$

Using the continuity of the lifting operator R_i from H^1 to $H^{\frac{1}{2}}(\Gamma)$, the continuity of the canonical injection from $H^1(\Omega_i)$ to $L^6(\Omega_i)$, and Poincaré-Friedrichs inequality, there exists a positive constant c_6 , depending only on Ω_i , such that

$$\begin{aligned} \left| \int_{\Omega_i} (\alpha_i(k_i^n) - \alpha_i(k_i^{n-1})) |\nabla \mathbf{u}_i^n|^2 R_i(k_i^{n+1} - k_i^n) \, dx \right| &\leq c_6 \delta_2 M \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)} \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}, \\ &\leq \frac{c_6 \beta \delta_2^2 M^2}{\nu} \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2 + \frac{\nu}{\beta} \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2. \end{aligned}$$

Summing on $i = 1, 2$, we deduce the following estimation of I_6 ,

$$I_6 \leq \frac{c_6 \beta \delta_2^2 M^2}{\nu} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2 + \frac{\nu}{\beta} \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2. \tag{3.15}$$

Estimation of I_7 . The estimation of I_7 is more involved. To achieve it, we use a result of continuity of the product of traces on Γ due to Grisvard [13,14].

Lemma 3.5. *Assume that Ω is a bounded Lipschitz-continuous open subset of \mathbb{R}^d . Let s, s_1 and s_2 be three non negative reals and p, p_1, p_2 be three real numbers in $[1, +\infty)$ such that $s_1 \geq s, s_2 \geq s$ and either*

$$s_1 + s_2 - s \geq d \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \right) \geq 0, \quad s_i - s > d \left(\frac{1}{p_i} - \frac{1}{p} \right) \quad i = 1, 2 \tag{3.16}$$

or

$$s_1 + s_2 - s > d \left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \right) \geq 0, \quad s_i - s \geq d \left(\frac{1}{p_i} - \frac{1}{p} \right) \quad i = 1, 2. \tag{3.17}$$

Then the mapping $(u, v) \rightarrow uv$ is a continuous bilinear map from $W^{s_1, p_1}(\Omega) \times W^{s_2, p_2}(\Omega)$ to $W^{s, p}(\Omega)$.

Using the results of Hebey [15], this lemma also holds for Sobolev spaces defined on compact Riemannian manifolds. This is the case of Γ .

We remind that

$$I_7 = \sum_{i=1}^2 \left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla(k_i^{n+1} - k_i^n) \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, dx \right|.$$

Using relation (2.1), Cauchy-Schwarz inequality, and the continuity of the lifting (3.7), we find

$$\left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla(k_i^{n+1} - k_i^n) \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right| \leq c_R \delta_1 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)} \|k_i^{n+1} - k_i^n\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)}.$$

The boundary condition equation (2.7) implies that $(k_i^{n+1} - k_i^n)|_\Gamma = \lambda(|\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|^2 - |\mathbf{u}_1^n - \mathbf{u}_2^n|^2)|_\Gamma$. Thus

$$\|k_i^{n+1} - k_i^n\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)} = \lambda \left\| [(\mathbf{u}_1^{n+1} - \mathbf{u}_1^n) - (\mathbf{u}_2^{n+1} - \mathbf{u}_2^n)] [(\mathbf{u}_1^{n+1} + \mathbf{u}_1^n) - (\mathbf{u}_2^{n+1} + \mathbf{u}_2^n)] \right\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)}.$$

Let us apply Lemma 3.5, by taking

$$s_1 = s = \frac{1}{2}, \quad s_2 = 1 - \frac{1}{3 + \varepsilon},$$

$$p_1 = p = 2, \quad p_2 = 3 + \varepsilon,$$

and $(\mathbf{u}_1^{n+1} - \mathbf{u}_1^n) - (\mathbf{u}_2^{n+1} - \mathbf{u}_2^n) = \mathbf{u} \in H^{\frac{1}{2}}(\Gamma) \quad (= W^{\frac{1}{2},2}(\Gamma) = W^{s_1,p_1}(\Gamma)),$

$$(\mathbf{u}_1^{n+1} - \mathbf{u}_1^n) - (\mathbf{u}_2^{n+1} - \mathbf{u}_2^n) = \mathbf{v} \in W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma) \quad (= W^{s_2,p_2}(\Gamma)).$$

We obtain

$$\left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla(k_i^{n+1} - k_i^n) \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right| \leq c_R \lambda \delta_1 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}$$

$$\times \left[\|\mathbf{u}_1^{n+1} - \mathbf{u}_1^n\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)} + \|\mathbf{u}_2^{n+1} - \mathbf{u}_2^n\|_{\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)} \right]$$

$$\times \left[\|\mathbf{u}_1^{n+1} + \mathbf{u}_1^n\|_{W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma)^d} + \|\mathbf{u}_2^{n+1} + \mathbf{u}_2^n\|_{W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma)^d} \right].$$

Using the continuity of the trace operators from $W^{1,3+\varepsilon}(\Omega_i)^d$ to $W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma)^d$ and from $\mathbf{H}^1(\Omega_i)$ to $\mathbf{H}_{00}^{\frac{1}{2}}(\Gamma)$, there exists a positive constant c'_7 , depending only on domains Ω_i and λ , such that

$$\left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla(k_i^{n+1} - k_i^n) \cdot \nabla R_i(k_i^{n+1} - k_i^n) \, d\mathbf{x} \right|$$

$$\leq c'_7 \delta_1 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)} \left(\sum_{i=1}^2 \|\mathbf{u}_i^{n+1} - \mathbf{u}_i^n\|_{\mathbf{H}^1(\Omega_i)^d} \right) \left(\sum_{i=1}^2 \|\mathbf{u}_i^{n+1} + \mathbf{u}_i^n\|_{W^{1,3+\varepsilon}(\Omega_i)^d} \right)$$

$$\leq 2M c'_7 \delta_1 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)} \sum_{i=1}^2 \|\mathbf{u}_i^{n+1} - \mathbf{u}_i^n\|_{\mathbf{H}^1(\Omega_i)^d} \quad (\text{by Hypothesis 3.1})$$

$$\leq 2M c'_7 \delta_1 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)} \sum_{i=1}^2 \|\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega)^d} \quad (\text{by Poincaré-Friedrichs inequality}).$$

According to relation (3.3) of Lemma 3.3, relation (3.9) and summing upon $i = 1, 2$, there exists a positive constant c_7 , depending only on Ω_i and λ such that

$$I_7 \leq \frac{c_7 \beta \delta_1^2 M^2}{\nu} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2 + \frac{\nu}{\beta} \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}. \tag{3.18}$$

Finally, using estimations (3.10)–(3.18), relation (3.8), and choosing $\beta = 14$ (for instance), there exists a positive constant c , depending only on Ω_i , κ_i and λ , such that

$$\sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2 \leq \frac{c(1 + \delta_1^2)\delta_2^2 M^2}{\nu^3} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2.$$

This finishes the proof of the Theorem 3.4. □

Corollary 3.6 (convergence of the iterative process). *Under the hypotheses of Theorem 3.4, there exists a positive constant c , depending only on Ω_i and on the data κ_i and λ , such that if $K = c \frac{(1 + \delta_1^2)\delta_2^2}{\nu^3} M^2 < 1$, then the sequences $(\mathbf{u}_i^n)_n$ and $(k_i^n)_n$ are contracting, in the sense of relation (3.1), i.e.*

$$\begin{aligned} \sum_{i=1}^2 \|\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i^n)\|_{L^2(\Omega_i)}^2 &\leq K \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2, \quad \text{and} \\ \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2 &\leq K \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_{L^2(\Omega_i)}^2. \end{aligned}$$

Furthermore, from estimate (3.3), as $K < 1$, $(\mathbf{u}_i^n)_n$ is a Cauchy sequence,

$$\sum_{i=1}^2 \|\nabla(\mathbf{u}_i^m - \mathbf{u}_i^n)\|_0^2 \leq \frac{1 - K^{m-n+1}}{1 - K} \sum_{i=1}^2 \|\nabla(k_i^n - k_i^{n-1})\|_0^2, \quad \forall n \leq m \in \mathbb{N}. \tag{3.19}$$

Then, since \mathbf{X}_i and $H^1(\Omega_i)$ are Banach spaces, the sequences $(\mathbf{u}_i^n)_n$ and $(k_i^n)_n$ have unique strong limits in \mathbf{X}_i and $H^1(\Omega_i)$, \mathbf{u}_i and k_i .

We next prove that the sequence $(p_i^n)_n$ is a Cauchy sequence. For that purpose, we use the following inf-sup condition proved in [3], Lemma 3.1.

Corollary 3.7 (inf-sup condition). *Assume that Ω_i is bounded Lipschitz-continuous open subset of \mathbb{R}^d . Then, there exists a positive constant $\beta_i > 0$, depending only on the domain Ω_i , such that*

$$\forall q_i \in L^2(\Omega_i), \quad \sup_{\mathbf{v}_i \in \mathbf{X}_i} \frac{b_i(\mathbf{v}_i, q_i)}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)}} \geq \beta_i \|q_i\|_{L^2(\Omega_i)}.$$

Theorem 3.8 (convergence of the pressure). *Assume that Hypothesis 3.1 holds, that $\mathbf{f}_i \in L^2(\Omega_i)^d$ and that $K < 1$. Then, $(p_i^n)_n$ is a Cauchy sequence. More specifically, there exists positive constants c' , depending only on Ω_i , α_i , and M , and c'' depending only on Ω_i , such that for any two non negative integers $m \geq n$,*

$$\begin{aligned} \sum_{i=1}^2 \|p_i^{m+1} - p_i^{n+1}\|_0^2 &\leq \frac{c'}{\beta^2(1 - K)} \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_0^2 \\ &\quad + \frac{c''}{\beta^2} \|\|\mathbf{u}_1^{m+1} - \mathbf{u}_2^{m+1}\|(\mathbf{u}_1^{m+1} - \mathbf{u}_2^{m+1}) - \|\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}\|(\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1})\|_{L^{\frac{3}{2}}(\Gamma)^d}^2, \end{aligned}$$

where $\beta = \min\{\beta_1, \beta_2\}$.

Proof. For all $\mathbf{v}_i \in \mathbf{X}_i$, we make the difference of equation (2.5) at steps m and n . This yields for $1 \leq i \neq j \leq 2$,

$$\begin{aligned} \frac{b_i(\mathbf{v}_i, p_i^{m+1} - p_i^{n+1})}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)^d}} &= -\kappa_i \int_{\Gamma} \left[|\mathbf{u}_i^{m+1} - \mathbf{u}_j^{m+1}|(\mathbf{u}_i^{m+1} - \mathbf{u}_j^{m+1}) - |\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \right] \cdot \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)^d}} \, d\tau \\ &\quad - \int_{\Omega_i} \alpha_i(k_i^m) \nabla(\mathbf{u}_i^{m+1} - \mathbf{u}_i^{n+1}) : \frac{\nabla \mathbf{v}_i}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)^d}} \, d\mathbf{x} \\ &\quad - \int_{\Omega_i} (\alpha_i(k_i^m) - \alpha_i(k_i^n)) \nabla \mathbf{u}_i^n : \frac{\nabla \mathbf{v}_i}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)^d}} \, d\mathbf{x}. \end{aligned}$$

We know that the sequence $(\mathbf{u}_i^n)_n$ belongs to $\mathbf{H}^1(\Omega_i)$, then its trace on γ belongs to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$, thus in $L^3(\Gamma)^d$ by injection, *i.e.*

$$\left| |\mathbf{u}_i^{m+1} - \mathbf{u}_j^{m+1}|(\mathbf{u}_i^{m+1} - \mathbf{u}_j^{m+1}) - |\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \right| \in L^{\frac{3}{2}}(\Gamma)^d, \quad \forall i \neq j \text{ and } n \leq m.$$

Using the Mean Value Theorem, relations (2.1)–(2.2) and Hölder inequality, we find

$$\begin{aligned} \frac{b_i(\mathbf{v}_i, p_i^{m+1} - p_i^{n+1})}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)}} &\leq \left\| |\mathbf{u}_i^{m+1} - \mathbf{u}_j^{m+1}|(\mathbf{u}_i^{m+1} - \mathbf{u}_j^{m+1}) - |\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \right\|_{L^{\frac{3}{2}}(\Gamma)^d} \frac{\|\mathbf{v}_i\|_{L^3(\Gamma)^d}}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)}} \\ &\quad + \delta_1 \|\nabla(\mathbf{u}_i^{m+1} - \mathbf{u}_i^{n+1})\|_{L^2(\Omega_i)^d} \frac{\|\nabla \mathbf{v}_i\|_{L^2(\Omega_i)^d}}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)}} \\ &\quad + \delta_2 \|k_i^m - k_i^n\|_{L^6(\Omega_i)} \|\nabla \mathbf{u}_i^n\|_{L^3(\Omega_i)^d} \frac{\|\nabla \mathbf{v}_i\|_{L^2(\Omega_i)^d}}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)}}. \end{aligned}$$

Applying the continuity of the trace operator from $\mathbf{H}^1(\Omega_i)$ to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$, and the continuity of the canonical injection from $\mathbf{H}^{\frac{1}{2}}(\Gamma)$ to $L^3(\Gamma)^d$, there exists a positive constant c , depending only on Ω_i , α_i and M , such that for all $\mathbf{v}_i \in \mathbf{X}_i$

$$\begin{aligned} \frac{b_i(\mathbf{v}_i, p_i^{m+1} - p_i^{n+1})}{\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)}} &\leq c \left(\left\| |\mathbf{u}_i^{m+1} - \mathbf{u}_j^{m+1}|(\mathbf{u}_i^{m+1} - \mathbf{u}_j^{m+1}) - |\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \right\|_{L^{\frac{3}{2}}(\Gamma)^d} \right. \\ &\quad \left. + \|\nabla(\mathbf{u}_i^{m+1} - \mathbf{u}_i^{n+1})\|_{L^2(\Omega_i)^d} + \|\nabla(k_i^m - k_i^n)\|_{L^2(\Omega_i)} \|\nabla \mathbf{u}_i^n\|_{L^3(\Omega_i)^d} \right). \end{aligned}$$

Using the inf-sup condition (Cor. 3.7), and summing on $i = 1, 2$, we obtain

$$\begin{aligned} \beta \sum_{i=1}^2 \|p_i^{m+1} - p_i^{n+1}\|_{L^2(\Omega_i)} &\leq c \left[\left\| |\mathbf{u}_1^{m+1} - \mathbf{u}_2^{m+1}|(\mathbf{u}_1^{m+1} - \mathbf{u}_2^{m+1}) - |\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|(\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}) \right\|_{L^{\frac{3}{2}}(\Gamma)^d} \right. \\ &\quad \left. + \sqrt{2} \left(\sum_{i=1}^2 \|\nabla(\mathbf{u}_i^{m+1} - \mathbf{u}_i^{n+1})\|_{L^2(\Omega_i)^d}^2 \right)^{\frac{1}{2}} + \sqrt{2} \left(\sum_{i=1}^2 \|\nabla(k_i^m - k_i^n)\|_{L^2(\Omega_i)}^2 \right)^{\frac{1}{2}} \right]. \end{aligned}$$

According to relation (3.19)

$$\begin{aligned} \sum_{i=1}^2 \|p_i^{m+1} - p_i^{n+1}\|_{L^2(\Omega_i)}^2 &\leq 4\sqrt{2} \frac{c^2}{\beta^2(1-K)} \sum_{i=1}^2 \|\nabla(k_i^{n+1} - k_i^n)\|_{L^2(\Omega_i)}^2 \\ &\quad + \frac{2c^2}{\beta^2} \left\| |\mathbf{u}_1^{m+1} - \mathbf{u}_2^{m+1}|(\mathbf{u}_1^{m+1} - \mathbf{u}_2^{m+1}) - |\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|(\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}) \right\|_{L^{\frac{3}{2}}(\Gamma)^d}^2. \quad (3.20) \end{aligned}$$

We have proved that the sequence $(\mathbf{u}_i^n)_n$ converges in $\mathbf{H}^1(\Omega_i)$ strong. Then, using the continuity of the trace operator from $\mathbf{H}^1(\Omega_i)$ to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$, and that of the canonical injection from $\mathbf{H}^{\frac{1}{2}}(\Gamma)$ into $L^3(\Gamma)^d$, we deduce that the sequence

$$\left(\left[(\mathbf{u}_1^n - \mathbf{u}_2^n) | \mathbf{u}_1^n - \mathbf{u}_2^n \right] \Big|_{\Gamma} \right)_n$$

is a Cauchy sequence in $L^{\frac{3}{2}}(\Gamma)^d$. Thus

$$\left\| |\mathbf{u}_1^{m+1} - \mathbf{u}_2^{m+1}| (\mathbf{u}_1^{m+1} - \mathbf{u}_2^{m+1}) - |\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}| (\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}) \right\|_{L^{\frac{3}{2}}(\Gamma)^d}^2 \longrightarrow 0.$$

We conclude that $(p_i^n)_n$ is a Cauchy sequence in $L^2(\Omega_i)$, $i = 1, 2$. □

We shall denote by p_i the limit of the sequence $(p_i^n)_n$.

4. IDENTIFICATION OF THE LIMIT

In this section, we show that the limit (\mathbf{u}_i, p_i, k_i) of the sequence $(\mathbf{u}_i^n, p_i^n, k_i^n)_n$ is a solution of the variational formulation (2.3)–(2.4).

Theorem 4.1 (identification of the limit). *Under the same assumptions of Theorem 3.6, the limit (\mathbf{u}_i, p_i, k_i) of the sequence $(\mathbf{u}_i^n, p_i^n, k_i^n)_n$ is a solution of the variational formulation (2.3)–(2.4).*

Proof. The proof is made of three steps:

- Stokes equation (2.3) is verified by (\mathbf{u}_i, p_i, k_i) ;
- The TKE equation (2.4) is verified by (\mathbf{u}_i, p_i, k_i) ; and
- The boundary conditions are verified, i.e. $k_i = \lambda |\mathbf{u}_1 - \mathbf{u}_2|^2$ on Γ and k_i vanishes on Γ_i .

First step. Stokes equation

Let $\mathbf{v}_i \in \mathbf{X}_i$, and let us show that

$$a_i(k_i, \mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) + \kappa_i \int_{\Gamma} |\mathbf{u}_i - \mathbf{u}_j| (\mathbf{u}_i - \mathbf{u}_j) \mathbf{v}_i \, d\tau = \int_{\Omega_i} \mathbf{f}_i \mathbf{v}_i \, d\mathbf{x}, \quad \forall 1 \leq i \neq j \leq 2.$$

We first focus on the bilinear form $a_i(\cdot; \cdot, \cdot)$. k_i is the limit of the sequence $(k_i^n)_n$ in $H^1(\Omega_i)$. Then, there exists a subsequence of $(k_i^n)_n$ that converges a.e. in Ω_i to k_i . As the global sequence is contracting, all the subsequences are convergent and converge to k_i . Thus, as the function α_i is continuous and bounded, $\lim_{n \rightarrow \infty} \alpha_i(k_i^n) = \alpha_i(k_i)$, a.e. in Ω_i . Moreover, the sequence $(\mathbf{u}_i^n)_n$ converges strongly to the unique limit \mathbf{u}_i in $\mathbf{H}^1(\Omega_i)$, then $\lim_{n \rightarrow \infty} \nabla \mathbf{u}_i^n = \nabla \mathbf{u}_i$ strongly in $L^2(\omega_i)^d$. Hence by the inverse Lebesgue theorem (see for instance [7], Thm. IV.9), there exists a further subsequence, still denoted by $(\nabla \mathbf{u}_i^n)_n$, which tends to $\nabla \mathbf{u}_i$ a.e. on Ω_i , and $\forall n \in \mathbb{N}$, $|\nabla \mathbf{u}_i^n| \leq g_i$, where the function g_i belongs to $L^2(\Omega_i)$. Thus $\forall \mathbf{v}_i \in \mathbf{X}_i$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_i(k_i^n) \nabla \mathbf{u}_i^n : \nabla \mathbf{v}_i &= \alpha_i(k_i) \nabla \mathbf{u}_i : \nabla \mathbf{v}_i, \text{ a.e. on } \Omega_i, \text{ and} \\ |\alpha_i(k_i^n) \nabla \mathbf{u}_i^n : \nabla \mathbf{v}_i| &\leq \delta_1 |g_i| |\nabla \mathbf{v}_i| \in L^1(\Omega_i). \end{aligned}$$

Again, using the uniqueness of the limit of all subsequences of $(\mathbf{u}_i^n)_n$, we obtain $\lim_{n \rightarrow \infty} a_i(k_i^n; \mathbf{u}_i^{n+1}, \mathbf{v}_i) = a_i(k_i; \mathbf{u}_i, \mathbf{v}_i)$.

Due to the strong convergence of the sequence (p_i^n) to p_i in $L^2(\Omega_i)$, we have $\lim_{n \rightarrow \infty} b_i(\mathbf{v}_i, p_i^{n+1}) = b_i(\mathbf{v}_i, p_i)$, for each vector \mathbf{v}_i that belongs to \mathbf{X}_i .

Concerning the boundary term, since the trace of \mathbf{u}_i^n on Γ belongs to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$ and using the compactness of the canonical injection of $\mathbf{H}^{\frac{1}{2}}(\Gamma)$ to $L^3(\Gamma)^d$, we have $\lim_{n \rightarrow \infty} \mathbf{u}_i^n = \mathbf{u}_i$ in $L^3(\Gamma)^d$, thus for $1 \leq i \neq j \leq 2$

$$\lim_{n \rightarrow \infty} |\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) = |\mathbf{u}_i - \mathbf{u}_j|(\mathbf{u}_i - \mathbf{u}_j), \quad \text{strongly in } L^{\frac{3}{2}}(\Gamma)^d.$$

Furthermore, the trace of the vector \mathbf{v}_i belongs to $L^3(\Gamma)^d$, then

$$\lim_{n \rightarrow \infty} \kappa_i \int_{\Gamma} |\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \mathbf{v}_i \, d\tau = \kappa_i \int_{\Gamma} |\mathbf{u}_i - \mathbf{u}_j|(\mathbf{u}_i - \mathbf{u}_j) \mathbf{v}_i \, d\tau.$$

We proved that the limit (\mathbf{u}_i, p_i, k_i) of the sequence $(\mathbf{u}_i^n, p_i^n, k_i^n)_n$ verifies the first equation of the Stokes problem (2.3). Concerning the second one, due to the fact that $b_i(\mathbf{u}_i^{n+1}, q_i) = 0, \forall q_i \in L^2(\Omega_i)$, and to the strong convergence in $\mathbf{H}^1(\Omega_i)$ of the sequence $(\mathbf{u}_i^n)_n$ to \mathbf{u}_i , we deduce that $b_i(\mathbf{u}_i, q_i) = 0, \forall q_i \in L^2(\Omega_i)$.

Second step. Equation for the TKE

We want to show that $\forall \varphi_i \in W_0^{1,r}(\Omega_i), \int_{\Omega_i} \gamma_i(k_i) \nabla k_i \cdot \nabla \varphi_i \, d\mathbf{x} = \int_{\Omega_i} \alpha_i(k_i) |\nabla \mathbf{u}_i|^2 \varphi_i \, d\mathbf{x}$. To achieve this result, we only need to show that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega_i} \gamma_i(k_i^n) \nabla k_i^{n+1} \cdot \nabla \varphi_i \, d\mathbf{x} &= \int_{\Omega_i} \gamma_i(k_i) \nabla k_i \cdot \nabla \varphi_i \, d\mathbf{x}, \quad \text{and} \\ \lim_{n \rightarrow \infty} \int_{\Omega_i} \alpha_i(k_i^n) |\nabla \mathbf{u}_i^{n+1}|^2 \varphi_i \, d\mathbf{x} &= \int_{\Omega_i} \alpha_i(k_i) |\nabla \mathbf{u}_i|^2 \varphi_i \, d\mathbf{x}. \end{aligned} \tag{4.1}$$

The idea consists in choosing test functions φ_i that belong to $\mathcal{D}(\Omega_i)$ at first, then by density we will consider all test functions in $W_0^{1,r}(\Omega_i)$.

Let us write $\gamma_i(k_i^n) \nabla k_i^{n+1} - \gamma_i(k_i) \nabla k_i$ using (4.1). For all $\varphi_i \in \mathcal{D}(\Omega_i)$:

$$\int_{\Omega_i} (\gamma_i(k_i^n) \nabla k_i^{n+1} - \gamma_i(k_i) \nabla k_i) \cdot \nabla \varphi_i \, d\mathbf{x} = \int_{\Omega_i} \gamma_i(k_i^n) \nabla(k_i^{n+1} - k_i) \cdot \nabla \varphi_i \, d\mathbf{x} + \int_{\Omega_i} (\gamma_i(k_i^n) - \gamma_i(k_i)) \nabla k_i \cdot \nabla \varphi_i \, d\mathbf{x}.$$

Using the Cauchy-Schwarz inequality, and the strong convergence of the sequence $(k_i^n)_n$ to k_i in $H^1(\Omega_i)$, we obtain

$$\left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla(k_i^{n+1} - k_i) \cdot \nabla \varphi_i \, d\mathbf{x} \right| \leq \delta_1 \|\nabla(k_i^{n+1} - k_i)\|_{L^2(\Omega_i)} \|\nabla \varphi_i\|_{L^2(\Omega_i)} \xrightarrow{n \rightarrow \infty} 0.$$

On the other hand, we apply the Mean Value Theorem to the function γ_i , and using Hölder inequality, we find

$$\left| \int_{\Omega_i} (\gamma_i(k_i^n) - \gamma_i(k_i)) \nabla k_i \cdot \nabla \varphi_i \, d\mathbf{x} \right| \leq \delta_2 \|k_i^n - k_i\|_{L^2(\Omega_i)} \|\nabla k_i\|_{L^2(\Omega_i)} \|\nabla \varphi_i\|_{\infty} \xrightarrow{n \rightarrow \infty} 0.$$

We deduce that for all $\varphi_i \in \mathcal{D}(\Omega_i), \lim_{n \rightarrow \infty} \int_{\Omega_i} \gamma_i(k_i^n) \nabla k_i^{n+1} \cdot \nabla \varphi_i \, d\mathbf{x} = \int_{\Omega_i} \gamma_i(k_i) \nabla k_i \cdot \nabla \varphi_i \, d\mathbf{x}$. Let η be a strictly positive real, such that for all integer $n \geq n_0$, we have

$$\left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla k_i^{n+1} \cdot \nabla \varphi_i \, d\mathbf{x} - \int_{\Omega_i} \gamma_i(k_i) \nabla k_i \cdot \nabla \varphi_i \, d\mathbf{x} \right| \leq \frac{\eta}{3}, \quad \forall \varphi_i \in \mathcal{D}(\Omega). \tag{4.2}$$

Let now φ_i belong to $W_0^{1,r}(\Omega_i)$. Since $\mathcal{D}(\Omega_i)$ is dense in $W_0^{1,r}(\Omega_i)$, there exists a sequence $(\varphi_i^m)_m \geq 0$ that belongs to $\mathcal{D}(\Omega_i)$, such that for $m \geq m_0$, we have $\|\varphi_i^m - \varphi_i\|_{W_0^{1,r}(\Omega_i)} \leq \frac{\eta}{3}$. Thanks to the Sobolev continuous

embedding from $W_0^{1,r}(\Omega_i)$ to $H_0^1(\Omega)$, we can write for $m \geq m_0$

$$\|\varphi_i^m - \varphi_i\|_{H_0^1(\Omega_i)} \leq \frac{\eta}{3}. \quad (4.3)$$

Thanks to the triangular inequality, we obtain

$$\begin{aligned} \left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla k_i^{n+1} \cdot \nabla \varphi_i \, d\mathbf{x} - \int_{\Omega_i} \gamma_i(k_i) \nabla k_i \cdot \nabla \varphi_i \, d\mathbf{x} \right| &\leq \left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla k_i^{n+1} \cdot \nabla \varphi_i \, d\mathbf{x} - \int_{\Omega_i} \gamma_i(k_i^n) \nabla k_i^{n+1} \cdot \nabla \varphi_i^m \, d\mathbf{x} \right| \\ &\quad + \left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla k_i^{n+1} \cdot \nabla \varphi_i^m \, d\mathbf{x} - \int_{\Omega_i} \gamma_i(k_i) \nabla(k_i) \cdot \nabla \varphi_i^m \, d\mathbf{x} \right| \\ &\quad + \left| \int_{\Omega_i} \gamma_i(k_i) \nabla(k_i) \cdot \nabla \varphi_i^m \, d\mathbf{x} - \int_{\Omega_i} \gamma_i(k_i) \nabla(k_i) \cdot \nabla \varphi_i \, d\mathbf{x} \right|. \end{aligned} \quad (4.4)$$

Hölder inequality implies

$$\begin{aligned} \left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla k_i^{n+1} \cdot \nabla \varphi_i \, d\mathbf{x} - \int_{\Omega_i} \gamma_i(k_i) \nabla k_i \cdot \nabla \varphi_i \, d\mathbf{x} \right| &\leq \delta_1 \|\nabla k_i^{n+1}\|_{L^2(\Omega_i)} \|\nabla(\varphi_i^m - \varphi_i)\|_{L^2(\Omega_i)} \\ &\quad + \left| \int_{\Omega_i} \gamma_i(k_i^n) \nabla k_i^{n+1} \cdot \nabla \varphi_i^m \, d\mathbf{x} - \int_{\Omega_i} \gamma_i(k_i) \nabla(k_i) \cdot \nabla \varphi_i^m \, d\mathbf{x} \right| + \delta_1 \|\nabla k_i\|_0 \|\nabla(\varphi_i^m - \varphi_i)\|_{L^2(\Omega_i)}. \end{aligned}$$

Using relations (4.2)–(4.3), and the strong convergence of the sequence $(k_i^n)_n$ in $H^1(\Omega_i)$, we deduce that for any function $\varphi_i \in W_0^{1,r}(\Omega_i)$, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega_i} \gamma_i(k_i^n) \nabla(k_i^{n+1}) \cdot \nabla \varphi_i \, d\mathbf{x} = \int_{\Omega_i} \gamma_i(k_i) \nabla(k_i) \cdot \nabla \varphi_i \, d\mathbf{x}.$$

To show the second equation of the relation (4.1), we write

$$\begin{aligned} \left| \int_{\Omega_i} \alpha_i(k_i^n) |\nabla \mathbf{u}_i^{n+1}|^2 - \alpha_i(k_i) |\nabla \mathbf{u}_i|^2 \varphi_i \, d\mathbf{x} \right| \\ \leq \int_{\Omega_i} \alpha_i(k_i^n) \left| |\nabla \mathbf{u}_i^{n+1}|^2 - |\nabla \mathbf{u}_i|^2 \right| |\varphi_i| \, d\mathbf{x} + \int_{\Omega_i} |\alpha_i(k_i^n) - \alpha_i(k_i)| |\nabla \mathbf{u}_i|^2 |\varphi_i| \, d\mathbf{x}, \end{aligned}$$

thanks to Hypothesis 3.1, the Sobolev embedding from $\mathbf{H}^1(\Omega_i)$ to $L^6(\Omega_i)$ and Hölder inequality, we can write for all $\varphi_i \in \mathcal{D}(\Omega_i)$,

$$\begin{aligned} \left| \int_{\Omega_i} \alpha_i(k_i^n) |\nabla \mathbf{u}_i^{n+1}|^2 - \alpha_i(k_i) |\nabla \mathbf{u}_i|^2 \varphi_i \, d\mathbf{x} \right| \\ \leq \delta_1 \|\varphi_i\|_\infty \|\nabla \mathbf{u}_i^{n+1}\|_{L^2(\Omega_i)^d} \|\nabla \mathbf{u}_i\|_{L^2(\Omega_i)^d} \|\nabla(\mathbf{u}_i^{n+1} - \mathbf{u}_i)\|_{L^2(\Omega_i)^d} + cM^2 \delta_2 \|\varphi_i\|_\infty \|k_i^n - k_i\|_{H^1(\Omega_i)}, \end{aligned}$$

where c is a positive constant, depending only on the Ω_i . Due to the strong convergence in \mathbf{H}^1 of the sequences $(\mathbf{u}_i^n)_n$ and $(k_i^n)_n$ to \mathbf{u}_i and k_i ,

$$\left| \int_{\Omega_i} (\alpha_i(k_i^n) |\nabla \mathbf{u}_i^{n+1}|^2 - \alpha_i(k_i) |\nabla \mathbf{u}_i|^2) \varphi_i \, d\mathbf{x} \right| \leq \frac{\eta}{3}.$$

Let now $\varphi_i \in W_0^{1,r}(\Omega_i)$. The density of $\mathcal{D}(\Omega_i)$ in $W_0^{1,r}(\Omega_i)$, implies that there exists a sequence $(\varphi_i^m)_m$ in $\mathcal{D}(\Omega_i)$, such that

$$\|\varphi_i^m - \varphi_i\|_{L^3(\Omega_i)} \leq c(\Omega_i)\|\varphi_i^m - \varphi_i\|_{W^{1,r}(\Omega_i)} \leq \frac{\eta}{3},$$

where $c(\Omega_i)$ only depends on Ω_i . Finally, using the triangular inequality, we obtain

$$\begin{aligned} \left| \int_{\Omega_i} (\alpha_i(k_i^n)|\nabla \mathbf{u}_i^{n+1}|^2 - \alpha_i(k_i)|\nabla \mathbf{u}_i|^2) \varphi_i \, d\mathbf{x} \right| &\leq \left| \int_{\Omega_i} (\alpha_i(k_i^n)|\nabla \mathbf{u}_i^{n+1}|^2 - \alpha_i(k_i)|\nabla \mathbf{u}_i|^2) \varphi_i^m \, d\mathbf{x} \right| \\ &+ \left| \int_{\Omega_i} \alpha_i(k_i^n)|\nabla \mathbf{u}_i^{n+1}|^2(\varphi_i^m - \varphi_i) \, d\mathbf{x} \right| + \left| \int_{\Omega_i} \alpha_i(k_i)|\nabla \mathbf{u}_i|^2(\varphi_i^m - \varphi_i) \, d\mathbf{x} \right|. \end{aligned}$$

Thus,

$$\begin{aligned} \left| \int_{\Omega_i} (\alpha_i(k_i^n)|\nabla \mathbf{u}_i^{n+1}|^2 - \alpha_i(k_i)|\nabla \mathbf{u}_i|^2) \varphi_i \, d\mathbf{x} \right| &\leq \|\alpha_i\|_\infty \|\nabla \mathbf{u}_i^{n+1}\|_{L^3(\Omega_i)^d}^2 \|\varphi_i^m - \varphi_i\|_{L^3(\Omega_i)} \\ &+ \frac{\varepsilon}{3} + \|\alpha_i\|_\infty \|\nabla \mathbf{u}_i\|_{L^3(\Omega_i)^d}^2 \|\varphi_i^m - \varphi_i\|_{L^3(\Omega_i)}. \end{aligned}$$

We deduce that $\forall \varphi_i \in W_0^{1,r}(\Omega_i)$, $\lim_{n \rightarrow \infty} \int_{\Omega_i} \alpha_i(k_i^n)|\nabla \mathbf{u}_i^{n+1}|^2 \varphi_i \, d\mathbf{x} = \int_{\Omega_i} \alpha_i(k_i)|\nabla \mathbf{u}_i|^2 \varphi_i \, d\mathbf{x}$.

Third step. Boundary conditions of TKE on Γ

In this step, we show that $k_i = \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2$ on Γ .

Consider the following triangular inequality

$$\|\lambda|\mathbf{u}_1 - \mathbf{u}_2|^2 - k_i\|_{H^{\frac{1}{2}}(\Gamma)} \leq \|\lambda|\mathbf{u}_1^n - \mathbf{u}_2^n|^2 - k_i\|_{H^{\frac{1}{2}}(\Gamma)} + \|\lambda|\mathbf{u}_1^n - \mathbf{u}_2^n|^2 - \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2\|_{H^{\frac{1}{2}}(\Gamma)}.$$

Due to the strong convergence of $(k_i^n)_n$ to k_i in $H^1(\Omega_i)$, and thanks to the continuity of the trace operator from $H^1(\Omega_i)$ to $H^{\frac{1}{2}}(\Gamma)$, the sequence k_i^n converges strongly to k_i in $H^{\frac{1}{2}}(\Gamma)$. Furthermore, $k_i^n = \lambda|\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|^2$ on Γ . Thus,

$$\lim_{n \rightarrow \infty} \lambda|\mathbf{u}_1^{n+1} - \mathbf{u}_2^{n+1}|^2 = k_i,$$

strongly in $H^{\frac{1}{2}}(\Gamma)$.

Let us now prove that

$$\lim_{n \rightarrow \infty} \|\lambda|\mathbf{u}_1^n - \mathbf{u}_2^n|^2 - \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2\|_{H^{\frac{1}{2}}(\Gamma)} = 0.$$

Using the identity $a^2 - b^2 = (a - b)(a + b)$ and Lemma 3.5, we obtain

$$\|\lambda|\mathbf{u}_1^n - \mathbf{u}_2^n|^2 - \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2\|_{H^{\frac{1}{2}}(\Gamma)} \leq \lambda \|(\mathbf{u}_1^n - \mathbf{u}_1) - (\mathbf{u}_2^n - \mathbf{u}_2)\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \|(\mathbf{u}_1^n + \mathbf{u}_1) - (\mathbf{u}_2^n + \mathbf{u}_2)\|_{W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma)^d},$$

so,

$$\begin{aligned} &\|\lambda|\mathbf{u}_1^n - \mathbf{u}_2^n|^2 - \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2\|_{H^{\frac{1}{2}}(\Gamma)} \\ &\leq \lambda \left[\|\mathbf{u}_1^n - \mathbf{u}_1\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} + \|\mathbf{u}_2^n - \mathbf{u}_2\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)} \right] \left[\|\mathbf{u}_1^n + \mathbf{u}_1\|_{W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma)^d} + \|\mathbf{u}_2^n + \mathbf{u}_2\|_{W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma)^d} \right]. \end{aligned}$$

Due to the continuity of the trace operators from $W^{1,3+\varepsilon}(\Omega_1)^d$ to $W^{1-\frac{1}{3+\varepsilon},3+\varepsilon}(\Gamma)^d$ and from $\mathbf{H}^1(\Omega_i)$ to $\mathbf{H}^{\frac{1}{2}}(\Gamma)$, and by Hypothesis 3.1, there exists a positive constant c , that only depends on the domains Ω_i , such that

$$\|\lambda|\mathbf{u}_1^n - \mathbf{u}_2^n|^2 - \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2\|_{H^{\frac{1}{2}}(\Gamma)} \leq cM\lambda (\|\mathbf{u}_1^n - \mathbf{u}_1\|_{\mathbf{H}^1(\Omega_1)} + \|\mathbf{u}_2^n - \mathbf{u}_2\|_{\mathbf{H}^1(\Omega_2)}).$$

Finally, due to the strong convergence of $(\mathbf{u}_i^n)_n$ to \mathbf{u}_i in $\mathbf{H}^1(\Omega_i)$, we deduce that

$$\lim_{n \rightarrow \infty} \|\lambda|\mathbf{u}_1^n - \mathbf{u}_2^n|^2 - \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2\|_{H^{\frac{1}{2}}(\Gamma)} = 0.$$

Consequently, $k_i = \lambda|\mathbf{u}_1 - \mathbf{u}_2|^2$ on Γ . This, finishes the proof of this theorem. \square

5. NUMERICAL EXPERIMENTS

To conclude this paper, we use the algorithm introduced to solve the interaction of the ocean and the atmosphere in a simplified geometry. The discretization is performed using a Spectral method based on Legendre polynomials (see [2,10] or [11] for instance) that we have implemented in **FreeFEM3D**⁴.

The algorithm presented in this paper is nonlinear, thus it cannot be used “as is”. Various strategies could be employed to treat this nonlinearity (Newton or fixed point algorithms, etc.). According to the monotonic nature of the nonlinear friction boundary condition on the boundary Γ , we choose to linearize the term

$$\int_{\Gamma} |\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \cdot \mathbf{v}_i d\tau$$

of Problem 1. We replace it by

$$\int_{\Gamma} |\mathbf{u}_i^n - \mathbf{u}_j^n|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \cdot \mathbf{v}_i d\tau.$$

This term is linear and still monotonic in the unknowns $(\mathbf{u}_1^{n+1}, \mathbf{u}_2^{n+1})$, due to property (3.4). The problems for $(\mathbf{u}_1^{n+1}, p_1^{n+1})$ and $(\mathbf{u}_2^{n+1}, p_2^{n+1})$ still are coupled but the overall problem is linear and admits a unique solution:

Obtain $(\mathbf{u}_i^{n+1}, p_i^{n+1}, k_i^{n+1}) \in \mathbf{X}_i \times L^2(\Omega_i) \times W^{1,r'}(\Omega_i)$, such that $\forall (\mathbf{v}_i, q_i, \varphi_i) \in \mathbf{X}_i \times L^2(\Omega_i) \times W_0^{1,r}(\Omega_i)$,

$$a_i(k_i^n; \mathbf{u}_i^{n+1}, \nabla \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i^{n+1}) + \kappa_i \int_{\Gamma} |\mathbf{u}_i^n - \mathbf{u}_j^n|(\mathbf{u}_i^{n+1} - \mathbf{u}_j^{n+1}) \cdot \mathbf{v}_i d\tau = \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{v}_i d\tau, \quad (5.1)$$

$$\text{and} \quad b_i(\mathbf{u}_i^{n+1}, q_i) = 0. \quad (5.2)$$

To perform a somewhat realistic computation, we consider turbulent viscosities α_i and γ_i with the structure $\nu_t + \ell\sqrt{k}$. We consider the data reported in [5]:

- Geometry:
 - $\Omega_1 =]0, 5[\times]0, 1[\times]0, 1[$ describes the atmosphere;
 - $\Omega_1 =]0, 5[\times]0, -1[\times]0, 1[$ is the ocean.
- Physical data (taken from [5]):
 - $\gamma_1(k_1) = 3 \times 10^{-3} + 0.277 \times 10^{-4} \sqrt{k_1}$;
 - $\gamma_2(k_2) = 3 \times 10^{-2} + 0.185 \times 10^{-5} \sqrt{k_2}$;
 - $\alpha_i(\cdot) = \gamma_i(\cdot)$, for $i = 1, 2$.
- Friction coefficients (coming also from [5]):
 - $\kappa_i = 10^{-3}$, for $i = 1, 2$; and
 - $\lambda = 5 \times 10^{-2}$.

These data correspond to an air-sea flow, each modeled by a simplified TKE-mixing layer turbulence model. The mixing lengths are calculated by wall laws. The friction coefficients, in their turn, are just tentative. The physical units are in MKS system. The velocity boundary conditions imposes that $\mathbf{u}_1 = \mathbf{0}$ on $\Gamma_1 \setminus \tilde{\Gamma}_1$, $\mathbf{u}_1 = (1, 0, 0)$ on $\tilde{\Gamma}_1$ and $\mathbf{u}_2 = \mathbf{0}$ on Γ_2 , where $\tilde{\Gamma}_1$ is the upper face ($y = 1$) of Ω_1 . These settings are chosen in order to create a driven cavity-like flow in Ω_1 . One expects to generate another driven cavity-like flow in Ω_2 rotating in the opposite sense.

⁴<http://www.freefem.org/ff3d/>.

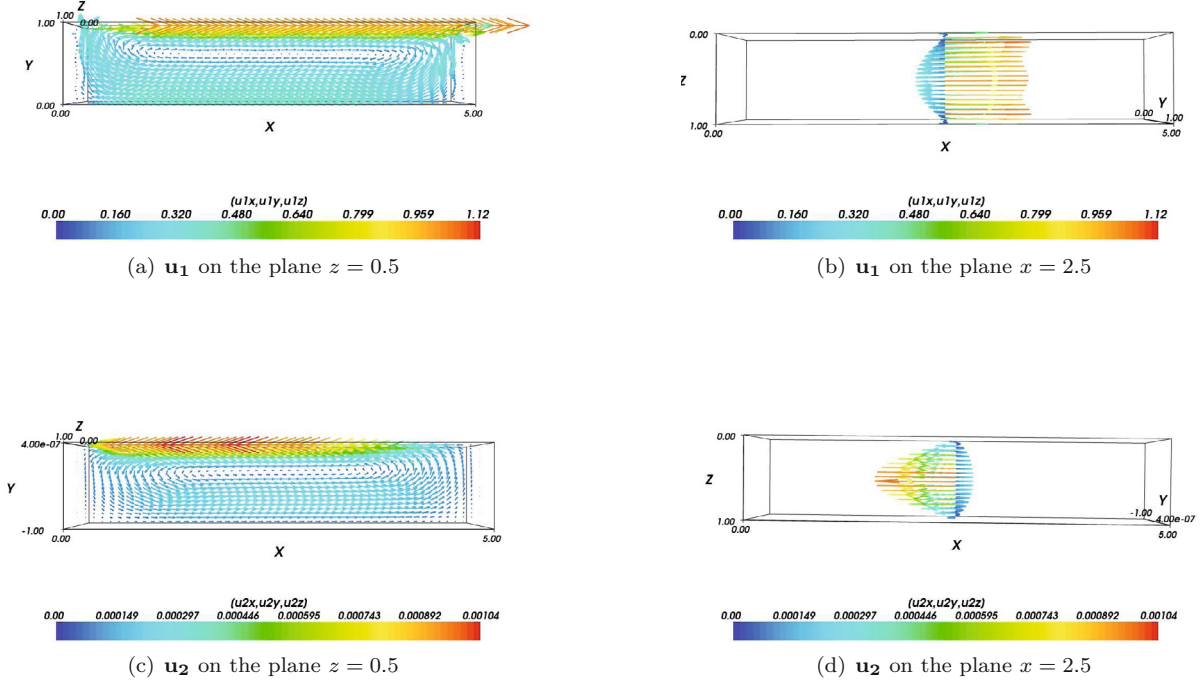


FIGURE 1. Velocity fields on cutting planes.

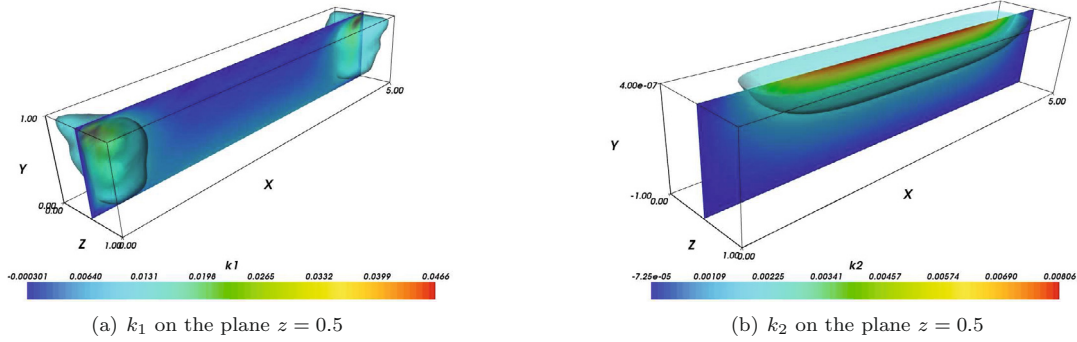


FIGURE 2. Turbulent Kinetic Energy on cutting plane.

The results are obtained using a $(\mathbb{P}_N)^3 \times \mathbb{P}_{N-2}$ discretization of (\mathbf{u}_i, p_i) to avoid spurious modes of the Stokes problem. The TKE is discretized using a \mathbb{P}_N space. For this particular simulation we have chosen the following degrees for \mathbf{u}_i and k_i : 28 in direction x and 8 in directions y and z .

Computed velocity fields \mathbf{u}_i and the TKE k_i are represented in Figures 1 and 2. The results are quantitatively correct: The atmosphere flow generates a driven-lid like flow in the ocean, due to the boundary conditions on the velocity at $z = 0$. Also, there is generation of TKE at $z = 0$, again due to the TKE generation boundary conditions.

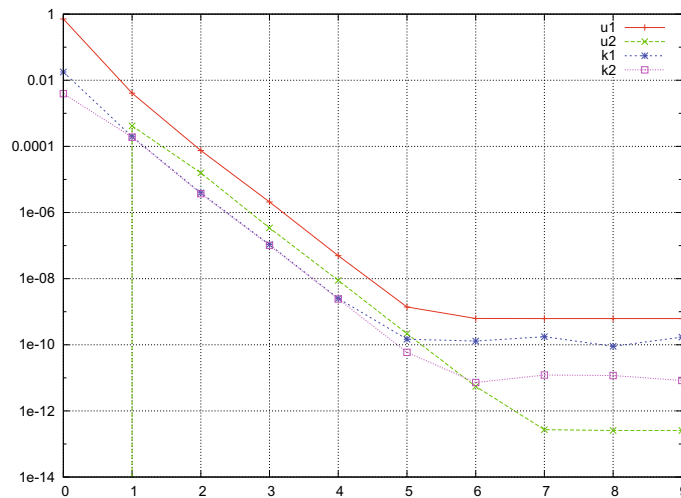


FIGURE 3. Convergence history: computed L^2 norm of the difference of successive iterates.

Figure 3 shows the expected exponential convergence rate of the algorithm due to its contractiveness. Note that this does not depend on the type of discretization (see [24] where Finite Element approximations have also been used).

6. CONCLUSION

In this paper, we have presented and analyzed a numerical scheme for the approximation of a model of two steady turbulent fluids with coupling at the interface. This is a simplified model for the atmosphere-ocean interaction, where we have neglected Coriolis forces and buoyancy effects, but have kept several non-linear interactions across the common boundary.

The proposed scheme is mainly linear, a monotone nonlinearity being just kept at the interface between the fluids. We showed the convergence of the triple $(\mathbf{u}_i^n, p_i^n, k_i^n)$, for reasonable hypothesis on the regularity of the velocity and the turbulent kinetic energy. This contribution ends with some numerical results, in good agreement with the theoretical expectations. Notably, the exponential convergence that appeared through the contractiveness of the sequences $(\mathbf{u}_i^n)_n$ and $(k_i^n)_n$ is found in our tests.

Several extensions to this work can be considered. To name a few, taking into consideration anisotropic diffusion and Coriolis forces (more realistic) are straightforward generalizations of the present analysis. Also, switching to unsteady incompressible flows should be possible using our approach. Taking into account buoyancy effects is more technically involved, but it should also be possible, similarly to the extension of the standard analysis for incompressible Navier–Stokes to buoyancy effects.

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REFERENCES

- [1] J.J.F. Adams and R.A. Fournier, *Sobolev spaces*. Second edition, Pure and Applied Mathematics Series, Elsevier/Academic Press (2003).
- [2] C. Bernardi and Y. Maday, *Approximations spectrales de problèmes aux limites elliptiques*, *Mathematics & Applications* **10**. Springer-Verlag (1992).
- [3] C. Bernardi, T. Chacón Rebollo, R. Lewandowski and F. Murat, A model for two coupled turbulent fluids. I. Analysis of the system, in *Nonlinear partial differential equations and their applications*, *Collège de France Seminar*, Vol. **XIV** (Paris, 1997/1998), *Stud. Math. Appl.* **31**, Amsterdam, North-Holland (2002) 69–102.

- [4] C. Bernardi, T. Chacón Rebollo, R. Lewandowski and F. Murat, A model for two coupled turbulent fluids. II. Numerical analysis of a spectral discretization. *SIAM J. Numer. Anal.* **40** (2003) 2368–2394.
- [5] C. Bernardi, T. Chacón Rebollo, M. Gómez Mármol, R. Lewandowski and F. Murat, A model for two coupled turbulent fluids. III. Numerical approximation by finite elements. *Numer. Math.* **98** (2004) 33–66.
- [6] C. Bernardi, T. Chacón Rebollo, F. Hecht and R. Lewandowski, Automatic insertion of a turbulence model in the finite element discretization of the Navier-Stokes Equations. *Math. Mod. Meth. Appl. Sci.* **19** (2009) 1139–1183.
- [7] H. Brezis, *Analyse Fonctionnelle : Théorie et Applications*. Collection “Mathématiques Appliquées pour la Maîtrise”, Masson (1983).
- [8] F. Brossier and R. Lewandowski, Impact of the variations of the mixing length in a first order turbulent closure system. *ESAIM: M2AN* **36** (2002) 345–372.
- [9] K. Bryan, A numerical method for the study of the circulation of the world ocean. *J. Comput. Phys.* **4** (1969) 347–369.
- [10] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral methods – Fundamentals in single domains*. Springer, Berlin, Germany (2006).
- [11] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, *Spectral methods – Evolution to complex geometries and applications to fluid dynamics*. Springer, Berlin, Germany (2007).
- [12] S. Del Pino and O. Pironneau, A fictitious domain based on general pde’s solvers, in *Proc. ECCOMAS 2001, Swansea*, K. Morgan Ed., Wiley (2002).
- [13] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*. Springer-Verlag, Germany (1986).
- [14] P. Grisvard, *Elliptic Problems in Nonsmooth Domains, Monographs and Studies in Mathematics* **24**. Pitman (Advanced Publishing Program), Boston, USA (1985).
- [15] E. Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities, Courant Lecture Notes* **5**. American Mathematical Society, USA (1999).
- [16] B.E. Launder and D.B. Spalding, *Mathematical Modeling of Turbulence*. Academic Press, London, UK (1972).
- [17] J. Lederer and R. Lewandowski, A RANS 3D model with unbounded eddy viscosities. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **24** (2007) 413–441.
- [18] R. Lewandowski, *Analyse Mathématique et Océanographie*. Collection Recherches en Mathématiques Appliquées, Masson (1997).
- [19] R. Lewandowski, The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier-Stokes equation with an eddy viscosity. *Nonlinear Anal.* **28** (1997) 393–417.
- [20] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications* **3**, *Travaux et Recherches Mathématiques* **20**. Dunod, Paris, France (1970).
- [21] B. Mohammadi and O. Pironneau, *Analysis of the k-epsilon turbulence model*. RAM: Research in Applied Mathematics. Masson, Paris (1994).
- [22] J. Piquet, *Turbulent Flows, Models and Physics*. Springer, Germany (1999).
- [23] D.C. Wilcox, *Turbulence Modeling for CFD*. Sixth edition, DCW Industries, inc. California, USA (2006).
- [24] D. Yakoubi, *Analyse et mise en œuvre de nouveaux algorithmes en méthodes spectrales*. Ph.D. Thesis, Université Pierre et Marie Curie, Paris, France (2007).